BRIESKORN SPHERES, CYCLIC GROUP ACTIONS AND THE MILNOR CONJECTURE

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Abstract. In this paper we further develop the theory of equivariant Seiberg–Witten–Floer cohomology of the two authors, with an emphasis on Brieskorn homology spheres. We obtain a number of applications. First, we show that the knot concordance invariants $\theta^{(c)}$ defined by the first author satisfy $\theta^{(c)}(T_{a,b}) = (a-1)(b-1)/2$ for torus knots, whenever c is a prime not dividing ab. Since $\theta^{(c)}$ is a lower bound for the slice genus, this gives a new proof of the Milnor conjecture. Second, we prove that a free cyclic group action on a Brieskorn homology 3-sphere $Y = \Sigma(a_1, \dots, a_r)$ does not extend smoothly to any homology 4-ball bounding Y. In the case of a non-free cyclic group action of prime order, we prove that if the rank of $HF^+_{red}(Y)$ is greater than p times the rank of $HF^+_{red}(Y/\mathbb{Z}_p)$, then the $\mathbb{Z}_p$-action on Y does not extend smoothly to any homology 4-ball bounding Y. Third, we prove that for all but finitely many primes a similar non-extension result holds in the case that the bounding 4-manifold has positive definite intersection form. Finally, we also prove non-extension results for equivariant connected sums of Brieskorn homology spheres.

1. Introduction

In [5], we introduced the theory of equivariant Seiberg–Witten–Floer cohomology and established its basic properties. In this paper we further develop this theory, with a particular emphasis on Brieskorn homology spheres. Applications include a new proof of the Milnor conjecture and obstructions to extending group actions over a bounding 4-manifold.

For pairwise coprime positive integers $a_1, \dots, a_r > 1$, the Brieskorn manifold $Y = \Sigma(a_1, \dots, a_r)$ is an integral homology Seifert 3-manifold. The Seifert structure defines a circle action on Y. Restricting the circle action to finite subgroups, we obtain an action of the cyclic group $\mathbb{Z}_p$ on Y for each integer $p > 1$. We obtain our main results by considering the equivariant Seiberg–Witten–Floer cohomology of Y with respect to such $\mathbb{Z}_p$-actions.

1.1. **Knot concordance invariants and the Milnor conjecture.** Let p be a prime number. One way of producing $\mathbb{Z}_p$-actions on rational homology 3-spheres is to take $Y = \Sigma_p(K)$, the cyclic p-fold cover of S^3 branched over a knot K. From the equivariant Seiberg–Witten–Floer cohomology of Y one may extract invariants of the knot K. In [4], this construction was used to obtain a series of knot concordance invariants $\theta^{(p)}(K)$. These invariants are lower bounds for the slice genus $g_4(K)$,

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that is, $g_4(K) \geq \theta^{(p)}(K)$ for all primes p. More generally, the invariants $\theta^{(p)}$ can be used to bound the genus of surfaces bounding K in negative definite 4-manifolds with S^3 boundary.

We are interested in the case that K is a torus knot. If $K = T_{a,b}$ is an (a,b) torus knot and c is a prime not dividing ab, then $\Sigma_c(T_{a,b})$ is the Brieskorn homology sphere $\Sigma(a,b,c)$ and the $\mathbb{Z}_c$-action arising from the branched covering construction coincides with the restriction to $\mathbb{Z}_c$ of the Seifert circle action. By studying the equivariant Seiberg–Witten–Floer homology of $\Sigma(a,b,c)$, we deduce the following:

Theorem 1.1. *Let $a, b > 1$ be coprime integers and let c be a prime not dividing ab. Then $\theta^{(c)}(T_{a,b}) = \frac{1}{2}(a-1)(b-1)$.*

Since $\theta^{(c)}$ is a lower bound for the slice genus, we obtain the Milnor conjecture as an immediate corollary:

Corollary 1.2. *Let $a, b > 1$ be coprime. Then $g_4(T_{a,b}) = \frac{1}{2}(a-1)(b-1)$.*

Proof. The Milnor fibre of the singularity $x^a = y^b$ has genus $(a-1)(b-1)/2$ [29], hence $g_4(T_{a,b}) \leq (a-1)(b-1)/2$. On the other hand, if we let c be any prime not dividing ab, then Theorem 1.1 gives $g_4(T_{a,b}) \geq (a-1)(b-1)/2$. □

The original proof of the Milnor conjecture due to Kronheimer and Mrowka uses gauge theory and adjunction inequalities [18]. The result was proven again by Ozsváth and Szabó using the τ-invariant of Knot Floer homology [34] and by Rasmussen using the s-invariant of Khovanov homology [37]. Although our proof uses gauge theory, it does not use adjunction inequalities but rather is based on Floer theoretic methods. Thus our proof has more in common with Ozsváth–Szabó and Rasmussen than with Kronheimer–Mrowka. It is interesting to note that our proof, like those of Ozsváth–Szabó and Rasmussen, is based on finding a knot concordance invariant which bounds the slice genus and equals $(a-1)(b-1)/2$ for the torus knot $T_{a,b}$.

1.2. **Equivariant delta invariants of Brieskorn homology spheres.** Our next result concerns the equivariant delta invariants of Brieskorn homology spheres. The equivariant delta invariants, introduced in [5], are a certain equivariant generalisation of the Ozsváth–Szabó d-invariant and are equivariant homology cobordism invariants. Given a rational homology 3-sphere Y, an action of $\mathbb{Z}_p$ on Y by orientation preserving diffeomorphisms and a $\mathbb{Z}_p$-invariant spin^c-structure $\mathfrak{s}$, we obtain a sequence of invariants $\delta_j^{(p)}(Y, \mathfrak{s}) \in \mathbb{Q}$ indexed by a non-negative integer j. We call $\delta_j^{(p)}(Y, \mathfrak{s})$ the equivariant delta invariants of $(Y, \mathfrak{s})$. When Y is an integral homology 3-sphere, it has a unique spin^c-structure which is automatically $\mathbb{Z}_p$-invariant. In this case we may write the invariants as $\delta_j^{(p)}(Y)$. The most important property of these invariants is that they satisfy an equivariant version of the Frøyshov inequality [5]. In particular, this implies that they are invariant under equivariant homology cobordism. Consequently, the $\delta_j^{(p)}$ define obstructions to extending the $\mathbb{Z}_p$-action over an integral or rational homology 4-ball bounding Y:

Proposition 1.3 (Proposition 7.6, [5])**.** *Let Y be an integral homology 3-sphere on which $\mathbb{Z}_p$ acts by orientation preserving diffeomorphisms. Suppose that Y is bounded by a smooth integer homology 4-ball W. If the $\mathbb{Z}_p$-action extends smoothly over W then $\delta_j^{(p)}(Y) = \delta_j^{(p)}(-Y) = 0$ for all $j \geq 0$.*

In fact, we can relax the assumption that W is an integer homology 4-ball to simply being a rational homology 4-ball provided that W admits a $\mathbb{Z}_p$-invariant spin^c-structure. This is automatically true if p does not divide the order of $H^2(W;\mathbb{Z})$, for then $\mathbb{Z}_p$ can't act freely on the set of spin^c-structures.

The sequence of invariants $\{\delta_j^{(p)}(Y)\}_{j\geq 0}$ is decreasing and eventually constant. We set $\delta_\infty^{(p)}(Y) = \lim_{j\to\infty} \delta_j^{(p)}(Y)$.

Let $Y = \Sigma(a_1, \dots, a_r)$ be a Brieskorn homology sphere and let p be any prime. We assume $a_1, \dots, a_r > 1$ and $r \geq 3$ so that $Y \neq S^3$. In §3.3 we prove the following results (see Proposition 3.6):

1. $\delta_j^{(p)}(Y) = \delta_\infty^{(p)}(Y)$ for all $j \geq 0$.
2. $\delta(Y) \leq \delta_\infty^{(p)}(Y) \leq -\lambda(Y)$.
3. $\lambda(Y) \leq \delta_j^{(p)}(-Y) \leq -\delta(Y)$ for all $j \geq 0$.

Here $\lambda(Y)$ is the Casson invariant of Y and $\delta(Y) = d(Y)/2$ is half the Ozsváth–Szabó d-invariant. In particular, if Y is a Brieskorn homology sphere which bounds a contractible 4-manifold, then $\delta(Y) = 0$, $\delta_j^{(p)}(Y) = \delta_\infty^{(p)}(Y)$ and

$$\delta_\infty^{(p)}(-Y) \leq \delta_j^{(p)}(-Y) \leq 0$$

for all $j \geq 0$. Thus $\delta_j^{(p)}(\pm Y) = 0$ for all j if and only if $\delta_\infty^{(p)}(\pm Y) = 0$. This justifies restricting attention to the invariants $\delta_\infty^{(p)}(\pm Y)$.

