# FRACTIONAL LOOP GROUP AND TWISTED K-THEORY

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Abstract. We study the structure of abelian extensions of the group $L_qG$ of $q$-differentiable loops (in the Sobolev sense), generalizing from the case of central extension of the smooth loop group. This is motivated by the aim of understanding the problems with current algebras in higher dimensions. Highest weight modules are constructed for the Lie algebra. The construction is extended to the current algebra of supersymmetric Wess-Zumino-Witten model. An application to the twisted K-theory on $G$ is discussed.

## 1. Introduction

The main motivation for the present paper comes from trying to understand the representation theory of groups of gauge transformations in higher dimensions than one. In the case of a circle, the relevant group is the loop group $LG$ of smooth functions on the unit circle $S^1$ taking values in a compact Lie group $G$. In quantum field theory one considers representations of a central extension $\widehat{LG}$ of $LG$; in case when $G$ is semisimple, this corresponds to an affine Lie algebra. The requirement that the energy is bounded from below leads to the study of highest weight representations of $\widehat{LG}$. This part of the representation theory is well understood, [6].

In higher dimensions much less is known. Quantum field theory gives us a candidate for an extension of the gauge group $Map(M, G)$, the group of smooth mappings from a compact manifold $M$ to a compact group $G$. The extension is not central, but by an abelian ideal. The geometric reason for this is that the curvature form of the determinant line bundle over the moduli space of gauge connections (the Chern class of which is determined by a quantum anomaly) is not homogeneous; it is not invariant under left (or right) translations, [11].

There are two main obstructions when trying to extend the representation theory of affine Lie algebras to the case of $Map(M, \mathfrak{g})$. The first is that there is no natural polarization giving meaning to the highest weight condition; on $S^1$ the polarization is given by the decomposition of loops to positive and negative Fourier modes. The second obstruction has to do with renormalization problems in higher dimensions. On the circle, with respect to the Fourier polarization, one can use methods of canonical quantization for producing representations of the loop group; the only renormalization needed is the normal ordering of quantities quadratic in the fermion field, [9], [17]. In higher dimensions further renormalization is needed, leading to an action of the gauge group, not in a single Hilbert space, but in a Hilbert bundle over the space of gauge connections, [12].

In this paper we make partial progress in trying to resolve the two obstructions above. We consider instead of $LG$ the group $L_qG$ of loops which are not smooth but only differentiable of order $0 < q < \infty$ in the Sobolev sense, the *fractional loop group.* In the range $\frac{1}{2} \leq q$ the usual theory of highest weight representations is valid, the cocycle determining the central extension is well defined down to the critical order $q = \frac{1}{2}$. However, for $q < \frac{1}{2}$ we have to use again a "renormalized" cocycle defining an abelian extension of the group $L_qG$, similar as in the case of $Map(M, G)$. The renormalization means that the restriction of the 2-cocycle to the smooth subgroup $LG \subset L_qG$ is equal to the 2-cocycle $c$ for the central extension (affine Kac-Moody algebra) plus a coboundary $\delta\eta$ of a 1-cochain $\eta$ (the renormalization 1-cochain). The 1-cochain is defined only on $LG$ and does not extend to $L_qG$ when $q < 1/2$, only the sum $c + \delta\eta$ is well-defined on $L_qG$.

The important difference between $L_qG$ and $Map(M, G)$ is that in the former case we still have a natural polarization of the Lie algebra into positive and negative Fourier modes and we can still talk about highest weight modules for the Lie algebra $L_q\mathfrak{g}$.

Because of the existence of the highest weight modules for $L_q\mathfrak{g}$ we can even define the supercharge operator $Q$ for the supersymmetric Wess-Zumino-Witten model. In the case of the central extension the supercharge is defined as a product of a fermion field on the circle and the gauge current; this is well defined because the vacuum is annihilated both by the negative frequencies of the fermion field and the current, thus when acting on the vacuum only the (finite number of) zero Fourier modes remain. This property is still intact in the case of the abelian extension of the current algebra.

We can also introduce a family of supercharges, by a "minimal coupling" to a gauge connection on the loop group. In the case of central extension the connections on $LG$ can be taken to be left invariant and they are written as a fixed connection plus a left invariant 1-form $A$ on $LG$. The form $A$ at the identity element is identified as a vector in the dual $L\mathfrak{g}^*$ which again is identified, through an invariant inner product, as a vector in $L\mathfrak{g}$. This vector in turn defines a $\mathfrak{g}$-valued 1-form on the circle. The left translations on $LG$ induce the gauge action on the potentials $A$. Modulo the action of the group $\Omega G$ of based loops, the set of vector potentials on the circle is equal to the group $G$ of holonomies. In this way the family of supercharges parametrized by $A$ defines an element in the twisted K-theory of $G$. Here the twist is equal to an integral 3-cohomology class on $G$ fixed by the level $k$ of the loop group representation, [13].

In the case of $L_qG$ and the abelian extension, the connections are not invariant under the action of $L_qG$ and thus we have to consider the larger family of supercharges parametrized by the space $\mathcal{A}$ of all connections of a circle bundle over $L_qG$. This is still an affine space, the extension of $L_qG$ acts on it. The family of supercharges transforms equivariantly under the extension and it follows that it can be viewed as an element in twisted K-theory of the moduli stack $\mathcal{A}//\widehat{L_qG}$. This replaces the G-equivariant twisted K-theory on $\mathcal{A}//\widehat{LG}$ in case of the central extension $\widehat{LG}$, the latter being equivalent to twisted G-equivariant K-theory on the group $G$ of gauge holonomies.

The paper is organized as follows. In Section 2 we introduce the fractional loop group and consider its role as the gauge group of a fractional Dirac-Yang-Mills system. It turns out that the natural setting is a spectral triple in the sense of non-commutative geometry. Interestingly enough, similar attempts have been made recently, [4]. We move on to discuss the embedding $L_qG \subset GL_p$ in Section 3 and the construction of Lie algebra cocycles in Section 4. Finally the last two sections are devoted to extending the current algebra of the supersymmetric WZW model to the fractional case and discussing its application to the twisted K-theory of $G$.

## 2. Fractional Loop Group

Let $G$ denote a compact semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Fix a faithful representation $\rho : G \to GL(V)$ in a finite dimensional complex vector space $V$.

**Definition 2.1.** *The fractional loop group $L_qG$ for real index $\frac{1}{2} < q$ is defined to be the Sobolev space,*

$$L_qG := H^q(S^1, G) = \{g \in Map(S^1, G) \mid \|g\|^2_{2,q} = \sum_{k\in\mathbb{Z}} (1+k^2)^q |\rho(g_k)|^2 < \infty\} \ ,$$

*where $|\rho(g_k)|$ is the standard matrix norm of the $k$:th Fourier component of $g : S^1 \to G$. The group operation is given by pointwise multiplication $(g_1g_2)(x) = g_1(x)g_2(x)$.*

There is a natural Hilbert Lie group structure on $L_qG$ for $\frac{1}{2} < q$. It is defined by the Hilbert space completion of the Lie algebra of smooth maps $C^\infty(S^1, \mathfrak{g})$ with respect to the Sobolev inner product. The exponential map $\exp : H^q(S^1, \mathfrak{g}) \to H^q(S^1, G)$ provides a local chart near the identity and is extended to an atlas by left translations.

For our purposes however, we will use a Banach topology on the Lie algebra,

$$L_q\mathfrak{g} = \{X \in L^\infty(S^1, \mathfrak{g}) \mid ||X|| = ||X||_\infty + ||X||_{2,q} < \infty \} ,$$

where $L^\infty$ is the set of measurable essentially bounded functions and $|| \, ||_\infty$ denotes the supremum norm. This induces a Banach Lie group structure on $L_qG$, [17] and is a more appropriate topology for the applications in this paper. In fact, we have natural inclusions of Lie groups $LG \subset L_qG \subset L^cG$, where $LG$ is the smooth loop group and $L^cG$ is the Banach-Lie group of continuous loops in $G$, see Section 6. In the next section, we will modify the definition of $L_qG$ with respect to a different norm, to allow for values $0 < q \leq \frac{1}{2}$.