We first consider the case of free $\mathbb{Z}_p$-actions. Given a prime p, the restriction of the Seifert circle action on $Y = \Sigma(a_1, \dots, a_r)$ to $\mathbb{Z}_p$ acts freely if and only if p does not divide $a_1 \cdots a_r$. Hence the action is free for all but finitely many primes. In fact any free action of a finite group on Y is conjugate to a finite subgroup of the Seifert circle action [27, Proposition 4.3]. In the free case we have:

Theorem 1.4. *Let $Y = \Sigma(a_1, \dots, a_r)$ be a Brieskorn homology sphere and let p be a prime not dividing $a_1 \cdots a_r$. Set $Y_0 = Y/\mathbb{Z}_p$. Then for any spin^c-structure $\mathfrak{s}_0$ on Y_0, we have*

$$\delta_\infty^{(p)}(Y) - \delta(Y) = \mathrm{rk}(HF^+_{red}(Y)) - \mathrm{rk}(HF^+_{red}(Y_0, \mathfrak{s}_0)).$$

Furthermore, we have:

Theorem 1.5. *We have that $\mathrm{rk}(HF^+_{red}(Y)) > \mathrm{rk}(HF^+_{red}(Y_0, \mathfrak{s}_0))$ except in the following cases:*

1. *$Y = \Sigma(2, 3, 5)$ and p is any prime.*
2. *$Y = \Sigma(2, 3, 11)$ and $p = 5$.*

In case (1) we have $\mathrm{rk}(HF^+_{red}(Y)) = \mathrm{rk}(HF^+_{red}(Y_0, \mathfrak{s}_0)) = 0$ and in case (2) we have $\mathrm{rk}(HF^+_{red}(Y)) = \mathrm{rk}(HF^+_{red}(Y_0, \mathfrak{s}_0)) = 1$.

Combining these two results gives:

Corollary 1.6. *Let $Y = \Sigma(a_1, a_2, \dots, a_r)$ be a Brieskorn homology sphere and let p be a prime not dividing $a_1 \cdots a_r$. Then $\delta^{(p)}_\infty(Y) > \delta(Y)$ except in the following cases:*

(1) $Y = \Sigma(2,3,5)$ *and p is any prime.*
(2) $Y = \Sigma(2,3,11)$ *and $p = 5$.*

In both cases we have $\delta^{(p)}_\infty(Y) = \delta(Y) = 1$.

Corollary 1.7. *Let $Y = \Sigma(a_1, a_2, \dots, a_r)$ be a Brieskorn homology sphere and let $m > 1$ be an integer not dividing $a_1 \cdots a_r$. Suppose that W is smooth rational homology 4-ball bounding Y and that m does not divide $|H^2(X;\mathbb{Z})|$. Then the $\mathbb{Z}_m$-action on Y does not extend smoothly to W.*

Proof. Let p be a prime divisor of m which does not divide $|H^2(X;\mathbb{Z})|$. It suffices to show that the subgroup $\mathbb{Z}_p \subseteq \mathbb{Z}_m$ does not extend over W. Since Y is bounded by a rational homology 4-ball we have $\delta(Y) = 0$. Then Corollary 1.6 implies that $\delta^{(p)}_\infty(Y) > 0$, unless $Y = \Sigma(2,3,5)$ or $Y = \Sigma(2,3,11)$ and $p = 5$. However these cases do not bound rational homology 4-balls as they have $\delta(Y) = 1$. So $\delta^{(p)}_\infty(Y) > 0$ which implies that the $\mathbb{Z}_p$-action does not extend smoothly to W. □

The $r = 3$ case of the above result was proven by Anvari–Hambleton [2], under the assumption that the bounding manifold is contractible. We note that there is a conjecture that Brieskorn spheres with $r > 3$ can not bound contractible manifolds (see, for example [40, Problem I]). On the other hand, there are many examples of Brieskorn spheres which bound rational homology balls, but not integer homology balls [14, 1, 39]. Thus Corollary 1.7 is a non-trivial result.

We can also show that for all sufficiently large primes, $\delta^{(p)}_\infty(Y)$ equals minus the Casson invariant $-\lambda(Y)$.

Theorem 1.8. *Let $Y = \Sigma(a_1, a_2, \dots, a_r)$ be a Brieskorn homology sphere and let p be a prime not dividing $a_1 \cdots a_r$. Suppose that $p > N$ where*

$$N = a_1 \cdots a_r \left((r-2) - \sum_{i=1}^{r} \frac{1}{a_i} \right).$$

Then $\delta^{(p)}_\infty(Y) = -\lambda(Y)$.

Proof. This follows from Theorem 1.4 and Proposition 6.2. □

We now consider the case of branched coverings. Let $Y = \Sigma(a_1, \dots, a_r)$ be a Brieskorn homology sphere and let p be a prime dividing $a_1 \cdots a_r$. Without loss of generality we may assume that p divides a_1. Then the quotient space $Y_0 = Y/\mathbb{Z}_p$ is the Brieskorn homology sphere $\Sigma(a_1/p, a_2, \dots, a_r)$ and $Y \to Y_0$ is a cyclic branched covering. Our main result is the following:

Theorem 1.9. *We have that*

$$\delta(-Y) - \delta^{(p)}_\infty(-Y) \geq \mathrm{rk}(HF^+_{red}(Y)) - p\,\mathrm{rk}(HF^+_{red}(Y_0)).$$

Remark 1.10. Karakurt–Lidman have shown that $\mathrm{rk}(HF^+_{red}(Y)) \geq p\,\mathrm{rk}(HF^+_{red}(Y_0))$. Thus the right hand side of the inequality in Theorem 1.9 is non-negative. Moreover, if $\delta^{(p)}_\infty(-Y) = \delta(-Y)$ then we must have an equality: $\mathrm{rk}(HF^+_{red}(Y)) = p\,\mathrm{rk}(HF^+_{red}(Y_0))$.

Theorem 1.11. *Let $Y = \Sigma(a_1, a_2, \dots, a_r)$ be a Brieskorn homology sphere and let p be a prime dividing $a_1 \cdots a_r$. Suppose that W is a rational homology 4-ball bounding Y and that p does not divide the order of $H^2(W;\mathbb{Z})$. If $\mathrm{rk}(HF^+_{red}(Y)) > p\,\mathrm{rk}(HF^+_{red}(Y_0))$, then the $\mathbb{Z}_p$-action on Y does not extend smoothly to W.*

The $r = 3$ case of this result was proved by Anvari–Hambleton [3] (in the integral homology case) without requiring the assumption that $\mathrm{rk}(HF^+_{red}(Y)) > p\,\mathrm{rk}(HF^+_{red}(Y_0))$.

We expect that the condition $\mathrm{rk}(HF^+_{red}(Y)) = p\,\mathrm{rk}(HF^+_{red}(Y_0))$ is rarely satisfied, however there are some cases where it does hold. One family of examples is given by $Y = \Sigma(2, 3, 30n + 5)$ and $p = 5$, in which case $Y_0 = \Sigma(2, 3, 6n + 1)$ and $\mathrm{rk}(HF^+_{red}(Y)) = 5\,\mathrm{rk}(HF^+_{red}(Y_0)) = 5n$. All of these examples have $\delta(Y) = 1$, so such a Y can not bound a contractible 4-manifold. We suspect that there are no examples where $\mathrm{rk}(HF^+_{red}(Y)) = p\,\mathrm{rk}(HF^+_{red}(Y_0))$ and Y bounds an integral homology 4-ball.

1.3. **Non-extension results for positive definite 4-manifolds.**

Our equivariant δ-invariants can also be used to obstruct the extension of the $\mathbb{Z}_p$-action over a positive definite 4-manifold bounding Y. First, we have the following result, which is a consequence of [5, Theorem 5.3]:

Proposition 1.12. *Let Y be an integral homology 3-sphere on which $\mathbb{Z}_p$ acts by orientation preserving diffeomorphisms. Suppose that Y is bounded by a smooth, compact, oriented, 4-manifold W with positive definite intersection form and with $b_1(W) = 0$. Suppose that the $\mathbb{Z}_p$ extends to a smooth, homologically trivial action on W. Then*

$$\min_c \left\{ \frac{c^2 - rk(H^2(W;\mathbb{Z}))}{8} \right\} \geq \delta^{(p)}_0(Y)$$

where the minimum is taken over all characteristic elements of $H^2(W;\mathbb{Z})$.