In Yang-Mills theory on the cylinder $S^1 \times \mathbb{R}$, the loop group $LG$ appears as the group of (time-independent) local gauge transformations. Quantization of massless chiral fermions in external Yang-Mills fields breaks the local gauge symmetry, leading to a central extension of $LG$. In order to make sense of this in the fractional setting, we need a notion of fractional differentiation. The study of fractional calculus dates back to early 18th century and a comprehensive review can be found in [18]. The transition to fractional calculus is by no means unique. There are several competing definitions, but many are known to coincide on overlapping domains. For functions on the real line the Riemann-Liouville fractional derivative is defined by

$$D^q_a\psi(x) = \frac{d^n}{dx^n}\left\{\frac{1}{\Gamma(n-q)}\int_a^x \frac{\psi(y)}{(x-y)^{q-n+1}}dy\right\}$$

for $n > q$ and $x > a$. On the circle however this definition proves inconvenient since periodic functions are not mapped onto periodic ones. An operator that do preserve periodicity is the Weyl fractional derivative,

$$\widetilde{D}^q\psi(x) = \sum_{k\in\mathbb{Z}}(ik)^q\psi_k e^{ikx} = \sum_{k\in\mathbb{Z}} e^{\frac{iq\pi}{2}\operatorname{sgn}(k)}|k|^q\psi_k e^{ikx}$$

for all $q \in \mathbb{R}$. In fact, by extending to the real line one shows that $\widetilde{D}^q$ and $D^q_a$ coincide for $-1 < q$ and $a = -\infty$ on an appropriate domain. For our purposes, we need a self-adjoint operator on a dense domain in $\mathcal{H} = L^2(S^1, V)$. The fractional Dirac operator on the circle is therefore defined by

$$D^q\psi(x) = \sum_{k\in\mathbb{Z}}\operatorname{sgn}(k)|k|^q\psi_k e^{ikx} ,$$

where the complex phase has been replaced by the sign function. This has the consequence that $D^q \circ D^r \neq D^{q+r}$, but we have instead $D^q \circ D^r = |D^{q+r}|$. For odd integers $q > 1$, the fractional Dirac operator is simply the $q$-th power of the rotation operator $-i\frac{d}{dx}$ on the circle. The domain of $D^q$ is the Sobolev space $H^q(S^1, V)$. It is by construction an unbounded, self-adjoint operator with discrete spectrum $\{\operatorname{sgn}(k)|k|^q\}_{k\in\mathbb{Z}}$

and a complete set of eigenstates in $\mathcal{H}$. However, the Leibniz rule is no longer satisfied

$$D^q(\psi\phi) = \sum_{k,m\in\mathbb{Z}} \mathrm{sgn}(k)|k|^q \psi_m \phi_{k-m} e^{ikx} \neq$$

$$\neq (D^q\psi)\phi + \psi(D^q\phi) = \sum_{k,m\in\mathbb{Z}} \Big( \mathrm{sgn}(m)|m|^q + \mathrm{sgn}(k-m)|k-m|^q \Big) \psi_m \phi_{k-m} e^{ikx} ,$$

unless $q = 1$.

We introduce interactions by imposing local gauge invariance. The covariant derivative $D^q_A = D^q + A$ should transform equivariantly under gauge transformations

$$g^{-1} D^q_A g = D^q + g^{-1}[D^q, g] + g^{-1} A g = D^q_{A^g} .$$

This motivates the following definition of fractional Yang-Mills connections on the circle;

$$A = \alpha[D^q, \beta], \quad \alpha, \beta \in H^q(S^1, \mathfrak{g}) .$$

The fractional loop group $L_q G$ acts on $A$ by

$$(g, A) \mapsto A^g = g^{-1} A g + g^{-1}[D^q, g]$$

and the infinitesimal gauge action is given by

$$(X, A) \mapsto \mathcal{L}_X A = [A, X] + [D^q, X] .$$

For values $\frac{1}{2} < q \leq 1$, there is a geometric interpretation of this data in the non-commutative geometry sense. What we have is precisely a spectral triple, namely a Dirac operator $D^q$, a Hilbert space $\mathcal{H}$ and an associative $*$-algebra $L_q\mathbb{C}$, [2]. Here $L_q\mathbb{C} = H^q(S^1, \mathbb{C})$ is an algebra for $\frac{1}{2} < q$ by the Sobolev multiplication theorem.

**Proposition 2.2.** $[D^q, X]$ *is a bounded operator for all* $X \in L_q\mathbb{C}$ *and* $0 < q \leq 1$,

$$\|[D^q, X]\| = \sup_{\substack{\psi\in\mathcal{H} \\ \|\psi\|=1}} \|[D^q, X]\psi\| < \infty .$$

*Proof.* By expanding in Fourier series,

$$[D^q, X]\psi = \sum_{k,m\in\mathbb{Z}} X_m \psi_{k-m} \Big( \mathrm{sgn}(k)|k|^q - \mathrm{sgn}(k-m)|k-m|^q \Big) e^{i(k+m)x}$$

it follows that

$$\begin{aligned} \|[D^q, X]\psi\|^2 &= \sum_{m,k\in\mathbb{Z}} |X_m \psi_{k-m}|^2 \Big( \mathrm{sgn}(k)|k|^q - \mathrm{sgn}(k-m)|k-m|^q \Big)^2 \\ &= \sum_{m,n\in\mathbb{Z}} |X_m \psi_n|^2 \Big( \mathrm{sgn}(n+m)|n+m|^q - \mathrm{sgn}(n)|n|^q \Big)^2 . \end{aligned}$$

Since the sequence $|\psi_n|^2$ belongs to $l^1$, the sum converges if

$$\sum_{m\in\mathbb{Z}} |X_m|^2 \Big( \operatorname{sgn}(n+m)|n+m|^q - \operatorname{sgn}(n)|n|^q \Big)^2 < C$$

is uniformly bounded by some constant $C$. To establish this, we rewrite

$$\sum_{m\in\mathbb{Z}} |X_m|^2 \Big( \operatorname{sgn}(n+m)|n+m|^q - \operatorname{sgn}(n)|n|^q \Big)^2 =$$

$$= \sum_{m\in\mathbb{Z}} |X_m|^2 m^{2q} \left( \operatorname{sgn}\left(1+\frac{m}{n}\right) \left|1+\frac{n}{m}\right|^q - \left|\frac{n}{m}\right|^q \right)^2 .$$

For $0 < q \leq 1$, this sum is bounded for all $n \in \mathbb{Z}$ since

$$f(x) = \operatorname{sgn}\left(1+\frac{1}{x}\right)|1+x|^q - |x|^q$$

is a bounded function on the real line. □

We have an immediate corollary:

**Corollary 2.3.** *For $\frac{1}{2} \leq q \leq 1$ the space $L_q\mathbb{C}$ is the algebra of essentially bounded measurable loops $X$ such that $[D^q, X]$ is a bounded operator.*

The inverse statement follows from the observation that taking $\psi(x) = 1$, the constant loop in the proof above, the norm $||[D^q, X]\psi||$ is equal to $||X||_{2,q}$.

Similarly one verifies that $[|D^q|, X]$ is bounded for all $X \in L_q\mathbb{C}$. Recall that a spectral triple is $p^+$-summable if $|D^q|^{-p}$ belongs to the Dixmier ideal $\mathcal{L}^{1+}$. This means that for some real number $p \geq 1$,

$$\lim_{N\to\infty} \frac{1}{\log(N)} \sum_{k=1}^{N} \lambda_k < \infty$$

where $\lambda_k \geq \lambda_{k+1} \geq \lambda_{k+2} \ldots$ are eigenvalues of $|D^q|^{-p}$ listed in descending order. In our case

$$\lambda_k = \frac{1}{k^{qp}}$$

for $k = 1, 2, \ldots$. This gives

$$\lim_{N\to\infty} \frac{1}{\log(N)} \sum_{k=1}^{N} \frac{1}{k^{qp}}$$

which is finite if and only if $qp \geq 1$. Moreover, since

$$\left\| \operatorname{ad}^n_{|D|^q}(X)\psi \right\|^2 = \sum_{m,k\in\mathbb{Z}} |X_m \psi_{k-m}|^2 \Big( |k|^q - |k-m|^q \Big)^{2n}$$

diverges for $n > 1$, we conclude that the spectral triple is not tame.

It is also interesting to note that even though the algebra $L_q\mathbb{C}$ is commutative, the spectral dimension $p$ of the circle is strictly larger than one when $q < 1$.

Fractional differentiability has been studied systematically in the more general context of $\theta$-summable spectral triples in [5]. However, their definition of fractional differentiability, although similar, differs from ours which is geared to the special case of loop groups and $L_p$-summable spectral triples.

## 3. Embedding of $L_qG$ in $GL_p$

When dealing with representations of the loop group $LG$, one is lead to consider central extensions by the circle

$$1 \to S^1 \to \widehat{LG} \to LG \to 1 \ .$$

In Fourier basis, the generators $S_n^a$ of the Lie algebra $\widehat{L\mathfrak{g}}$ satisfy

$$[S_n^a, S_m^b] = \lambda^{abc} S_{n+m}^c + kn\delta^{ab}\delta_{n,-m}$$

where $k$ is the central element (represented as multiplication by a scalar in an irreducible representation). Here the upper index refers to a normalized basis (with respect to an invariant nondegenerate bilinear form) of the Lie algebra $\mathbf{g}$ of the group $G$. The $\lambda^{abc}$'s are the structure constants in this basis. We have adopted the Einstein summation convention meaning that an index appearing twice in a term is summed over all its possible values. When trying to extend the central extension to the fractional setting, one runs immediately into a problem. For infinite linear combinations of the Fourier modes $S_n^a$ the central term blows up. A precise condition for the divergence can be formulated by regarding $L_qG$ as a group of operators in a Hilbert space. Let $\rho : G \to GL(V)$ denote a representation of $G$ as in the previous section. Elements in the fractional loop group act as multiplication operators in $\mathcal{H} = L^2(S^1, V)$ by pointwise multiplication,

$$(M_g\psi)(x) = \rho(g(x))\psi(x)$$

for all $g \in L_qG, \ \psi \in \mathcal{H}$. In fact, $M : L_qG \to GL(\mathcal{H}), \ g \mapsto M_g$ defines a continuous embedding into the general linear group. This statement is somewhat crude however. Below we show that $L_qG$ is actually contained in a subgroup $GL_p$ of $GL$. Recall that the sign operator $\epsilon = \frac{D^q}{|D^q|}$ defines an orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into positive and negative Fourier modes. We use the convention that the zero mode of $D^q$ is on the positive side of the spectrum. Introduce the Schatten class

$$L_{2p} = \{A \in \mathcal{B}(\mathcal{H}) \mid \ \|A\|_{2p} = \left[\mathrm{Tr}(A^\dagger A)^p\right]^{\frac{1}{2p}} < \infty\}$$

which is a two-sided ideal in the algebra of bounded operators $\mathcal{B}(\mathcal{H})$. In particular, the first Schatten class $L_1$ is the space of trace class operators and $L_2$ is the space of Hilbert-Schmidt operators. We will use an equivalent norm which is better suited for computations

$$\|A\|_{2p} = \left[\sum_{k\in\mathbb{Z}} \|A\phi_k\|^{2p}\right]^{\frac{1}{2p}}$$

where $\{\phi_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis in $\mathcal{H}$. The subgroup $GL_p \subset GL(\mathcal{H})$ is defined by

$$GL_p = \{A \in GL(\mathcal{H}) \mid [\epsilon, A] \in L_{2p}\} \ .$$

Writing elements in $GL(\mathcal{H})$ in block form with respect to the Hilbert space polarization

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} \ ,$$

the condition

$$[\epsilon, A] = 2\begin{pmatrix} 0 & A_{+-} \\ -A_{-+} & 0 \end{pmatrix} \in L_{2p}$$

simply means that the off-diagonal blocks are not "too large". Given the topology defined by the norm

$$|||A|||_p = \|A_{++}\| + \|A_{+-}\|_{2p} + \|A_{-+}\|_{2p} + \|A_{--}\| \ ,$$

where

$$\|a\| = \sup_{\|\psi\|=1} \|a\psi\|$$

denotes the operator norm, $GL_p$ is a Banach Lie group with the Lie algebra

$$\mathfrak{gl}_p = \{X \in \mathcal{B}(\mathcal{H}) \mid [\epsilon, X] \in L_{2p}\} \ .$$

**Proposition 3.1.** *If $p \geq \frac{1}{2q}$, then $L_qG$ is contained in $GL_p$.*

*Proof.* In order to avoid cumbersome notation, we write $g(x)$ instead of $\rho(g(x))$. Expanding in Fourier series $g(x) = \sum_{k\in\mathbb{Z}} g_k e^{ikx}$, we have

$$M_g e^{ikx} = \sum_{m\in\mathbb{Z}} g_m e^{i(m+k)x} = \sum_{m\in\mathbb{Z}} g_{m-k} e^{imx}$$

so that $(M_g)_{mk} = g_{m-k}$. Moreover

$$\epsilon e^{ikx} = \begin{cases} \frac{k}{|k|} e^{ikx} & k \neq 0 \\ e^{ikx} & k = 0 \end{cases}$$

We check that $\|[\epsilon, M_g]\|_{2p}$ is finite;

$$\begin{aligned}
\|[\epsilon, M_g]\|_{2p}^{2p} &= \sum_{k\in\mathbb{Z}} \left\|[\epsilon, M_g]e^{ikx}\right\|^{2p} \\
&= \sum_{m,k\in\mathbb{Z}} \left|\frac{m}{|m|} - \frac{k}{|k|}\right|^{2p} |g_{k-m}e^{imx}|^{2p} \\
&= \sum_{n,k\in\mathbb{Z}} \left|\frac{(n+k)}{|n+k|} - \frac{k}{|k|}\right|^{2p} |g_n|^{2p} \\
&= 2^{2p}\sum_{n\in\mathbb{Z}} |n||g_n|^{2p} \\
&= 2^{2p}\sum_{n\in\mathbb{Z}} ||n|^{\frac{1}{2p}} g_n|^{2p} \\
&\leq 2^{2p}\sum_{n\in\mathbb{Z}} (n^2)^{\frac{1}{2p}} |g_n|^2 + C_g
\end{aligned}$$

where $C_g$ is the finite part of the sum containing all terms where $|n|^{\frac{1}{2p}}|g_n| \geq 1$. Thus the sum converges if the Sobolev norm

$$\|g\|_{2,\frac{1}{2p}}^2 = \sum_{n\in\mathbb{Z}} (1+n^2)^{\frac{1}{2p}} |g_n|^2$$

is finite, which is the case when

$$p \geq \frac{1}{2q} \ .$$

□

The same arguments apply without modification to the Lie algebra $L_q\mathfrak{g} \hookrightarrow \mathfrak{gl}_p$. We see here precisely how the degree of differentiability $q$ is related to the Schatten index $p$. Furthermore, this allows us to extend the definition of $L_qG$ to all real values $0 < q < \infty$:

**Definition 3.2.** *The fractional loop group $L_qG$ is defined to be the group of continuous loops contained in $GL_p$, where $p = max\left\{\frac{1}{2}, \frac{1}{2q}\right\}$. We shall use the induced Banach structure on the Lie algebra $L_q\mathfrak{g}$ coming from the embedding, defined by the norm*

$$\|X\|_\infty + \||X\||_p \ .$$

*This endows $L_qG$ with a Banach Lie group structure.*

**Remark 3.3.** By abuse of notation, we use the same label $L_qG$ as in Definition 2.1 for this slightly larger fractional loop group. Actually, it follows by Proposition 3.1 that the group in Definition 2.1 is continuously embedded in $L_qG$, for $q > \frac{1}{2}$. For the remainder of this paper, we shall adopt Definition 3.2 for the fractional loop group. Also, note that for $q \geq 1$, $L_qG$ consists of continuous loops contained in $GL_{\frac{1}{2}}$, that is operators whose off-diagonal blocks are trace class. This is the critical value for the Schatten index $p$, since for lower values than $\frac{1}{2}$ the spaces $L_{2p}$ are no longer ideals.

The cocycle defining the central extension $\widehat{L\mathfrak{g}}$ can be written

$$c_0(X,Y) = \frac{1}{8}\mathrm{Tr}\Big(\epsilon[[\epsilon,X],[\epsilon,Y]]\Big) = \mathrm{Tr}\Big(X_{+-}Y_{-+} - Y_{+-}X_{-+}\Big)$$

for $X,Y \in L\mathfrak{g}$. This is finite so long as the off-diagonal blocks are Hilbert-Schmidt, i.e. $[\epsilon,X]$ and $[\epsilon,Y]$ belong to $L_2$. However for $p>1$, corresponding to differentiability $q$ less than $\frac{1}{2}$, the operators $X_{+-}Y_{-+}$ and $Y_{+-}X_{-+}$ are no longer trace-class, because $a,b \in L_{2p}$ implies that $ab \in L_p$ by Hölder inequality. This is the reason behind the divergence. In order to make sense of the cocycle for higher $p$, regularization is required.

We introduce the Grassmannian $Gr_p$ [11], which is a smooth Banach manifold parametrized by idempotent hermitian operators $F$ such that $F-\epsilon \in L_{2p}$. Points on $Gr_p$ may also be thought of as closed subspaces $W \subset \mathcal{H}$ such that the orthogonal projection $\mathrm{pr}_+ : W \to \mathcal{H}_+$ is Fredholm and $\mathrm{pr}_- : W \to \mathcal{H}_-$ is in $L_{2p}$. Let $\eta(X;F)$ denote a 1-cochain parametrized by points on $Gr_p$. For a suitable choice of $\eta$, adding the coboundary to the original cocycle

$$c_p(X,Y;F) = c_0(X,Y) + (\delta\eta)(X,Y;F)\ ,$$

we obtain a well-defined cocycle on $\mathfrak{gl}_p$ for some fixed $p$. The Lie algebra cohomology coboundary operator $\delta$ is defined by Palais' formula in Section 4. Although each term on the right diverges separately, the sum will be finite. In the case $p=2$, we could take

$$\eta(X;F) = -\frac{1}{16}\mathrm{Tr}\Big([X,\epsilon][F,\epsilon]\Big)$$

which gives

$$c_2(X,Y;F) = c_0(X,Y) + (\delta\eta)(X,Y;F) = \frac{1}{8}\mathrm{Tr}\Big([[\epsilon,X],[\epsilon,Y]](\epsilon - F)\Big)\ .$$

An important consequence of this so called "infinite charge renormalization" is that the central extension $\widehat{L\mathfrak{g}}$ will be replaced by an abelian extension by the infinite-dimensional ideal $\mathrm{Map}(Gr_p,\mathbb{C})$,

$$0 \to \mathrm{Map}(Gr_p,\mathbb{C}) \to \widehat{L_q\mathfrak{g}} \to L_q\mathfrak{g} \to 0\ .$$