Proof. Suppose $\mathbb{Z}_p$ extends smoothly and homologically trivially to W. Take any characteristic $c \in H^2(X;\mathbb{Z})$. Then there is a unique spin^c-structure $\mathfrak{s}$ with $\mathfrak{s} = c$. Since the action is homologically trivial it follows that $\mathfrak{s}$ is $\mathbb{Z}_p$-invariant. Now we apply [5, Theorem 5.3] to $-W$ with Y regarded as an ingoing boundary to obtain: $(c^2 - rk(H^2(W;\mathbb{Z})))/8 \geq \delta^{(p)}_0(Y)$. Taking the minimum over all characteristics gives the result. □

Corollary 1.13. *Let W and Y be as in Proposition 1.12. If $\delta^{(p)}_0(Y) > 0$ or $\delta^{(p)}_\infty(-Y) < 0$. Then the $\mathbb{Z}_p$-action on Y does not extend smoothly and homologically trivially to W.*

Proof. First note that $\delta^{(p)}_0(Y) + \delta^{(p)}_\infty(-Y) \geq \delta^{(p)}_\infty(Y) + \delta^{(p)}_\infty(-Y) \geq 0$ [5]. So if $\delta^{(p)}_\infty(-Y) < 0$, then $\delta^{(p)}_0(Y) > 0$. So we can assume that $\delta^{(p)}_0(Y) > 0$. If the

$\mathbb{Z}_p$-action extends smoothly and homologically trivially to W, then

$$\min_c \left\{ \frac{c^2 - rk(H^2(W;\mathbb{Z}))}{8} \right\} \geq \delta_0^{(p)}(Y) > 0.$$

But Y is an integral homology 3-sphere, so $H^2(W;\mathbb{Z})$ is a unimodular integral lattice. A result of Elkies [13] implies that $\min_c\{(c^2 - rk(H^2(W;\mathbb{Z})))/8\} \leq 0$, which is a contradiction. □

Combined with our calculation of $\delta_0(Y), \delta_\infty(-Y)$ for Brieskorn spheres, we obtain the following non-extension result:

Corollary 1.14. *Let $Y = \Sigma(a_1, \dots, a_r)$ be a Brieskorn homology sphere and p any prime. If $\delta_\infty(Y) > 0$ or $\delta_\infty(-Y) > 0$, then the $\mathbb{Z}_p$-action on Y does not extend smoothly and homologically trivial to any smooth, compact, oriented, 4-manifold W with positive definite intersection form and with $b_1(W) = 0$ bounding Y.*

In particular, if p does not divide $a_1 \cdots a_r$ and $p > N = a_1 \cdots a_r \left((r-2) - \sum_{i=1}^r \frac{1}{a_i}\right)$, then $\delta_\infty(Y) = -\lambda(Y) > 0$ by Theorem 6.4. Hence for a given Y, the above non-extension result applies to all but finitely many primes.

Note that such a non-extension result does not exist if W has a negative definite intersection form. Indeed for any p, the $\mathbb{Z}_p$-action on Y extends smoothly and homologically trivially to any negative definite star-shaped plumbing bounding Y [33, §2].

1.4. **Non-extension results for connected sums.** Suppose that $Y_1, \dots, Y_m$ are Brieskorn homology spheres and p is a prime such that for each i, the $\mathbb{Z}_p$-action on Y_i is not free. Then we can form an equivariant connected sum $Y = Y_1 \# \cdots \# Y_m$ by attaching the summands to each other along fixed points. From [4, Proposition 3.1] we have that $\delta^{(p)}_{j_1+\cdots+j_m}(-Y) \leq \sum_{k=1}^m \delta^{(p)}_{j_k}(-Y_k)$. Taking $j_1, \dots, j_m$ sufficiently large, we obtain

$$\delta^{(p)}_\infty(-Y) \leq \sum_{k=1}^m \delta^{(p)}_\infty(-Y_k).$$

Then Theorem 1.9 implies that

$$\delta(-Y) - \delta^{(p)}_\infty(-Y) \geq \sum_{k=1}^m \left(\mathrm{rk}(HF^+_{red}(Y_k)) - p\,\mathrm{rk}(HF^+_{red}(Y_k/\mathbb{Z}_p)) \right), \tag{1.1}$$

which gives us the following result:

Corollary 1.15. *Let $Y_1, \dots, Y_m$ be Brieskorn homology spheres and let p be a prime such that for each i, the $\mathbb{Z}_p$-action on Y_i is not free. Let $Y = Y_1 \# \cdots \# Y_m$ be the equivariant connected sum. Suppose that W is a rational homology 4-ball bounding Y and that p does not divide the order of $H^2(W;\mathbb{Z})$. If $\mathrm{rk}(HF^+_{red}(Y_i)) > p\,\mathrm{rk}(HF^+_{red}(Y_i/\mathbb{Z}_p))$ for some i, then the $\mathbb{Z}_p$-action on Y does not extend smoothly to W.*

Corollary 1.13 and (1.1) also give us a non-extension result over positive-definite 4-manifolds:

Corollary 1.16. *Let $Y_1, \dots, Y_m$ be Brieskorn homology spheres and let p be a prime such that for each i, the $\mathbb{Z}_p$-action on Y_i is not free. Let $Y = Y_1 \# \cdots \# Y_m$ be the equivariant connected sum. Suppose that W is a smooth, compact, oriented, 4-manifold bounding Y, with positive definite intersection form and with $b_1(W) = 0$. If*

$$\delta(Y) + \sum_{k=1}^{m} \left(\mathrm{rk}(HF^+_{red}(Y_k)) - p\,\mathrm{rk}(HF^+_{red}(Y_k/\mathbb{Z}_p)) \right) > 0$$

then the $\mathbb{Z}_p$-action on Y does not extend smoothly and homologically trivially to W.

1.5. **Structure of the paper.** The paper is structured as follows. In §2 we recall the basic results on equivariant Seiberg–Witten–Floer cohomology from [5] and the associated knot concordance invariants. In §3 we examine in great detail the spectral sequence relating equivariant and non-equivariant Floer cohomology and use this to deduce more refined information about the equivariant delta invariants. In §4, we study the Floer homology of Brieskorn spheres $\Sigma(a, b, c)$ and use this to compute the invariants $\theta^{(c)}(T_{a,b})$ leading to the proof of Theorem 1.1. In §5, we consider the case where $\mathbb{Z}_p$ acts non-freely on $\Sigma(a_1, \dots, a_r)$ and prove Theorem 1.9. Finally in §6, we consider the case where $\mathbb{Z}_p$ acts freely on $\Sigma(a_1, \dots, a_r)$ and prove Theorems 1.4, 1.5 and 6.4.

2. Equivariant Seiberg–Witten–Floer cohomology and knot concordance invariants

2.1. **Seiberg–Witten–Floer cohomology.** Let Y be a rational homology 3-sphere and $\mathfrak{s}$ a spin^c-structure. For such a pair $(Y, \mathfrak{s})$, Manolescu constructed an S^1-equivariant stable homotopy type whose equivariant (co)homology groups are isomorphic to the Heegaard Floer or Monopole Floer (co)homology groups of $(Y, \mathfrak{s})$ [28]. We denote the S^1-equivariant reduced cohomology groups with coefficients in $\mathbb{F}$ by $HSW^*(Y, \mathfrak{s}; \mathbb{F})$ and refer to them as the *Seiberg–Witten–Floer cohomology* of $(Y, \mathfrak{s})$. If the coefficient group is understood then we will write $HSW^*(Y, \mathfrak{s})$. If Y is an integral homology 3-sphere, then it has a unique spin^c-structure and in this case we simply write $HSW^*(Y)$. We have that $HSW^*(Y, \mathfrak{s})$ is a graded module over the ring $H^*_{S^1} = H^*_{S^1}(pt; \mathbb{F})$. Note that $H^*_{S^1} \cong \mathbb{F}[U]$, where $deg(U) = 2$.

There exists a chain of isomorphisms relating Seiberg–Witten–Floer homology to monopole Floer homology [25] and to Heegaard Floer homology [19, 20, 21, 22, 23, 7, 8, 9, 41]. In particular we have isomorphisms

$$HSW^*(Y, \mathfrak{s}) \cong HF^*_+(Y, \mathfrak{s})$$

where $HF^*_+(Y, \mathfrak{s})$ denotes the plus version of Heegaard Floer cohomology with coefficients in $\mathbb{F}$. Unless stated otherwise, we will take our coefficient group $\mathbb{F}$ to be a field. Then the universal coefficient theorem implies that the Heegaard Floer cohomology $HF^*_+(Y, \mathfrak{s})$ is isomorphic to the Heegaard Floer homology $HF^+_*(Y, \mathfrak{s})$, except that the action of $\mathbb{F}[U]$ on $HF^+_*(Y, \mathfrak{s})$ gets replaced by its dual, so $deg(U) = 2$ whereas in Heegaard Floer homology one has $deg(U) = -2$. We will frequently identify $HSW^*(Y, \mathfrak{s})$ with $HF^+_*(Y, \mathfrak{s})$, equipped with the dual $\mathbb{F}[U]$-module structure.

Let $d(Y,\mathfrak{s})$ denote the Ozsváth–Szabó d-invariant. Due to the isomorphism $HSW^*(Y,\mathfrak{s}) \cong HF^+_*(Y,\mathfrak{s})$, we have that $d(Y,\mathfrak{s})$ is the minimal degree i for which there exists an $x \in HSW^i(Y,\mathfrak{s})$ with $U^k x \neq 0$ for all $k \geq 0$. For notational convenience we define $\delta(Y,\mathfrak{s}) = d(Y,\mathfrak{s})/2$.

We define $HSW^*_{red}(Y,\mathfrak{s}) = \{x \in HSW^*(Y,\mathfrak{s}) \mid U^k x = 0 \text{ for some } k \geq 0\}$. More generally, given an $\mathbb{F}[U]$-module M, we write M_{red} for the submodule of elements $x \in M$ such that $U^k x = 0$ for some $k \geq 0$.