## 4. Lie algebra cocycles

In this section we will construct Lie algebra cocycles for $\mathfrak{gl}_p$ for all $p \geq \frac{1}{2}$, which by restriction yield cocycles on $L_q\mathfrak{g}$. In particular, we show that the cocycles respect the decomposition of $L_q\mathfrak{g} = L_q\mathfrak{g}_+ \oplus L_q\mathfrak{g}_-$ into positive and negative Fourier modes on the circle. The computation is done in the non-commutative BRST bicomplex, where the classical de Rham complex is replaced by a graded differential algebra $(\Omega, d)$. Let $\epsilon = \frac{D^q}{|D^q|}$ denote the sign operator on the circle satisfying $\epsilon = \epsilon^*$ and $\epsilon^2 = 1$. We introduce a family of vector spaces

$$\begin{aligned}
\Omega^0 &= \{X \in \mathcal{B}(\mathcal{H}) \mid X - \epsilon X \epsilon \in L_{2p}\} \\
\Omega^{2k-1} &= \{X \in \mathcal{B}(\mathcal{H}) \mid X + \epsilon X \epsilon \in L_{\frac{2p}{2k}},\ X - \epsilon X \epsilon \in L_{\frac{2p}{2k-1}}\} \\
\Omega^{2k} &= \{X \in \mathcal{B}(\mathcal{H}) \mid X + \epsilon X \epsilon \in L_{\frac{2p}{2k}},\ X - \epsilon X \epsilon \in L_{\frac{2p}{2k+1}}\}
\end{aligned}$$

and set $\Omega = \oplus_{k=0}^{\infty} \Omega^k$. The exterior differentiation is defined by

$$dX = \begin{cases} [\epsilon, X], & \text{if } X \in \Omega^{2k} \\ \{\epsilon, X\}, & \text{if } X \in \Omega^{2k-1} \end{cases}$$

and satisfies $d^2 = 0$. Here $\{X, Y\} = XY + YX$ is the anticommutator. The space $\Omega^k$ consists of linear combinations of $k$-forms $X_0 dX_1 dX_2 \dots dX_k$ with $X_i \in \Omega^0$. If $X \in \Omega^k$ and $Y \in \Omega^l$, then

$$XY \in \Omega^{k+l}, \quad dX \in \Omega^{k+1}, \quad d(XY) = (dX)Y + (-1)^k X dY$$

which follows by the generalized Hölder inequality for Schatten ideals. This ensures that $(\Omega, d)$ is a $\mathbb{N}$-graded differential algebra. Integration of forms is substituted by a graded trace functional

$$\mathrm{Str}(X) = \mathrm{Tr}_C(\Gamma X), \text{ for } X \in \Omega^k \text{ and } k \geq p,$$

where $\Gamma$ is a grading operator on $\mathcal{H}$ and $\mathrm{Tr}_C(X) = \frac{1}{2}\mathrm{Tr}(X + \epsilon X \epsilon)$ is the conditional trace, [2]. In case of an even Fredholm module ($k$ even), $\Gamma$ anticommutes with $\epsilon$ and in case of an odd Fredholm module ($k$ odd), $\Gamma = 1$.

Next we define the Lie algebra chain complex $(C, \delta)$. Since $\Omega^0 = \mathfrak{gl}_p$ and $Gr_p \subset \Omega^1$, we interpret $B \in \Omega^1$ as generalized connection 1-forms and define the infinitesimal gauge action by

$$\Omega^0 \times \Omega^1 \to \Omega^1, \quad (X, B) \mapsto \mathcal{L}_X B = [B, X] + dX \ .$$

Indeed for any $F \in Gr_p$ we have $F = g^{-1} \epsilon g$ for some $g \in GL_p$, since $GL_p$ acts transitively on the Grassmannian. Thus $F - \epsilon = g^{-1} dg$ corresponds to flat connections. The abelian group $Map(\Omega^1, \Omega)$ is naturally a $\Omega^0$-module under the action

$$\Omega^0 \times Map(\Omega^1, \Omega) \to Map(\Omega^1, \Omega), \quad (X, f) \mapsto \mathcal{L}_X f(B) = \frac{d}{dt} f\Big(e^{-tX} B e^{tX} + t dX\Big)\Big|_{t=0} .$$

Define the space of $k$-chains $C^k$ as alternating multilinear maps

$$\omega : \underbrace{\Omega^0 \times \ldots \times \Omega^0}_{k} \to Map(\Omega^1, \Omega)$$

and set $C = \oplus_{k=0}^{\infty} C^k$. The coboundary operator is given by Palais' formula

$$\begin{aligned}\delta\omega(X_1,\ldots,X_k;B) &= \sum_{j=1}^{k}(-1)^{j+1}\mathcal{L}_{X_j}\omega(X_1,\ldots,\hat{X}_j,\ldots,X_k;B)\\ &\quad+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k;B)\end{aligned}$$

and satisfies $\delta^2 = 0$. Here $\hat{X}_j$ means that the variable $X_j$ is omitted.

This provides us with a double complex. For our purposes, we need cocycles of degree 2 in the Lie algebra cohomology. There is a straightforward way to compute such cocycles parametrized by *flat* connections $B = F - \epsilon$. Given a bicomplex with commuting differentials, there is an associated singly graded complex with differential $D = \delta + (-1)^k d$. The Lie algebra 2-cocycle is given by

$$\begin{aligned}\tilde{c}_p(X,Y;B) &= \frac{2^{2p}}{(2p+1)}\,\mathrm{Str}\big(g^{-1}Dg\big)^{2p+3}_{[2p+1,2]}(X,Y)\\ &= 2^{2p}\,\mathrm{Str}\left(\sum_{k=0}^{p}(g^{-1}\mathrm{d}g)^{2p+1-k}(g^{-1}\delta g)(g^{-1}\mathrm{d}g)^{k}(g^{-1}\delta g)\right)(X,Y)\\ &= 2^{2p}\,\mathrm{Str}\left(\sum_{k=0}^{p}(-1)^k\big(B^{2p+1-k}XB^kY - B^{2p+1-k}YB^kX\big)\right)\end{aligned}$$

for all $p \geq 0$. By $(\ldots)_{[j,k]}$ we mean the component of degree $(j,k)$ in $(d,\delta)$ cohomology. That this is a cocycle follows from

$$\delta\mathrm{Str}(g^{-1}Dg)^{2p+3} = \mathrm{Str}D(g^{-1}Dg)^{2p+3} = \mathrm{Str}(g^{-1}Dg)^{2p+4} = 0$$

where we have used that $\mathrm{Str}(g^{-1}Dg)^{even} = 0$. Although $\tilde{c}_p(X,Y;B)$ does not vanish when $X, Y \in L_q\mathfrak{g}_-$, we have:

**Theorem 4.1.** *The cocycle $\tilde{c}_p$ is cohomologous to a cocycle $c_p$ which has the property that $c_p(X,Y;B) = 0$ if both $X$ and $Y$ are in $L_q\mathfrak{g}_-$ (or both in $L_q\mathfrak{g}_+$).*

**Remark 4.2.** Note that we can replace the abelian ideal of smooth functions of the variable $B$ by the space of functions on $L_qG$ by using the embedding of $L_qG$ to the Grassmannian $Gr_p$ given by $g \mapsto B = g^{-1}[\epsilon, g]$.

The proof of Theorem 4.1 is by direct computation and is shifted to the Appendix.

**Theorem 4.3.** *The cocycle $c_p$ in the Appendix, when restricted to the Lie algebra of smooth loops, is cohomologous to the cocycle defining the standard central extension of the loop algebra $L\mathfrak{g}$.*

*Proof.* Define $\eta_p(X;B) = 2^{2p+1}\ \mathrm{Tr}\big(\epsilon B^{2p+1} dX\big)$. Then by direct computation, similar to the one in the Appendix and which will not be repeated here,

$$c_{p+1}(X,Y;B) = c_p(X,Y;B) - (\delta\eta_p)(X,Y;B)\ .$$

Thus $c_p$ is cohomologous to $c_{p-1}$ and by induction to the cocycle $c_0$. But

$$c_0(X,Y;B) = \frac{1}{2}\mathrm{Tr}\big(XdY\big)$$

which is precisely the cocycle defining the central extension of the loop algebra. □

We note that the 1-cochain $\eta(X;F)$ in Section 3 is simply the sum $-\sum_{k=0}^{p-1}\eta_k(X;B)$, where $B = F - \epsilon$.

**Remark 4.4.** Since the Lie algebra cocycle vanishes on $L_q\mathfrak{g}_-$ one can define generalized Verma modules as the quotient of the universal enveloping algebra $\mathcal{U}(\widehat{L_q\mathfrak{g}})$ by the left ideal generated by $L_q\mathfrak{g}_-$ and by the elements $h - \lambda(h)$ where the $h$'s are elements in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and $\lambda \in \mathfrak{h}^*$ is a weight. In the case of the central extension of the smooth loop algebra, for dominant integral weights one can construct an invariant hermitean form in the Verma module; there is a subquotient (including the highest weight vector) which carries an irreducible unitarizable representation of the Lie algebra. However, in the case of $L_q\mathfrak{g}$ for $q < 1/2$ this is not possible: due to the large abelian ideal in the extension of $L_q\mathfrak{g}$ we cannot construct any invariant hermitean semidefinite form on the Verma module. Whether this is possible at all is an open question, although we conjecture that the answer is negative.

## 5. Generalized supersymmetric WZW model

Let us recall the construction of a family of supercharges $Q(A)$ for the supersymmetric Wess-Zumino-Witten model in the setting of representations of the smooth loop algebra, [13], [8], [3].