Recall that for any $(Y,\mathfrak{s})$, there is an isomorphism of $\mathbb{F}[U]$-modules

$$HF^+(Y,\mathfrak{s}) \cong \mathbb{F}[U]_{d(Y,\mathfrak{s})} \oplus HF^+_{red}(Y,\mathfrak{s}),$$

where for any $\mathbb{F}[U]$-module M^* and any $d \in \mathbb{Q}$, we define M^*_d by $(M^i_d) = M^{i-d}$. It follows that we similarly have an isomorphism

$$HSW^*(Y,\mathfrak{s}) \cong \mathbb{F}[U]_{d(Y,\mathfrak{s})} \oplus HSW_{red}(Y,\mathfrak{s}).$$

2.2. **Equivariant Seiberg–Witten–Floer cohomology.** Let Y be a rational homology 3-sphere and suppose that $\tau\colon Y \to Y$ an orientation preserving diffeomorphism of order p, where p is prime. This gives an action of the finite group $G = \mathbb{Z}_p$ on Y generated by τ. Let $\mathfrak{s}$ be a spin^c-structure preserved by τ. In [5], the authors constructed the equivariant Seiberg–Witten–Floer cohomology groups $HSW^*_G(Y,\mathfrak{s})$. Except where stated otherwise, we take Floer cohomology with respect to the coefficient field $\mathbb{F} = \mathbb{Z}_p$. Then $HSW^*_G(Y,\mathfrak{s})$ is a module over the ring $H^*_{S^1\times G} = H^*_{S^1\times G}(pt;\mathbb{F})$. If $p = 2$, then $H^*_{S^1\times G} \cong \mathbb{F}[U,Q]$, where $deg(U) = 2$, $deg(Q) = 1$. If p is odd, then $H^*_{S^1\times G} \cong \mathbb{F}[U,R,S]/(R^2)$, where $deg(U) = 2$, $deg(R) = 1$, $deg(S) = 2$. As in the non-equivariant case, the grading on $HSW^*_G(Y,\mathfrak{s})$ can in general take rational values. However, if Y is an integral homology sphere then the grading is integer-valued.

The localisation theorem in equivariant cohomology implies that the localisation $U^{-1}HSW^*_G(Y,\mathfrak{s})$ is a free $U^{-1}H^*_{S^1\times G}$-module of rank 1. Letting μ denote a generator of $U^{-1}HSW^*_G(Y,\mathfrak{s})$, we have an isomorphism of the form

$$\iota\colon U^{-1}HSW^*_G(Y,\mathfrak{s}) \to \mathbb{F}[Q,U,U^{-1}]\mu$$

for $p = 2$ and

$$\iota\colon U^{-1}HSW^*_G(Y,\mathfrak{s}) \to \frac{\mathbb{F}[R,S,U,U^{-1}]}{(R^2)}\mu$$

for p odd. Following [5, §3] we define a sequence of equivariant δ-invariants as follows. The cases $p = 2$ and $p \neq 2$ need to be treated separately. First suppose $p = 2$. For each $j \geq 0$, we define $\delta^{(p)}_j(Y,\mathfrak{s},\tau)$ to be $i/2 - j/2$, where i is the least degree for which there exists an element $x \in HSW^i_G(Y,\mathfrak{s})$ and a $k \in \mathbb{Z}$ such that

$$\iota x = Q^j U^k \mu \pmod{Q^{j+1}}.$$

If $p \neq 2$, then for each $j \geq 0$, we define $\delta^{(p)}_j(Y,\mathfrak{s},\tau)$ to be $i/2 - j$, where i is the least degree for which there exists an element $x \in HSW^i_G(Y,\mathfrak{s})$ and a $k \in \mathbb{Z}$ such that

$$\iota x = S^j U^k \mu \pmod{S^{j+1}, RS^j}.$$

When the diffeomorphism τ is understood we will omit it from the notation and simply write the delta invariants as $\delta_j^{(p)}(Y,\mathfrak{s})$.

Various properties of the δ-invariants are shown in [5]. In particular, we have:

(1) $\delta_0^{(p)}(Y,\mathfrak{s}) \geq \delta(Y,\mathfrak{s})$, where $\delta(Y,\mathfrak{s}) = d(Y,\mathfrak{s})/2$ and $d(Y,\mathfrak{s})$ is the Ozsváth–Szabó d-invariant.
(2) $\delta_{j+1}^{(p)}(Y,\mathfrak{s}) \leq \delta_j^{(p)}(Y,\mathfrak{s})$ for all $j \geq 0$.
(3) The sequence $\{\delta_j^{(p)}(Y,\mathfrak{s})\}_{j\geq 0}$ is eventually constant.

Using property (3), we may define two additional invariants of $(Y,\mathfrak{s},\tau)$ as follows. We define $\delta_\infty^{(p)}(Y,\mathfrak{s},\tau) = \lim_{j\to\infty} \delta_j^{(p)}(Y,\mathfrak{s},\tau)$ and we define $j^{(p)}(Y,\mathfrak{s},\tau)$ to be the smallest j such that $\delta_j^{(p)}(Y,\mathfrak{s},\tau) = \delta_\infty^{(p)}(Y,\mathfrak{s},\tau)$. If τ is understood we will simply write $\delta_\infty^{(p)}(Y,\mathfrak{s})$ and $j^{(p)}(Y,\mathfrak{s})$.

2.3. **Knot concordance invariants.** Given a knot $K \subset S^3$ and a prime number p, we let $Y = \Sigma_p(K)$ denote the degree p cyclic cover of S^3 branched over K. Then Y is a rational homology 3-sphere [26, Corollary 3.2] and it comes equipped with a natural $\mathbb{Z}_p$-action. Let $\pi\colon Y \to S^3$ denote the covering map. From [16, Corollary 2.2], any spin^c-structure on $Y \setminus \pi^{-1}(K)$ uniquely extends to Y. Then since $H^2(S^3\setminus K;\mathbb{Z}) = 0$, there is a unique spin^c-structure on $S^3\setminus K$. The pullback of this spin^c-structure under π extends uniquely to a spin^c-structure on Y. Following [16], we denote this spin^c-structure by $\mathfrak{s}_0 = \mathfrak{s}_0(K,p)$. Uniqueness of the extension implies that $\mathfrak{s}_0$ is $\mathbb{Z}_p$-invariant. Thus for any prime p and any $j \geq 0$, we obtain a knot invariant $\delta_j^{(p)}(K)$ by setting

$$\delta_j^{(p)}(K) = 4\delta_j^{(p)}(\Sigma_p(K),\mathfrak{s}_0).$$

The invariants $\delta_j^{(p)}(K)$ are knot concordance invariants and they satisfy a number of properties [5], [4]. In particular, for $p = 2$ we have:

(1) $\delta_0^{(2)}(K) \geq \delta^{(2)}(K)$, where $\delta^{(2)}(K)$ is the Manolescu–Owens invariant [30].
(2) $\delta_{j+1}^{(2)}(K) \leq \delta_j^{(2)}(K)$ for all $j \geq 0$.
(3) $\delta_j^{(2)}(K) \geq -\sigma(K)/2$ for all $j \geq 0$ and $\delta_j^{(2)}(K) = -\sigma(K)/2$ for $j \geq g_4(K) - \sigma(K)/2$.

Here $g_4(K)$ is the slice genus of K and $\sigma(K)$ is the signature. Similarly for $p \neq 2$, we have:

(1) $\delta_0^{(p)}(K) \geq \delta^{(p)}(K)$, where $\delta^{(p)}(K)$ is the Jabuka invariant [16].
(2) $\delta_{j+1}^{(p)}(K) \leq \delta_j^{(p)}(K)$ for all $j \geq 0$.
(3) $\delta_j^{(p)}(K) \geq -\sigma^{(p)}(K)/2$ for all $j \geq 0$ and $\delta_j^{(p)}(K) = -\sigma^{(p)}(K)/2$ for $2j \geq (p-1)g_4(K) - \sigma^{(p)}(K)/2$.

Here $\sigma^{(p)}(K)$ is defined as

$$\sigma^{(p)}(K) = \sum_{j=1}^{p-1} \sigma_K(e^{2\pi i j/p}),$$

where $\sigma_K(\omega)$ is the Levine–Tristram signature of K. Define $j^{(p)}(K)$ to be the smallest j such that $\delta_j^{(p)}(K)$ attains its minimum. Thus

$$j^{(p)}(K) = j^{(p)}(\Sigma_p(K), \mathfrak{s}_0).$$

From property (3), we see that the minimum value of $\delta_j^{(2)}(K)$ is precisely $-\sigma(K)/2$ and that

$$j^{(2)}(K) \leq g_4(K) - \sigma(K)/2.$$

This can be re-arranged into a lower bound for the slice genus:

$$g_4(K) \geq j^{(2)}(K) + \frac{\sigma(K)}{2}.$$

Similarly for $p \neq 2$, the minimum value of $\delta_j^{(p)}(K)$ is $-\sigma^{(p)}(K)/2$ and that

$$2j^{(p)}(K) \leq (p-1)g_4(K) - \frac{\sigma^{(p)}(K)}{2}.$$

This can be re-arranged to

$$g_4(K) \geq \frac{2j^{(p)}(K)}{(p-1)} + \frac{\sigma^{(p)}(K)}{2(p-1)}.$$

Replacing K by its mirror $-K$, we obtain the following slice genus bounds:

$$g_4(K) \geq j^{(2)}(-K) - \frac{\sigma(K)}{2}$$

for $p = 2$, and

$$g_4(K) \geq \frac{2j^{(p)}(-K)}{(p-1)} - \frac{\sigma^{(p)}(K)}{2(p-1)}$$

for $p \neq 2$. Following [4], we define knot concordance invariants $\theta^{(p)}(K)$ for all primes p by setting

$$\theta^{(2)}(K) = \max\left\{0, j^{(2)}(-K) - \frac{\sigma(K)}{2}\right\}$$

for $p = 2$, and

$$\theta^{(p)}(K) = \max\left\{0, \frac{2j^{(p)}(-K)}{(p-1)} - \frac{\sigma^{(p)}(K)}{2(p-1)}\right\}$$

for $p \neq 2$. Then we have the slice genus bounds $g_4(K) \geq \theta^{(p)}(K)$ for all p.