Here we assume that $G$ is a connected, simply connected simple compact Lie group of dimension $N$. Let $\mathcal{H}_b$ denote the "bosonic" Hilbert space carrying an irreducible unitary highest weight representation of the loop algebra $\widehat{L\mathfrak{g}}$ of level $k$. The level $k$ is a non-negative integer and we introduce $k' = 2\theta^2 k$, where $\theta$ is the length of the longest root of $G$. The generators of the loop algebra in Fourier basis $T_n^a$ satisfy

$$[T_n^a, T_m^b] = \lambda^{abc} T_{n+m}^c + \frac{k'}{4} n\delta^{ab}\delta_{n,-m}$$

where $n \in \mathbb{Z}$ and $a = 1, 2, \ldots, N$. We fix an orthonormal basis $T^a$ in $\mathfrak{g}$, with respect to the Killing form, so that the structure constants $\lambda^{abc}$ are completely antisymmetric and the Casimir invariant $C_2 = \lambda^{abc}\lambda^{acb}$ equals $-N$. Moreover, we have

$$(T_n^a)^* = -T_{-n}^a \ .$$

The fermionic Hilbert space $\mathcal{H}_f$ carries an irreducible representation of the canonical anticommutation relations (CAR)

$$\{\psi_n^a, \psi_m^b\} = 2\delta^{ab}\delta_{n,-m}$$

where $(\psi_n^a)^* = \psi_{-n}^a$. The Fock vacuum is a subspace of $\mathcal{H}_f$ of dimension $2^{[\frac{N}{2}]}$. It carries an irreducible representation of the Clifford algebra spanned by the zero modes $\psi_0^a$ and lies in the kernel of all $\psi_n^a$ with $n < 0$. The loop algebra $\widehat{L\mathfrak{g}}$ acts in $\mathcal{H}_f$ through the minimal representation of level $h^\vee$, the dual Coxeter number of $G$. The operators are explicitly realized as bilinears in the Clifford generators

$$K_n^a = -\frac{1}{4}\lambda^{abc} : \psi_{n-m}^b \psi_m^c :$$

and satisfy

$$[K_n^a, K_m^b] = \lambda^{abc} K_{n+m}^c + \frac{h'^\vee}{4} n\delta^{ab}\delta_{n,-m} \ ,$$

where $h'^\vee = 2\theta^2 h^\vee$. The normal ordering : : indicates that operators with positive Fourier index are placed to the left of those with negative index. In case of fermions there is a change of sign, $: \psi_{-n}^a \psi_n^b := -\psi_n^b \psi_{-n}^a$ if $n > 0$. The full Hilbert space $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$ carries a tensor product representation of $\widehat{L\mathfrak{g}}$ of level $k + h^\vee$, with generators $S_n^a = T_n^a \otimes \mathbf{1} + \mathbf{1} \otimes K_n^a$.

The supercharge operator is defined by

$$Q = i\psi_n^a \left( T_{-n}^a + \frac{1}{3} K_{-n}^a \right)$$

and squares to the free Hamilton operator $h = Q^2$ of the supersymmetric WZW model

$$h = - : T_n^a T_{-n}^a : + \frac{\bar{k}}{2} : n\psi_n^a \psi_{-n}^a : + \frac{N}{24} = h_b + 2\bar{k} h_f + \frac{N}{24}$$

where $\bar{k} = \frac{k' + h'^\vee}{4}$. Interaction with external $\mathfrak{g}$-valued 1-forms $A$ on the circle is introduced by minimal coupling

$$Q(A) = Q + i\bar{k}\psi_n^a A_{-n}^a$$

where $A_n^a$ are the Fourier coefficients of $A$ in the basis $T_n^a$, satisfying $(A_n^a)^* = -A_{-n}^a$. This provides us with a family of self-adjoint Fredholm operators $Q(A)$ that is equivariant with respect to the action of $\widehat{LG}$,

$$S(g)^{-1} Q(A) S(g) = Q(A^g)$$

with $A^g = g^{-1}Ag + g^{-1}dg$. Infinitesimally this translates to

$$[S_n^a, Q(A)] = i\bar{k}(n\psi_n^a + \lambda^{abc}\psi_{n+m}^c A_{-m}^b) = -\mathcal{L}_n^a Q(A) \ .$$

where $\mathcal{L}_n^a$ denotes the Lie derivative (infinitesimal gauge transformation) in the direction $X = S_n^a$. The interacting Hamiltonian $h(A) = Q(A)^2$ is given by

$$h(A) = h - \bar{k}\left(2S_n^a A_{-n}^a + \bar{k} A_n^a A_{-n}^a\right) = h + h_{int} \ .$$

Next we consider extending this construction to the fractional setting. As previously mentioned, this necessarily entails certain regularization. Let us first denote by $S_0$ the representation of the smooth loop algebra with commutation relations

$$[S_0(X), S_0(Y)] = S_0([X, Y]) + c_0(X, Y) \ .$$

We proceed by adding a 1-cochain

$$\begin{aligned} S(X) &= S_0(X) + \eta(X; B) \\ Q &= Q_0 + \eta(\psi; B) \end{aligned}$$

where in component notation $\eta(\psi; B) = \psi_n^a \eta(T_{-n}^a; B) = \psi_n^a \eta_{-n}^a$ and we denote the original supercharge by $Q_0$ from now on. Here $B = g^{-1}[\epsilon, g]$ parametrizes points on the Grassmannian, as in Section 4, with $g \in L_q G$. We write also $\mathcal{L}_X = X_n^a \mathcal{L}_{-n}^a$ for an element $X \in L_q\mathfrak{g}$. Adding cochains of the type $\eta$, which are functions of the variable $B$, means that we are extending the original loop algebra by Fréchet differentiable functions of $B$. Since gauge transformations are acting on $B$ by the formula $B \mapsto g^{-1}Bg + g^{-1}[\epsilon, g]$, or infinitesimally as $B \mapsto [B, X] + [\epsilon, X]$ for $X \in L_q\mathfrak{g}$, the commutator of $S(X)$ by any Fréchet differentiable function $f$ of $B$ is given as

$$[S(X), f(B)] = \mathcal{L}_X f(B) \ .$$

The other new commutation relations will be

$$\begin{aligned} [S(X), S(Y)] &= S([X, Y]) + c(X, Y; B) \\ [S(X), \psi(Y)] &= \psi([X, Y]) \\ [S(X), Q] &= ic(X, \psi; B) \\ \{Q, \psi(Y)\} &= 2iS(Y) \end{aligned}$$

where $c(X, Y; B) = c_0(X, Y) + (\delta\eta)(X, Y; B)$ converges for an appropriate choice of $\eta$, according to Theorem 4.3. Moreover, we set $h = Q^2$ where

$$Q^2 = Q_0^2 - 2S(\eta) - \eta^2 + i\mathcal{L}_\psi \eta(\psi; B) \ ,$$

$\eta^2 = \eta^a_n \eta^a_{-n}$ and $S(\eta) = \eta^a_n S^a_{-n}$. In Fourier basis, the generators $\{\psi^a_n, S^b_m, Q, h\}$ satisfy the following commutation relations,

$$\begin{aligned}
\{\psi^a_n, \psi^b_m\} &= 2\delta^{ab}\delta_{n,-m} \\
[S^a_n, S^b_m] &= \lambda^{abc} S^c_{n+m} + c^{a,b}_{n,m}(B) \\
[S^a_n, \psi^b_m] &= \lambda^{abc}\psi^c_{n+m} \\
\{\psi^a_n, Q\} &= 2iS^a_n \\
[S^a_n, Q] &= ic^{a,b}_{n,-m}(B)\psi^b_m \\
[\psi^a_n, h] &= 2c^{a,b}_{n,-m}(B)\psi^b_m \\
[h, S^a_n] &= 2S^b_m c^{a,b}_{n,-m}(B) - \psi^b_m \psi^c_p \mathcal{L}^c_{-p} c^{a,b}_{n,-m}(B) \\
[Q, h] &= 0
\end{aligned}$$

where $c^{a,b}_{n,m}(B) = c(S^a_n, S^b_m; B)$. For any Fréchet differentiable function $f = f(B)$,

$$\begin{aligned}
[S^a_n, f] &= \mathcal{L}^a_n f \\
[Q, f] &= i\psi^a_n \mathcal{L}^a_{-n} f \\
[h, f] &= -2S^a_n \mathcal{L}^a_{-n} f + \psi^a_n \psi^d_q \mathcal{L}^d_{-q} \mathcal{L}^a_{-n} f \ .
\end{aligned}$$

In the smooth case, $c^{a,b}_{n,m}(B) = kn\delta^{ab}\delta_{n,-m}$, one recovers the corresponding subalgebra of the superconformal current algebra, [7]. Let us consider highest weight representations of the loop algebra generated by $S$ and $S_0$ respectively. Since they differ by a coboundary, one can explicitly relate their vacua by restricting to the subalgebra of smooth loops $L\mathfrak{g} \subset L_q\mathfrak{g}$. Indeed, we have

$$S(X)|\Omega >= 0, \quad S_0(X)|\Omega_0 >= 0$$

for all $X \in L\mathfrak{g}_-$, which implies

$$S(X)|\Omega_0 >= \Big(S_0(X) + \eta(X;B)\Big)|\Omega_0 >= \eta(X;B)|\Omega_0 > \neq 0 \ .$$

However if $(\delta\eta)(X,Y;B) = 0$ for all $X, Y \in L\mathfrak{g}_-$, then $\eta$ restricts to a 1-cocycle on $L\mathfrak{g}_-$ and can in fact be written $\eta(X;B) = \mathcal{L}_X\Phi(B)$ for some function $\Phi$ of the variable $B$ on the smooth Grassmannian consisting of points $g^{-1}[\epsilon, g]$ for $g \in LG$. For the cochain $\eta$ in Theorem 4.3 one can choose $\Phi(B) \sim \mathrm{Tr}\big(\epsilon B^{2p+1}\big)$ and the vacua are linked according to

$$|\Omega >= e^{-\Phi(B)}|\Omega_0 > \ .$$

Indeed for all $X \in L\mathfrak{g}_-$, we have

$$\begin{aligned}
S(X)|\Omega > &= e^{-\Phi(B)}\Big(S(X) - \mathcal{L}_X\Phi(B)\Big)|\Omega_0 > \\
&= e^{-\Phi(B)}\Big(S_0(X) + \eta(X;B) - \mathcal{L}_X\Phi(B)\Big)|\Omega_0 >= 0 \ .
\end{aligned}$$

## 6. Twisted K-theory and the group $L_qG$

We want to make sense of a family of supercharges $Q(A)$ which transforms equivariantly under the action of the abelian extension $\widehat{L_qG}$ of the fractional loop group $L_qG$. This should generalize the construction of the similar family in the case of central extension of the smooth loop group. Let us recall the relevance of the latter for twisted K-theory over $G$ of level $k+h^\vee$. We fix $G$ to be a simple compact Lie group throughout this section. One can think of elements in $K^*(G, k+h^\vee)$ as maps $f: \mathcal{A} \to Fred(\mathcal{H})$, to Fredholm operators in a Hilbert space $\mathcal{H}$, with the property $f(A^g) = \hat{g}^{-1} f(A) \hat{g}$. Here $g \in LG$ and $\mathcal{A}$ is the space of smooth $\mathfrak{g}$-valued vector potentials on the circle. The moduli space $\mathcal{A}/\Omega G$ (where $\Omega G$ is the group of based loops) can be identified as $G$. Actually, one can still use the equivariantness under constant loops so that we really deal with the case of $G$-equivariant twisted K-theory $K^*_G(G, k+h^\vee)$. For odd/even dimensional groups one gets elements in $K^1/K^0$.

The real motivation here is to try to understand the corresponding supercharge operator $Q$ arising from Yang-Mills theory in higher dimensions. If $M$ is a compact spin manifold the gauge group $Map(M, G)$ can be embedded in $U_p$ for any $2p > \dim M$; this was used in [10] for constructing a geometric realization for the extension of $Map(M, G)$ arising from quantization of chiral Dirac operators in background gauge fields. This is an analogy for our embedding of $L_qG$ in $U_p$ for $p > 1/2q$, the index $1/2q$ playing the role of the dimension of $M$.

The more modest aim here is to show that there is a true family of Fredholm operators which transforms covariantly under $L_qG$. The operators are parametrized by 1-forms on $L_qG$ and generalize the family of Fredholm operators $Q(A)$ from the smooth setting to the fractional case. Hopefully, this will help us understand the renormalizations needed for the corresponding problem in gauge theory on a manifold $M$.

Let us denote by $L^cG$ the Banach-Lie group of continuous loops in $G$. The natural topology of $L^cG$ is the metric topology defined as

$$d(f,g) = \sup_{x \in S^1} d_G(f(x), g(x))$$

where $d_G$ is the distance function on $G$ determined by the Riemann metric. Local charts on $L^cG$ are given by the inverse of the exponential function; at any point $f_0 \in L^cG$ we can map a sufficiently small open ball around $f_0$ to an open ball at zero in $L^c\mathfrak{g}$ by $f \mapsto \log(f_0^{-1} f)$.

In the smooth version $LG$ the topology is locally given by the topology on $L\mathfrak{g}$; the topology of the vector space $L\mathfrak{g}$ is defined by the family of seminorms $||X||_n =$

$\sup_{x\in S^1}|X^{(n)}(x)|$ for a fixed norm $|\cdot|$ on $\mathfrak{g}$. More precisely, we can define a family of distance functions on $LG$ by

$$d_0(f,g)=\sup_{x\in S^1} d_G(f(x),g(x))$$

for $n=0$, and

$$d_n(f,g)=\sup_{x\in S^1}|f^{(n)}(x)-g^{(n)}(x)|$$

for $n>0$, where $f^{(n)}$ is the n:th derivative with respect to the loop parameter: We identify the first derivative as a function with values in $\mathfrak{g}$ by left translation $f'\mapsto f^{-1}f'$ and then all the higher derivatives are $\mathfrak{g}$-valued functions on the circle.

The metric is then defined as

$$d(f,g)=\sum_{n\geq 0}\frac{d_n(f,g)}{1+d_n(f,g)}2^{-n}\ .$$

The subgroup $LG\subset L^cG$ is dense in the topology of the latter. For this reason the cohomology of $L^cG$ is completely determined by restriction to $LG$. Actually, we have a stronger statement:

**Lemma 6.1.** *(Carey-Crowley-Murray) The group $L^cG$ of continuous loops is homotopy equivalent to the smooth loop group $LG$.*

*Proof.* We may assume that $G$ is connected; otherwise, one repeats the proof for each component of $G$. When $G$ is connected the full loop group is a product of $G$ and the group $\Omega G$ of based loops (whether continuous, smooth, or of type $L_q$). So we restrict to the group of based loops. As shown in [1], the groups $\Omega^cG$ and $\Omega G$ are weakly homotopic, i.e. the inclusion $\Omega G\subset\Omega^cG$ induces an isomorphism of the homotopy groups. According to the Theorem 15 by R. Palais, [15] a weak homotopy equivalence of metrizable manifolds implies homotopy equivalence. Actually, in [1] the authors use the CW property of the loop groups for the last step. The CW property in the case of $\Omega^cG$ is a direct consequence of Theorem 3 in [14]. □

**Lemma 6.2.** *The group $L_qG$ is homotopy equivalent to $LG$ and thus also to $L^cG$.*

*Proof.* The proof in [1] can be directly adapted from the smooth setting to the larger group $L_qG$. Let $M$ be a compact manifold with base point $m_0$ and $C((I^n,\partial I^n),(M,m_0))$ the set of continuous maps from the n-dimensional unit cube to $M$ such that the boundary of the cube is mapped to $m_0$. The key step in their proof is the observation that in the homotopy class of any map $g\in C((I^n,\partial I^n),(M,m_0))$ there exists a smooth map; in addition, a homotopy can be given in terms of a differentiable map. Taking

$M = G$ and thinking of $g$ as a representative for an element in the homotopy group $\pi_{n-1}(\Omega^c G)$. Since $g$ is homotopic to a smooth map, it also represents an element in the smooth homotopy group of $\Omega G$ and thus also an element in the $(n-1)$:th homotopy group of $\Omega_q$. In addition, a continuous homotopy is equivalent to a smooth homotopy. The embedding of $LG \subset L_q G$ is continuous in their respective topologies [this follows from the Sobolev norm estimates in the proof of Proposition 3.1] and therefore the representatives for the homotopy groups of the former are mapped to representatives of the homotopy groups of the latter. □

Let $F : L^c G \to LG$ be a smooth homotopy equivalence. We define

$$\theta(f; g) = F(f)^{-1} F(fg)$$

for $f, g \in L^c G$. This is a 1-cocycle in the sense of

$$\theta(f; gg') = \theta(f; g)\theta(fg; g') \ .$$

For any $g \in LG$ we then have

$$\hat\theta(f; g)^{-1} Q_0 \hat\theta(f; g) = Q_0 + i\bar k < \psi, \theta(f; g)^{-1} \partial\theta(f; g) >$$

where $\hat\theta$ is the lift of $\theta$ to the central extension $\widehat{LG}$. Here $\partial$ is the differentiation with respect to the loop parameter.

Since the homotopy $F$ is smooth we can define a Lie algebra cocycle

$$d\theta(f; X) = \frac{d}{dt}|_{t=0} \theta(f; e^{tX})$$

with values in $L\mathfrak{g}$ for $X \in L^c\mathfrak{g}$. This is a 1-cocycle in the sense that

$$d\theta(f; [X, Y]) + \mathcal{L}_X d\theta(f; Y) - \mathcal{L}_Y d\theta(f; X) = [d\theta(f; X), d\theta(f; Y)] \ .$$

Let next $Q(A) = Q_0 + i\bar k < \psi, A >$ be a perturbation of $Q_0$ by a function $A : L^c G \to L^c\mathfrak{g}$. The group $L^c G$ acts on $A$ by right translation, $(g \cdot A)(f) = A(fg)$. Denote by $\hat\Theta(g)$ the operator consisting of the right translation on functions of $f$ and of $\hat\theta(\cdot; g)$ acting on values of functions in the Hilbert space $\mathcal{H}$. Then

$$\hat\Theta(g)^{-1} Q(A) \hat\Theta(g) = Q(A^g) \ ,$$

where

$$(A^g)(f) = \theta(f; g)^{-1} A(fg)\theta(f; g) + \theta(f; g)^{-1}\partial\theta(f; g) \ . \tag{1}$$

Since the group $LG$ acts in $\mathcal{H}$ through its central extension $\widehat{LG}$, the Lie algebra $L^c\mathfrak{g}$ acts through its abelian extension by $Map(L^c G, i\mathbb{R})$, the extension being defined by the 2-cocycle

$$\omega(f; X, Y) = [\widehat{d\theta}(f; X), \widehat{d\theta}(f; Y)] - \widehat{d\theta}(f; [X, Y]) - \mathcal{L}_X \widehat{d\theta}(f; Y) + \mathcal{L}_Y \widehat{d\theta}(f; X) \ .$$

For an infinitesimal gauge transformations $X$, the formula (1) leads to

$$\delta_X A = [A, d\theta(f;X)] + \partial\theta(f;X) + \mathcal{L}_X A \tag{2}$$

which should be compared with $\delta_X A = [A, X] + \partial X$ in the smooth case, for constant functions $A : LG \to L\mathfrak{g}$.