3. Further properties of the delta invariants

In this section we will use the spectral sequence of [5] relating equivariant and non-equivariant Seiberg–Witten–Floer cohomologies in order to deduce more precise information on the δ-invariants. In particular, we will be able to determine the value of $j^{(p)}(Y)$ under certain assumptions on the Floer homology of Y.

3.1. **Spectral sequence.** Let Y be a rational homology 3-sphere and $\tau\colon Y \to Y$ an orientation preserving diffeomorphism of prime order p. Let $\mathfrak{s}$ be a spin^c-structure preserved by τ. Let the coefficient group be $\mathbb{F} = \mathbb{Z}_p$ and let $G = \langle \tau \rangle = \mathbb{Z}_p$.

Recall that H^*_G is isomorphic to $\mathbb{F}[Q]$, where $deg(Q) = 1$ if $p = 2$ and is isomorphic to $\mathbb{F}[R, S]/(R^2)$ where $deg(R) = 1$, $deg(S) = 2$ if p is odd. Thus in either case we have $H^{ev}_G \cong \mathbb{F}[S]$, where in the $p = 2$ case we set $S = Q^2$.

Let $\{E^{p,q}_r, d_r\}$ denote the spectral sequence for the G-equivariant Seiberg–Witten–Floer cohomology $HSW^*_G(Y, \mathfrak{s})$ [5, Theorem 3.2]. In detail, this means there is a filtration $\{\mathcal{F}^*_j\}_{j\geq 0}$ on $HSW^*_G(Y, \mathfrak{s})$ such that E_∞ is isomorphic to the associated graded group

$$E_\infty \cong Gr(HSW^*_G(Y, \mathfrak{s})) = \bigoplus_{j\geq 0} \mathcal{F}^*_j/\mathcal{F}^*_{j+1}$$

as $H^*_{S^1\times G}$-modules. In terms of bigrading this means that $E^{p,q}_\infty \cong \mathcal{F}^{p+q}_p/\mathcal{F}^{p+q}_{p+1}$. Furthermore, we have that

$$E^{p,q}_2 = H^p(\mathbb{Z}_p; HSW^q(Y, \mathfrak{s})).$$

3.2. **Behaviour of the delta invariants.** Let $(Y, \mathfrak{s}, \tau)$ be as in Section 3.1. We will examine the spectral sequence $\{E^{p,q}_r\}$ to deduce some properties of the delta invariants $\{\delta^{(p)}_j(Y, \mathfrak{s})\}$.

To simplify notation, we will set $H^i = HSW^i(Y, \mathfrak{s})$ and $\widehat{H}^i = HSW^i_G(Y, \mathfrak{s})$. Let $d = d(Y, \mathfrak{s})$ and $\delta = d/2$. We will make the assumption that $H^i = 0$ unless $i = d \pmod 2$. This assumption is satisfied if Y is a Seifert homology sphere oriented so that $-Y$ is the boundary of a negative definite plumbing.

Lemma 3.1. *If $p = 2$ and if $HSW^*(Y, \mathfrak{s})$ is concentrated in degrees equal to $d(Y, \mathfrak{s})$ mod 2, then for all $j \geq 0$ we have $\delta^{(2)}_{2j+1}(Y, \mathfrak{s}) = \delta^{(2)}_{2j}(Y, \mathfrak{s})$.*

Proof. Adapting the proof of [5, Lemma 5.7] to the $p = 2$ case, one can show that $Q\colon E^{p,q}_\infty \to E^{p+1,q}_\infty$ is surjective for all p, q. This implies that $Q\colon \mathcal{F}_j \to \mathcal{F}_{j+1}$ is surjective. Now let $x \in HSW^i_{\mathbb{Z}_2}(Y, \mathfrak{s})$ satisfy $\iota x = Q^{2j+1}U^k\mu \pmod{Q^{2j+2}}$ for some k and assume that x has the minimal possible degree, so

$$\delta^{(2)}_{2j+1}(Y, \mathfrak{s}) = deg(x)/2 - (2j + 1)/2 = i/2 - j - 1/2.$$

Note that since $deg(Q) = 1$, $deg(U) = 2$, we have that $deg(x) = 1+d(Y, \mathfrak{s}) \pmod 2$. This means that the image of x in $HSW^i_{\mathbb{Z}_2}(Y, \mathfrak{s})/\mathcal{F}_1 = E^{0,q}_\infty$ is zero, because $E^{0,q}_\infty \subseteq E^{0,q}_2 = H^0(\mathbb{Z}_2; HSW^q(Y, \mathfrak{s}))$ is concentrated in degrees equal to $d(Y, \mathfrak{s})$ mod 2. Hence $x \in \mathcal{F}_1$. This implies that $x = Qy$ for some $y \in HSW^{i-1}_{\mathbb{Z}_2}(Y, \mathfrak{s})$. Then since $x = Qy$, we have

$$\iota x = Q\iota y = Q^{2j+1}U^k\mu \pmod{Q^{2j+2}}.$$

Moreover, Q is injective on $U^{-1}HSW^*_{\mathbb{Z}_2}(Y, \mathfrak{s}) \cong \mathbb{F}[Q, U, U^{-1}]$, so we deduce that

$$\iota y = Q^{2j}U^k\mu \pmod{Q^{2j+1}}.$$

Therefore, from the definition of $\delta^{(2)}_{2j}(Y, \mathfrak{s})$, it follows that

$$\delta^{(2)}_{2j}(Y, \mathfrak{s}) \leq deg(y)/2 - j = deg(x)/2 - 1/2 - j = \delta^{(2)}_{2j+1}(Y, \mathfrak{s}).$$

But we always have $\delta^{(2)}_{2j+1}(Y,\mathfrak{s}) \le \delta^{(2)}_{2j}(Y,\mathfrak{s})$, so we must have an equality $\delta^{(2)}_{2j}(Y,\mathfrak{s}) = \delta^{(2)}_{2j+1}(Y,\mathfrak{s})$. □

Remark 3.2. By Lemma 3.1, if $p = 2$ and $HSW^*(Y,\mathfrak{s})$ is concentrated in degrees equal to $d(Y,\mathfrak{s})$ mod 2, then $j^{(2)}(Y,\mathfrak{s})$ is even.

In what follows, we find it more convenient to consider not the whole equivariant Seiberg–Witten–Floer cohomology but only the part concentrated in degrees equal to $d(Y,\mathfrak{s})$ mod 2. For this reason we consider even counterparts of the relevant ingredients, such as the spectral sequence and filtration. One advantage is that it allows us to treat the $p = 2$ and $p \neq 2$ cases simultaneously.

Let $H^{ev}_{S^1\times G}$ denote the subring of $H^*_{S^1\times G}$ given by elements of even degree. Then $H^{ev}_{S^1\times G} \cong \mathbb{F}[U,S]$. Let $\widehat{H}^{ev}$ denote the $H^{ev}_{S^1\times G}$-submodule of $\widehat{H}^*$ given by elements of degree equal to d mod 2. The filtration $\{\mathcal{F}^*_j\}$ defines a filtration $\{\mathcal{F}^{ev}_j\}$ on $\widehat{H}^{ev}$, where $\mathcal{F}^{ev}_j = \mathcal{F}_{2j} \cap \widehat{H}^{ev}$. The associated graded $H^{ev}_{S^1\times G}$-module of this filtration is isomorphic to

$$E^{ev}_\infty = \bigoplus_{p,q} E^{2p,q}_\infty.$$

By [5, Lemma 5.7], the map $S\colon E^{2p,q}_\infty \to E^{2p+2,q}_\infty$ is surjective for all p,q. It follows that $S\colon \mathcal{F}^{ev}_j \to \mathcal{F}^{ev}_{j+1}$ is surjective for each $j \ge 0$. In particular, since $\mathcal{F}^{ev}_0 = \widehat{H}^{ev}$, we see that $\mathcal{F}^{ev}_j = S^j\widehat{H}^{ev}$.

Recall the localisation isomorphism

$$\iota\colon U^{-1}\widehat{H}^* \cong H^*_G[U,U^{-1}]\mu$$

for some μ. Restricting to $\widehat{H}^{ev}$, we get a localisation isomorphism

$$\iota\colon U^{-1}\widehat{H}^{ev} \cong H^{ev}_G[U,U^{-1}]\mu \cong \mathbb{F}[S,U,U^{-1}]\mu.$$

Let δ_j denote $\delta^{(p)}_j(Y,\mathfrak{s})$ if p is odd and $\delta^{(2)}_{2j}(Y,\mathfrak{s})$ if $p=2$. Since $S = Q^2$ in the $p=2$ case, it follows that regardless of whether p is even or odd, we have $\delta_j = a/2 - j$, where a is the least degree such that there exists an element $x \in \widehat{H}^a$ with

$$\iota x = S^jU^k\mu \ (\mathrm{mod}\ S^{j+1})$$

for some $k \in \mathbb{Z}$.

Observe that $E^{0,q}_2 = H^0(\mathbb{Z}_p; H^q)$ is a submodule of H^q. Then since $E^{0,q}_{r+1}$ is a submodule of $E^{0,q}_r$ for each $r \ge 2$, we see that $E^{0,*}_\infty = \mathcal{F}^{ev}_0/\mathcal{F}^{ev}_1$ can be identified with an $\mathbb{F}[U]$-submodule of H^*. We will denote this submodule by $J^* \subset H^*$. Now since $H^* \cong \mathbb{F}[U]\lambda \oplus H^*_{red}$, where λ has degree d and $H^*_{red} = \{x \in H^* \mid U^mx = 0 \text{ for some } m \ge 0\}$. It follows that $J^* \cong \mathbb{F}[U]\theta \oplus J^*_{red}$, where θ has degree at least d and $J^*_{red} \subseteq H^*_{red}$. Since $U^m\theta \neq 0$ for all $m \ge 0$, we see that the image of θ under the localisation map ι is non-zero. More precisely,

$$\iota\theta = U^\alpha\mu \ (\mathrm{mod}\ S)$$

for some $\alpha \in \mathbb{Z}$. On the other hand, any $x \in J^*$ has $U^mx = 0$ (mod $\mathcal{F}^{ev}_1$) for some $m \ge 0$. This means that $U^mx = Sy$ for some y and thus ιx is a multiple of S. It follows that $deg(\theta) = \delta_0$.

By [5, Proposition 3.14], we then have that $\delta_j = a/2 - j$, where a is the least degree such that there exists an element $x \in \widehat{H}^a$ such that

$$U^m x = S^j U^k \theta \ (\mathrm{mod}\ \mathcal{F}^{ev}_{j+1})$$

for some $m, k \geq 0$.

Consider again the sequence $\delta_0 \geq \delta_1 \geq \delta_2 \geq \cdots$. This sequence is eventually constant, hence there exists a finite set of indices $0 < j_1 < j_2 < \cdots < j_r$ such that for each $j > 0$, $\delta_j < \delta_{j-1}$ if and only if $j = j_i$ for some i. We let $n_i = \delta_{j_{i-1}} - \delta_{j_i} > 0$ for $1 < i \leq r$ and $n_1 = \delta_{j_1} - \delta_0$ for $i = 1$. Thus $\delta_{j_i} = \delta_0 - (n_1 + \cdots + n_i)$.

Lemma 3.3. *There exists $x_1, \ldots, x_r \in J^*_{red}$ such that:*

(1) *The set $\{U^a x_i \mid 1 \leq i \leq r, 0 \leq a \leq n_i - 1\}$ is linearly independent over $\mathbb{F}$. In particular, $U^a x_i \neq 0$ for $a < n_i$.*
(2) *We have $\delta_{j_i} = deg(x_i)/2 - j_i$.*
(3) *Each x_i has a lift to an element $y_i \in \widehat{H}^i$ satisfying*

$$U^{m_i} y_i = S^{j_i} U^{k_i} \theta \ (\mathrm{mod}\ \mathcal{F}^{ev}_{j_i+1})$$

for some $m_i, k_i \geq 0$.

Proof. By the definition of δ_{j_i}, there exists $y_i \in \widehat{H}^*$ such that $\delta_{j_i} = deg(y_i)/2 - j_i$ and $U^{m_i} y_i = S^{j_i} U^{k_i} \theta$ (mod $\mathcal{F}^{ev}_{j_i+1}$) for some m_i, k_i. Let $x_i \in H^*$ be the image of y_i under the natural map $\widehat{H}^{ev} \to \widehat{H}^{ev}/\mathcal{F}^{ev}_1 \cong J^*$. We claim that $x_i \in J^*_{red}$. If not, then $x_i = cU^k\theta + w$ for some $c \in \mathbb{F}^*$, $k \geq 0$ and $w \in J^*_{red}$. Then $U^m x_i = cU^{k+m}\theta$ for some $m, k \geq 0$ and hence $U^m y_i = cU^{k+m}\theta$ (mod $\mathcal{F}^{ev}_1$). This contradicts $U^{m_i} y_i = S^{j_i} U^{k_i} \theta$ (mod $\mathcal{F}^{ev}_{j_i+1}$) as $S^{j_i} U^{k_i} \theta = 0$ (mod $\mathcal{F}^{ev}_1$) (since $j_i > 0$). So $x_i \in J^*_{red}$. It is now evident that (2) and (3) are satisfied. We claim that (1) also holds, that is, the set $\{U^a x_i \mid 1 \leq i \leq r, 0 \leq a \leq n_i - 1\}$ is linearly independent over $\mathbb{F}$.

Suppose we have a non-trivial linear relation amongst elements of $\mathcal{B} = \{U^a x_i \mid 1 \leq i \leq r, 0 \leq a \leq n_i - 1\}$ of degree s. So

$$\sum_i c_i U^{\beta_i} x_i = 0$$

for some $c_i \in \mathbb{F}$, not all zero and $\beta_i = (s - deg(x_i))/2$. Furthermore, the sum is restricted to those i such that $0 \leq \beta_i \leq n_i - 1$ (so that $U^{\beta_i} x_i$ belongs to $\mathcal{B}$). Lifting this to $\widehat{H}^{ev}$, we have $\sum_i c_i U^{\beta_i} y_i \in \mathcal{F}^{ev}_1$ and hence

$$\sum_i c_i U^{\beta_i} y_i = Sw \tag{3.1}$$

for some $w \in \widehat{H}^{s-2}$. Let u be the smallest value of i such that $c_u \neq 0$. Multiplying by a sufficiently large power of U and taking the result modulo $\mathcal{F}^{ev}_{j_u+1}$, we see that

$$SU^m w = c_i S^{j_u} U^b \theta \ (\mathrm{mod}\ \mathcal{F}^{ev}_{j_u+1})$$

for some $m, b \geq 0$. Hence $S\iota(U^m w) = S\iota(c_i S^{j_u - 1} U^b \theta)$ (mod S^{j_u+1}), where ι is the localisation map. Since S is injective on $\mathbb{F}[S, U, U^{-1}]\mu$, we also have $\iota w = c_i S^{j_u-1} U^{b-m} \iota\theta$ (mod S^{j_u}). Now we recall that $\iota\theta = U^\alpha \mu$ (mod S) for some α, hence $\iota(c_i^{-1} w) = S^{j_u - 1} U^{b-m+\alpha} \mu$ (mod S^{j_u}). By the definition of $\delta_{j_u - 1}$, we must therefore have $deg(w)/2 \geq \delta_{j_u-1} + (j_u - 1)$. But $deg(w) = s - 2$, so $s \geq \delta_{j_u-1} + j_u$.

On the other hand, equating degrees in Equation (3.1) gives $s = \beta_u + deg(x_u)/2 = \beta_u + \delta_{j_u} + j_u$. This gives

$$\beta_u = s - \delta_{j_u} - j_u \geq \delta_{j_u - 1} - \delta_{j_u} = n_u.$$

But this contradicts $\beta_u \leq n_u - 1$. So no such linear relation can exist. □

Corollary 3.4. *We have that $\delta_0 - \delta_j \leq dim_{\mathbb{F}}(J^*_{red}) \leq dim_{\mathbb{F}}(HF^*_{red}(Y, \mathfrak{s}))$ for all $j \geq 0$.*

Proof. Since $\delta_j = \delta_{j_r}$ for all $j \geq j_r$, we have that $\delta_0 - \delta_j \leq \delta_0 - \delta_{j_r}$ for all j. But the subspace $span_{\mathbb{F}}\{U^a x_i \mid 1 \leq i \leq r, 0 \leq a \leq n_i - 1\} \subseteq J^*$ of Lemma 3.3 has dimension $n_1 + \cdots + n_r = \delta_0 - \delta_{j_r}$. □

Let $j'(Y, \mathfrak{s})$ denote the smallest j such that δ_j attains its minimum. Thus $j'(Y, \mathfrak{s}) = j^{(p)}(Y, \mathfrak{s})$ if p is odd and $j'(Y, \mathfrak{s}) = j^{(2)}(Y, \mathfrak{s})/2$ if $p = 2$, by Remark 3.2.

Proposition 3.5. *Let Y be a rational homology 3-sphere, $\tau \colon Y \to Y$ an orientation preserving diffeomorphism of order p, where p is prime. Let $\mathfrak{s}$ be a spin^c-structure preserved by τ. Suppose that the following conditions hold:*

1. *$HF^+(Y, \mathfrak{s})$ is non-zero only in degrees $i = d(Y, \mathfrak{s}) \pmod 2$.*
2. *$\delta_0 - \delta_j = dim_{\mathbb{F}}(HF^*_{red}(Y, \mathfrak{s}))$ for some $j \geq 0$.*
3. *$HF^*_{red}(Y, \mathfrak{s}) \neq 0$.*
4. *Let $\ell^+(Y, \mathfrak{s})$ denote the highest non-zero degree in $HF^+_{red}(Y, \mathfrak{s})$. Any non-zero element in the image of $U \colon HF^+_{red}(Y, \mathfrak{s}) \to HF^+_{red}(Y, \mathfrak{s})$ has degree strictly less than $\ell^+(Y, \mathfrak{s})$.*

Then

$$j'(Y, \mathfrak{s}) = \ell^+(Y, \mathfrak{s})/2 - \delta_0(Y, \mathfrak{s}) + dim_{\mathbb{F}}(HF^*_{red}(Y, \mathfrak{s})).$$

Moreover, if Y is an integral homology sphere and $\delta(Y) \in 2\mathbb{Z}$, then

$$j'(Y) = \ell^+(Y)/2 - \delta_0(Y) + \delta(Y) + \lambda(Y),$$

where $\lambda(Y)$ is the Casson invariant of Y.