The Lie algebra cohomology of any Lie algebra, with coefficients in the module of smooth functions on the Lie group $G$, is by definition the same as the de Rham cohomology of $G$. Indeed, take a cocycle $c$ in de Rham cohomology on $G$. It is an alternating multilinear form on on the space of vector fields on $G$, with values in the space of smooth functions on $G$. We can restrict it to left invariant vector fields on $G$; but the left invariant vector fields are just elements of the Lie algebra of $G$. So we obtain an alternating multilinear form on the Lie algebra of $G$, with values in the space of smooth functions on $G$. Looking at the definition of a de Rham cocycle (in terms of smooth vector fields) one sees that the cocycle condition is exactly the same as the Lie algebra cocycle condition, with values in $C^\infty(G)$, with the standard action of the Lie algebra (as derivations) on functions. Thus we get a linear map from the space of de Rham cocycles to Lie algebra cocycles. Exact cocycles on the de Rham side map to exact Lie algebra cocycles, so we have a map between the cohomologies. This is an isomorphism since the de Rham forms are uniquely determined by the restriction to left invariant vector fields (at each point on G the left invariant vector fields form a basis).

We apply this to the loop group $LG$ together with the fact that the standard central extension of $L\mathfrak{g}$, when viewed as a left invariant 2-form on $LG$, generates $H^2(LG, \mathbb{R})$. Thus the Lie algebra cohomology of $L\mathfrak{g}$ with coefficients in the module of smooth functions on $LG$ is one dimensional.

By the Lemma 6.2 we have

**Lemma 6.3.** *For a simple compact Lie group $G$ the Lie algebra cohomology in degree 2 of $L^c\mathfrak{g}$ with coefficients in the module of smooth functions on $L^cG$ (with respect to the Banach manifold structure) is one dimensional and the class of a cocycle is fixed by the restriction to the smooth version $LG$.*

**Remark 6.4.** We can easily produce an explicit homotopy connecting the standard 2-cocycle on $L\mathfrak{g}$ to the restriction of the 2-cocycle on $L^c\mathfrak{g}$ to the smooth subalgebra $L\mathfrak{g}$. Let $F_s : LG \to LG$ be any one-parameter family of maps connecting the identity $F_1$ to the map $F_0 = F \circ i : LG \to LG$ where $i : LG \to L^cG$ is the inclusion and $F : L^cG \to LG$ is the homotopy equivalence used before, $0 \leq s \leq 1$. On $LG$ we define

$$\eta = \int_0^1 \mathrm{Tr}\Big([\epsilon, F_s^{-1}\delta F_s] F_s^{-1}\partial_s F_s\Big) \ .$$

Denote $c_0(X,Y) = \frac{1}{2}\mathrm{Tr}\big(X[\epsilon, Y]\big)$, the standard Lie algebra 2-cocycle on $L\mathfrak{g}$, and $c_1(X,Y) = \frac{1}{2}\mathrm{Tr}\big(d\theta(f;X)[\epsilon, d\theta(f;Y)]\big)$ the 2-cocycle coming from the homotopy $F_0$. Then one checks that, for the restriction of $c_1$ to $LG$,

$$c_0 - c_1 = \delta\eta \ .$$

**Proposition 6.5.** *The unitary subgroup $U_p \subset GL_p$ for each $p \geq \frac{1}{2}$ is smoothly homotopic to $U_{\frac{1}{2}}$.*

*Proof.* We prove the claim inductively by constructing a homotopy equivalence from $U_p$ to $U_{p/2}$ for all $p \geq 1$. Let $g \in U_p$ with

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} .$$

Let us denote $x = \alpha\gamma^* - \beta\delta^* \in L_{2p}$. Define the unitary operator $h(g)$

$$h(g) = \exp \begin{pmatrix} 0 & -x/2 \\ x^*/2 & 0 \end{pmatrix} .$$

Let $F(g) = h(g)^{-1}g$. Using $x^2 \in L_p$, by a direct computation one checks that the upper right block in this operator is equal to

$$\beta + \frac{1}{2}(\alpha\gamma^* - \beta\delta^*)\delta \bmod L_p \ .$$

Using the unitarity relations $\alpha^*\beta + \gamma^*\delta = 0$ and $\beta^*\beta + \delta^*\delta = 1$ we see that the above operator is equal to $\beta\beta^*\beta$. Since $\beta \in L_{2p}$, by the operator Hölder inequalities $\beta\beta^*\beta$ is in $L_{2p/3} \subset L_p$. Thus $F(g) \in U_{p/2}$. This map is homotopic to the identity map in $U_p$ : one just needs to replace the operator $x$ by $tx$ where $0 \leq t \leq 1$. At $t = 0$ we then have $F_t(g) = g$. Since the blocks of $F(g)$ are rational functions in the blocks of $g$ without singularities, the map $g \mapsto F(g)$ is smooth in the natural Banach manifold structures of the groups $U_p$ and $U_{p/2}$. The same argument shows that the identity map on $U_r$ contracts to a map to the subgroup $U_{p/2}$ for any $p \geq r \geq p/2$. □

According to [17] the restriction of the standard 2-cocycle of the Lie algebra of $U_1$ (denoted by $U_{res}$ in [17]) to the subalgebra $L\mathfrak{g}$ is the cocycle defining an affine Kac-Moody algebra. In combination with Lemma 6.3 and Proposition 6.5 we conclude:

**Proposition 6.6.** *The restriction from $L^cG$ to $L_qG$ of the Lie algebra cocycle $\omega$ is equivalent to the Lie algebra cocycle of Theorem 4.1 (obtained by restriction of $c_p$ from $GL_p$ to $L_qG$ with $p = max\left\{\frac{1}{2}, \frac{1}{2q}\right\}$).*

**Remark 6.7.** The discussion in Remark 6.4 applies here as well. By a similar formula one can produce an explicit homotopy between the standard 2-cocycle on the Lie algebra of $U_{\frac{1}{2}}$ and the cocycle obtained from the restriction from $U_p$ to the subgroup $U_{\frac{1}{2}}$, for $p \geq \frac{1}{2}$.

We close this section by showing that in the fractional setting, the transformation (2) of the field $A : L_qG \to L_q\mathfrak{g}$ under $L_qG$ has a more geometric interpretation.

Using the inner product $< X, Y >= \int_{S^1} (X(x), Y(x))_{\mathfrak{g}}\, dx$ in $L_q\mathfrak{g}$, where $(\cdot,\cdot)_{\mathfrak{g}}$ is an invariant inner product in $\mathfrak{g}$, and the fact a Lie group is a parallelizable manifold we can think of $A$ as a 1-form on the loop group $L_qG$. We denote by $\mathcal{A}$ the space of all Fréchet differentiable 1-forms on $L_qG$. There is a circle bundle $P$ over $L_qG$ with connection and curvature; the curvature $\omega$ is given by the cocycle of the abelian extension, $\omega(g; X, Y)$ where the elements $X, Y \in T_g(L_qG)$ are identified as left invariant vector fields (elements of the Lie algebra of $L_qG$). Let us denote by $\widehat{L_qG}$ the extension of $L_qG$ by the abelian normal subgroup $Map(L_qG, S^1)$ corresponding to the given Lie algebra extension.

Conversely, starting from the abelian extension $\widehat{L_qG}$, viewed as a principal bundle over $L_qG$ with fiber $Map(L_qG, S^1)$, we can recover the geometry of the the circle bundle $P$. The connection in $P$ is obtained as follows. First, the connection form in $\widehat{L_qG}$ is given as

$$\Psi = Ad_{\hat{g}}^{-1}\mathrm{pr}_c(d\hat{g}\hat{g}^{-1})$$

where $\mathrm{pr}_c$ is the projection onto the abelian ideal $Map(L_qG, i\mathbb{R})$.

**Remark 6.8.** In the case of the central extension $\widehat{LG}$ of $LG$ the adjoint action on $\mathrm{pr}_c(\bullet)$ is trivial but in the case of the abelian extension of $L_qG$ it is needed in order to guarantee that the connection form is tautological in the vertical directions in the tangent bundle $TP$.

The connection $\nabla$ on $P$ is then defined using the identification of $P$ as the subbundle of $\widehat{L_qG} \to L_qG$ with fiber $S^1$ consisting of constant functions in $Map(L_qG, S^1)$, that is, at each base point $g \in L_qG$ we have the homomorphism $\gamma$ sending a function $f \in Map(L_qG, S^1)$ to the value of $f$ at the neutral element. This induces a homomorphism $d\gamma : Map(L_qG, i\mathbb{R}) \to i\mathbb{R}$ defining the $i\mathbb{R}$-valued 1-form $\psi = d\gamma \circ \Psi$, defining the covariant differentiation $\nabla$ in the associated complex line bundle $L$.

An arbitrary connection in the bundle $P$ is then written as a sum $\nabla + A$ with $A \in \mathcal{A}$.

**Theorem 6.9.** *The transformation of $\nabla + A$ under an infinitesimal right action of $(X, \alpha) \in \widehat{L_q\mathfrak{g}}$ induces the action $\delta A = \omega_g(X, \bullet) + \mathcal{L}_X A + \mathcal{L}_\bullet \alpha$. Here the Lie derivative acting on the from $A$ is composed from the directional derivative (by the left invariant* *vector field $X$) on the argument of $A$ and the commutator $[A, \theta(g; X)]$. That is, under the infinitesimal transformation $\delta_{(X,0)}$ the transformation is the same as in (2) using the identification of elements in $L_q\mathfrak{g}$ as elements in its dual through the inner product $< \cdot, \cdot >$ . The function $\alpha$ can be interpreted as an infinitesimal gauge transformation in the line bundle $L$ over $L_qG$.*

*Proof.* Let $(X, \alpha) \in \widehat{L_q\mathfrak{g}}$. Infinitesimally, the right action is generated by left invariant vector fields. Thus the infinitesimal shift in the direction $(X, \alpha)$, of the connection evaluated at the tangent vector $Y$ is the commutator $[\nabla_Y, \nabla_X + \alpha]$. The first term in the commutator is equal to the curvature of the circle bundle evaluated in the directions $X, Y$. This is in turn equal to $\omega_g(X, Y)$. Since the covariant derivative $\nabla_Y$ acts as the Lie derivative $\mathcal{L}_Y$ on functions, we obtain

$$\delta_{(X,\alpha)}(\nabla + A) = \omega_g(X, \bullet) + \mathcal{L}_X A + \mathcal{L}_\bullet \alpha \ , \tag{3}$$

□

It follows that we can morally interpret $Q(A)$ as a Dirac operator on the loop group $L_qG$ twisted by a complex line bundle.