Proof. Let $x_1, \ldots, x_r \in J^*_{red}$ be as in Lemma 3.3. It follows that $j'(Y, \mathfrak{s}) = j_r$. Choose j such that $\delta_0 - \delta_j = dim_{\mathbb{F}}(HF^*_{red}(Y, \mathfrak{s}))$. From Corollary 3.4, we have

$$dim_{\mathbb{F}}(HF^*_{red}(Y, \mathfrak{s})) = \delta_0 - \delta_j \leq \delta_0 - \delta_{j_r} = n_1 + \cdots + n_r \leq dim_{\mathbb{F}}(J^*_{red}) \leq dim_{\mathbb{F}}(HF^*_{red}(Y, \mathfrak{s})).$$

It follows that we must have equalities throughout. So $\delta_0 - \delta_{j_r} = n_1 + \cdots + n_r = dim_{\mathbb{F}}(HF^*_{red}(Y, \mathfrak{s}))$. So $\{U^a x_i \mid 1 \leq i \leq r, 0 \leq a \leq n_i - 1\}$ is a basis for $HF^*_{red}(Y, \mathfrak{s})$. Set $\alpha_i = deg(x_i)/2$. Lemma 3.3 (2) implies that $\delta_{j_i} = \alpha_i - j_i$ for $1 \leq i \leq r$. Setting $\alpha_0 = \delta_0$ and $j_0 = 0$, we also have $\delta_{j_i} = \alpha_i - j_i$ for $i = 0$. Further, we have that $\delta_{j_i} - \delta_{j_{i-1}} = n_i$ for $1 \leq i \leq r$. Hence we obtain $(\alpha_i + n_i - 1) - \alpha_{i-1} = j_i - j_{i-1} - 1$. But $j_i > j_{i-1}$, so $j_i - j_{i-1} - 1 \geq 0$, giving

$$(\alpha_i + n_i - 1) \geq \alpha_{i-1}. \tag{3.2}$$

Now we observe that $2(\alpha_i + n_i - 1)$ is the degree of $U^{n_i - 1} x_i$ and $2\alpha_{i-1}$ is the degree of α_{i-1}. Consider the space $A = HF^{\ell^+(Y,\mathfrak{s})}_{red}(Y, \mathfrak{s})$ of top degree elements in $HF^+_{red}(Y,\mathfrak{s})$. Let $a = dim_{\mathbb{F}}(A) > 0$. By assumption (4), no non-zero element of A is in the image of U. Hence there must exist indices $k_1 < k_2 < \cdots < k_a$ such that $x_{k_1}, \dots, x_{k_a}$ is a basis for A. In fact, $K = \{k_1, \dots, k_a\}$ is precisely the set of indices i such that $2\alpha_i = \ell^+(Y,\mathfrak{s})$. Suppose $i \in K$ and $i < r$. Then (3.2) gives $2(\alpha_{i+1} + n_{i+1} - 1) \geq 2\alpha_i = \ell^+(Y,\mathfrak{s})$. But $2(\alpha_{i+1} + n_{i+1} - 1)$ is the degree of $U^{n_{i+1}-1}x_{i+1}$ and $2\ell^+(Y,\mathfrak{s})$ is the highest degree in $HF^+_{red}(Y,\mathfrak{s})$. So we must have $2(\alpha_{i+1} + n_{i+1} - 1) = \ell^+(Y,\mathfrak{s})$. But from assumption (4), this can only happen if $n_{i+1} = 1$, hence $2\alpha_{i+1} = \ell^+(Y,\mathfrak{s})$ and so $i+1 \in K$. So if $i \in K$ and $i < r$, then $i+1 \in K$. It follows that $K = \{r-a+1, r-a+2, \dots, r\}$. In particular, $r \in K$ and $\alpha_r = \ell^+(Y,\mathfrak{s})/2$. Therefore

$$j'(Y,\mathfrak{s}) = j_r = \alpha_r - \delta_{j_r} = \ell^+(Y,\mathfrak{s})/2 - \delta_0(Y,\mathfrak{s}) + dim_{\mathbb{F}}(HF^*_{red}(Y,\mathfrak{s})),$$

where the last equality holds since $\delta_0 - \delta_{j_r} = dim_{\mathbb{F}}(HF^*_{red}(Y,\mathfrak{s}))$.

Lastly, suppose that Y is an integral homology sphere and $\delta(Y) \in 2\mathbb{Z}$. From assumption (1), it follows that $HF^+_{red}(Y)$ is non-zero only in even degrees. Then from [36, Theorem 1.3], it follows that $dim_{\mathbb{F}}(HF^*_{red}(Y)) = \delta(Y) + \lambda(Y)$, where $\lambda(Y)$ is the Casson invariant. Hence we have that $j'(Y) = \ell^+(Y)/2 - \delta_0(Y) + \delta(Y) + \lambda(Y)$. □

3.3. **Delta invariants of Brieskorn spheres.** Given pairwise coprime integers $a_1, \dots, a_r$ with $r \geq 3$ define the Brieskorn homology sphere $\Sigma(a_1, \dots, a_r)$ to be the link of the singularity at the origin of the variey $\{(z_1, \dots, z_r) \in \mathbb{C}^n \mid b_{i1}z_1^{a_1} + b_{i2}z_2^{a_2} + \cdots + b_{ir}z_r^{a_r} = 0,\ 1 \leq i \leq r-2\}$, where (b_{ij}) is a sufficiently generic $(r-2) \times r$ matrix. Then $\Sigma(a_1, a_2, \dots, a_r)$ is the Seifert homology sphere $M(e_0, (a_1, b_1), \dots, (a_r, b_r))$, where $e_0, b_1, \dots, b_r$ are uniquely determined by the conditions that $0 < b_j < a_j$ and

$$e_0 + \sum_{j=1}^{r} \frac{b_j}{a_j} = -\frac{1}{a_1 a_2 \cdots a_r}. \tag{3.3}$$

Note that $e_0 < 0$ because $b_j > 0$ for all j.

Let $Y = \Sigma(a_1, \dots, a_r)$ be a Brieskorn homology sphere. The Seifert structure of Y gives a circle action. For any prime p, the restriction of the circle action to $\mathbb{Z}_p \subset S^1$ defines an action of the finite cyclic group $G = \mathbb{Z}_p$. We are interested in studying the equivariant Seiberg–Witten–Floer cohomology groups of Y and $-Y$ with respect to this action. Since Y has a unique spin^c-structure we will omit it from the notation and write $HSW^*_G(Y)$ and $HSW^*_G(-Y)$. Similarly the δ-invariants will be denoted $\delta^{(p)}_j(Y)$ and $\delta^{(p)}_j(-Y)$.

To understand the equivariant Seiberg–Witten–Floer cohomology groups of Y and $-Y$, we will use the spectral sequence relating it to the corresponding non-equivariant groups. We have

$$HSW^*(Y) \cong HF^+(Y) \cong \mathbb{F}[U]_{d(Y)} \oplus HF^+_{red}(Y)$$
$$HSW^*(-Y) \cong HF^+(-Y) \cong \mathbb{F}[U]_{-d(Y)} \oplus HF^+_{red}(-Y)$$

From [35], we have that $HF^+(-Y)$ is concentrated in even degrees. It follows that $d(Y)$ is even and $HF^+_{red}(Y)$ is concentrated in *odd* degrees. Furthermore $HF^+(-Y)$ can be computed from the graded roots algorithm [31]. This algorithm implies that $HF^+_{red}(-Y)$ is concentrated in degrees at least $-d(Y)$. Dually this implies that $HF^+_{red}(Y)$ is concentrated in degrees at most $d(Y)-1$.

Proposition 3.6. *Let $Y = \Sigma(a_1,\dots,a_r)$ be a Brieskorn homology sphere. We have:*

1. $\mathrm{rk}(HF^+_{red}(Y)) = -\delta(Y) - \lambda(Y)$ *where $\lambda(Y)$ is the Casson invariant of Y.*
2. $\delta^{(p)}_j(Y) = \delta^{(p)}_\infty(Y)$ *for all $j \geq 0$.*
3. $\delta(Y) \leq \delta^{(p)}_\infty(Y) \leq -\lambda(Y)$ *and* $\lambda(Y) \leq \delta^{(p)}_j(-Y) \leq -\delta(Y)$ *for all $j \geq 0$.*

Proof. From [36, Theorem 1.3] we have that $\chi(HF^+_{red}(Y)) = \delta(Y) + \lambda(Y)$. But $HF^+_{red}(Y)$ is concentrated in odd degrees, which gives (1).