In the case of a central extension the above formulas (on level $k + h^\vee$) reproduce the classical gauge action, after identification of the dual $L\mathbf{g}^*$ as $L\mathbf{g}$ using the Killing form,

$$A \mapsto [A, X] + \frac{1}{4}(k' + h'^\vee)dX, \tag{4}$$

for a left invariant 1-form $A$. Namely, the right Lie derivative in (3), when acting on left invariant forms, produces the commutator term $[A, X]$ and whereas the shift by $dX$ is coming from the central extension $\omega_g(X, \bullet) = c_0(X, \bullet)$. The third term on the right-hand-side of the equation (3) is absent since in case of central extension we can take $\alpha =$ constant.

Going the other way, starting from the action on $\mathcal{A}$ in terms of the cocycle $\omega$, we may take the restriction to the Lie algebra of smooth loops and the connection $\nabla$ becomes a sum

$$\nabla = \nabla_0 + \eta$$

where $\nabla_0$ is the connection in $P$ defined by the standard central extension of the loop group and $\eta$ is a 1-form on $LG$. The form $\eta$ is actually the 1-cochain relating the central extension to the abelian extension of $LG$. Writing now $a = \eta + A$ and $S(X) = S_0(X) + \eta(X)$ we recover the gauge action formula $a \mapsto [a, X] + \frac{1}{4}(k' + h'^\vee)dX$

with respect to $S_0(X)$. Thus the addition of $\eta$ can be viewed as a renormalization needed for extending from the case of smooth loop algebra to the fractional loops.

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## Appendix: Proof of Theorem 4.1

Define a 1-cochain by

$$\tilde{\eta}_p(X;B) = \mathrm{Str}\Big(B^{2p+1}X\Big)$$

for $p \geq 0$. Using Palais' formula the coboundary is given by

$$\begin{aligned}(\delta\tilde{\eta}_p)(X,Y;B) &= \sum_{k=0}^{2p}\mathrm{Str}\Big(B^k[B,X]B^{2p-k}Y - B^k[B,Y]B^{2p-k}X\Big)\\ &+\sum_{k=0}^{2p}\mathrm{Str}\Big(B^k\mathrm{d}XB^{2p-k}Y - B^k\mathrm{d}YB^{2p-k}X\Big) - \mathrm{Str}\Big(B^{2p+1}[X,Y]\Big)\ .\end{aligned}$$

By cyclicity of trace we can rewrite the first sum,

$$\sum_{k=0}^{2p}\mathrm{Str}\Big(B^k[B,X]B^{2p-k}Y - B^k[B,Y]B^{2p-k}X\Big) = \tag{5}$$

$$= \mathrm{Str}\Big(B[X,B^{2p}Y] - B[Y,B^{2p}X]\Big) + \sum_{k=1}^{2p}\mathrm{Str}\Big(B^k[B,X]B^{2p-k}Y - B^k[B,Y]B^{2p-k}X\Big)\ .$$

This can be simplified further. Using $[a,bc] = [a,b]c + b[a,c]$ repeatedly we get

$$B[X,B^{2p}Y] = B^{2p+1}[X,Y] + \sum_{k=0}^{2p-1}B^{2p-k}[X,B]B^kY \tag{6}$$

and inserting (6) into (5) yields

$$\begin{aligned}&2\mathrm{Str}\Big(B^{2p+1}[X,Y]\Big) + \sum_{k=0}^{2p-1}\mathrm{Str}\Big(B^{2p-k}[X,B]B^kY - B^{2p-k}[Y,B]B^kX\Big)\\ &+\sum_{k=1}^{2p}\mathrm{Str}\Big(B^k[B,X]B^{2p-k}Y - B^k[B,Y]B^{2p-k}X\Big) = 2\mathrm{Str}\Big(B^{2p+1}[X,Y]\Big)\ .\end{aligned}$$

Thus the expression for the coboundary $\delta\tilde{\eta}_p$ reduces to

$$(\delta\tilde{\eta}_p)(X,Y;B) = \mathrm{Str}\left(B^{2p+1}[X,Y] + \sum_{k=0}^{2p} B^k \mathrm{d}X B^{2p-k} Y - B^k \mathrm{d}Y B^{2p-k} X\right) . \tag{7}$$

We split the remaining sum into even and odd powers. When $k = 2m$ is even,

$$\sum_{m=0}^{p} \mathrm{Str}\Big(B^{2m}\mathrm{d}X B^{2p-2m} Y - B^{2m}\mathrm{d}Y B^{2p-2m} X\Big) =$$

$$= \sum_{m=0}^{p} \mathrm{Str}\Big(B^{2m}\mathrm{d}X B^{2p-2m} Y + B^{2m} Y B^{2p-2m}\mathrm{d}X\Big) \tag{8}$$

and when $k = 2m-1$ is odd,

$$\sum_{m=1}^{p} \mathrm{Str}\Big(B^{2m-1}\mathrm{d}X B^{2p-2m+1} Y - B^{2m-1}\mathrm{d}Y B^{2p-2m+1} X\Big) =$$

$$= \sum_{m=1}^{p} \mathrm{Str}\Big(B^{2m-1} X B^{2p-2m+2} Y - B^{2m} X B^{2p-2m+1} Y\Big) , \tag{9}$$

where we have used $\mathrm{d}B^{2m} = 0$, $\mathrm{d}B^{2m+1} = -B^{2m+2}$ and "integration by parts" using

$$\begin{aligned} \mathrm{d}(B^{2m-1} X B^{2p-2m+1} Y) &= -B^{2m} X B^{2p-2m+1} Y - B^{2m-1}\mathrm{d}X B^{2p-2m+1} Y + \\ &\quad + B^{2m-1} X B^{2p-2m+2} Y + B^{2m-1} X B^{2p-2m+1}\mathrm{d}Y . \end{aligned}$$

Finally we show that the sum in $\tilde{c}_p(X,Y;B)$ for values $k \geq 1$ equals (9), up to the normalization factor $2^{2p}$;

$$\sum_{k=1}^{p} (-1)^k \mathrm{Str}\Big(B^{2p-k+1} X B^k Y - B^{2p-k+1} Y B^k X\Big) =$$

$$= \sum_{m=1}^{p/2} \mathrm{Str}\Big(B^{2p-2m+2} Y B^{2m-1} X - B^{2p-2m+2} X B^{2m-1} Y\Big)$$

$$+ \sum_{m=1}^{p/2} \mathrm{Str}\Big(B^{2p-2m+1} X B^{2m} Y - B^{2p-2m+1} Y B^{2m} X\Big)$$

$$= \sum_{m=1}^{p/2} \mathrm{Str}\Big(B^{2m-1} X B^{2p-2m+2} Y - B^{2m} X B^{2p-2m+1} Y\Big) \tag{10}$$

$$+ \sum_{m=1}^{p/2} \mathrm{Str}\Big(B^{2p-2m+1} X B^{2m} Y - B^{2p-2m+2} X B^{2m-1} Y\Big) .$$

Shifting the index $m = n - \frac{p}{2}$ in the last sum in (10) we get

$$\sum_{n=(p+2)/2}^{p} \mathrm{Str}\Big(B^{3p-2n+1} X B^{2n-p} Y - B^{3p-2n+2} X B^{2n-p-1} Y\Big)$$

and reversing the summation order using

$$\sum_{\alpha}^{\beta} x^{\pm 2n} = \sum_{\alpha}^{\beta} x^{\pm 2(\alpha+\beta)\mp 2n}$$

this becomes

$$\sum_{n=(p+2)/2}^{p} \mathrm{Str}\Big(B^{2n-1}XB^{2p-2n+2}Y - B^{2n}XB^{2p-2n+1}Y\Big) .$$

Altogether this is precisely the odd part of the sum in (7). Now we define a new cocycle

$$\begin{aligned} c_p(X,Y;B) &= \tilde{c}_p(X,Y;B) - 2^{2p}(\delta\tilde{\eta}_p)(X,Y;B) \\ &= -2^{2p}\sum_{m=0}^{p} \mathrm{Str}\Big(B^{2m}\mathrm{d}XB^{2p-2m}Y - B^{2m}\mathrm{d}YB^{2p-2m}X\Big) \\ &= -2^{2p}\sum_{m=0}^{p} \mathrm{Str}\Big(B^{2m}\mathrm{d}XB^{2p-2m}Y + B^{2m}YB^{2p-2m}\mathrm{d}X\Big) . \end{aligned}$$

We note that this cocycle vanishes when both $X, Y$ have only negative (or positive) Fourier components, because the trace of $B^n\mathrm{d}XB^{2m}\mathrm{d}Y$ and $B^{2m}\mathrm{d}XB^{2n}Y$ are zero when $X, Y$ are upper (or lower) triangular matrices with respect to the $\epsilon$ grading.

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