We will prove (2) and (3) in the case that p is odd. The case $p=2$ is similar and omitted for brevity. Recall that for odd primes we have $H^*_{\mathbb{Z}_p} \cong \mathbb{F}[R,S]/(R^2)$ with $deg(R)=1$, $deg(S)=2$.

As in Section 3.1, let $\{E^{p,q}_r(Y), d_r\}$ denote the spectral sequence for the equivariant Seiberg–Witten–Floer cohomology $HSW^*_{\mathbb{Z}_p}(Y)$. We have a filtration $\{\mathcal{F}^*_j(Y)\}_{j\geq 0}$ on $HSW^*_{\mathbb{Z}_p}(Y)$ such that $E_\infty(Y)$ is isomorphic to the associated graded group

$$E_\infty(Y) \cong Gr(HSW^*_{\mathbb{Z}_p}(Y)) = \bigoplus_{j\geq 0} \mathcal{F}_j(Y)/\mathcal{F}_{j+1}(Y)$$

as $H^*_{S^1\times\mathbb{Z}_p}$-modules. Furthermore, we have that

$$E^{p,q}_2(Y) = H^p(\mathbb{Z}_p; HF^+_q(Y)).$$

Now since the $\mathbb{Z}_p$-action on Y is contained in a circle action, the generator of the $\mathbb{Z}_p$-action is smoothly isotopic to the identity and acts trivially on $HF^+(Y)$. Hence we further have

$$E^{p,q}_2(Y) \cong HF^+_q(Y) \otimes_{\mathbb{F}} H^p_{\mathbb{Z}_p}.$$

Choose an element $\theta \in HF^+_{d(Y)}(Y)$ such that $U^k\theta \neq 0$ for all $k \geq 0$. Then $HF^+(Y) \cong \mathbb{F}[U]\theta \oplus HF^+_{red}(Y)$ and hence

$$\begin{aligned} E_2(Y) &\cong H^*_{\mathbb{Z}_p}[U]\theta \oplus \Big(HF^+_{red}(Y) \otimes_{\mathbb{F}} H^*_{\mathbb{Z}_p}\Big) \\ &\cong \frac{\mathbb{F}[U,R,S]}{(R^2)}\theta \oplus \frac{HF^+_{red}(Y)[R,S]}{(R^2)}. \end{aligned}$$

We have a similar spectral sequence $E^{p,q}_r(-Y)$ and filtration $\{\mathcal{F}_j(-Y)\}$ on $HSW^*_{\mathbb{Z}_p}(-Y)$ such that $E^{*,*}_\infty(-Y)$ is isomorphic to the associated graded $H^*_{S^1\times\mathbb{Z}_p}$-module of the filtration. We have

$$E_2(-Y) \cong \frac{\mathbb{F}[U,R,S]}{(R^2)}\omega \oplus \frac{HF^+_{red}(-Y)[R,S]}{(R^2)}$$

for some ω of bi-degree $(0,-d(Y))$. Now recall that $HF^+_{red}(-Y)$ is concentrated in degrees at least $-d(Y)$. Hence $d_r(\omega)=0$ for all $r\geq 2$. It immediately follows that $\delta^{(p)}_0(-Y) = -\delta(Y)$ and hence $\delta^{(p)}_j(-Y) \leq -\delta(Y)$ for all $j \geq 0$.

Define $E_r(Y)_{red}$ to be the subgroup of $E_r(Y)$ consisting of elements x such that $U^k x = 0$ for some $k \geq 0$. So for $r=2$ we have $E_2(Y)_{red} \cong HF^+_{red}(Y)[R,S]/(R^2)$

and

$$E_2(Y) \cong \frac{\mathbb{F}[U,R,S]}{(R^2)}\theta \oplus E_2(Y)_{red}.$$

As explained in [5, §5], the image of the differential d_r is contained in $E_r(Y)_{red}$. It follows that for each $r \geq 2$, $E_r(Y)$ is of the form

$$E_r(Y) \cong \frac{\mathbb{F}[U,R,S]}{(R^2)}U^{m_r}\theta \oplus E_r(Y)_{red}$$

for some increasing sequence $0 = m_0 \leq m_1 \leq \dots$. It is further shown in [5, §5] that the sequence m_r must be eventually constant. Hence we have

$$E_\infty(Y) \cong \frac{\mathbb{F}[U,R,S]}{(R^2)}U^{m}\theta \oplus E_\infty(Y)_{red}$$

where $m = \lim_{r\to\infty} m_r$.

Since we work over a field $\mathbb{F}$, it is possible to (non-canonically) split the filtration $\{\mathcal{F}_j(Y)\}$ giving an isomorphism of $\mathbb{F}$-vector spaces

$$HSW^*_{\mathbb{Z}_p}(Y) \cong E^{*,*}_\infty(Y).$$

Under this isomorphism $\mathcal{F}_j(Y)$ corresponds to the subspace of $E_\infty(Y)$ spanned by homogeneous elements of bi-degree (p,q) where $p \geq j$. We fix a choice of such an isomorphism $\varphi\colon HSW^*_{\mathbb{Z}_p}(Y) \to E^{*,*}_\infty(Y)$ and henceforth identify $HSW^*_{\mathbb{Z}_p}(Y)$ with $E^{*,*}_\infty(Y)$. This isomorphism will typically not be an isomorphism of $H^*_{S^1\times\mathbb{Z}_p}$-modules. Nevertheless we can use the isomorphim φ to induce a new $H^*_{S^1\times\mathbb{Z}_p}$-module action on $E^{*,*}_\infty(Y)$ corresponding to the one on $HSW^*_{\mathbb{Z}_p}(Y)$. To be precise, if $c \in H^*_{S^1\times\mathbb{Z}_p}$, then we define $\widehat{c}\colon E_\infty \to E_\infty$ by $\widehat{c}\,x = \varphi(c\varphi^{-1}(x))$. Consider in particular $\widehat{U}$. The action of $\widehat{U}$ will typically not respect the bi-grading, but we can decompose it into homogeneous components as

$$\widehat{U} = U_{(0,2)} + U_{(1,1)} + U_{(2,0)} + U_{(3,-1)} + \cdots$$

where $U_{j,2-j}\colon E^{p,q}_\infty(Y) \to E^{p+j,q+2-j}_\infty(Y)$. Note that there are only terms of bi-degree $(j, 2-j)$ for $j \geq 0$ as $\widehat{U}$ respects the filtration $\{\mathcal{F}_j(Y)\}$. Note also that $U_{(0,2)} = U$ because taking the associated graded module of the filtration recovers the original $H^*_{S^1\times\mathbb{Z}_p}$-module structure on $E_\infty(Y)$.

Recall that $\delta^{(p)}_j(Y)$ is given by $i/2 - j$ where i is the least degree such that there exists an $x \in HSW^i_{\mathbb{Z}_p}(Y)$ with $U^k x = S^j U^l\theta \pmod{\mathcal{F}_{2j+1}(Y)}$ for some $k, l \geq 0$.

Recall that $E_\infty(Y) \cong \mathbb{F}[U,R,S]/(R^2)U^m\theta \oplus (E_\infty(Y))_{red}$. Since θ has bi-degree $(0, d(Y))$, each homogeneous element in $\mathbb{F}[U,R,S]/(R^2)U^m\theta$ has bi-degree (p,q) where $q \geq d(Y) + 2m$. On the other hand, since $E_\infty(Y)_{red}$ is a subquotient of $HF^+_{red}(Y)[R,S]/(R^2)$ and $HF^+_{red}(Y)$ is concentrated in degrees at most $d(Y) - 1$, we see that each homogeneous element x in $E_\infty(Y)_{red}$ has bi-degree (p,q), where $q \leq d(Y) - 1$. Then $Ux = U_{(0,2)}x + U_{(1,1)}x + \cdots$ is a sum of homogeneous terms $U_{(j,2-j)}x$ of bidegree $(p+j, q+2-j)$. Consider first $U_{(0,2)}x$. Since $U_{(0,2)} = U$, we see that $U_{(0,2)}x \in E_\infty(Y)_{red}$. Each of the remaining terms $U_{(j,2-j)}x$ for $j \geq 1$ has bi-degree of the form $(p', q') = (p+j, q+2-j)$, hence $q' = q+2-j \leq q+1 \leq d(Y)$. Thus if $m > 0$, then $U_{(j,2-j)}x$ must belong again to $E_\infty(Y)_{red}$. This implies that

$E_\infty(Y)_{red}$ is an $\mathbb{F}[\widehat{U}]$-submodule of $E_\infty(Y)$. It follows easily from this that for any fixed $j \geq 0$, there exists a $k \geq 0$ such that $U^k E_\infty(Y)_{red} \subseteq \mathcal{F}_{2j+1}(Y)$. This further implies that $x = S^j U^m \theta$ is a minimal degree element such that $U^k x = S^j U^l \theta \pmod{\mathcal{F}_{2j+1}(Y)}$ for some $k, l \geq 0$, hence $\delta_j^{(p)}(Y) = m + d(Y)/2$ for all $j \geq 0$. So $\delta_j^{(p)}(Y) = \delta_\infty^{(p)}(Y) = m + \delta(Y) > \delta(Y)$. From [5, Proposition 5.10], we also have that $m \leq \mathrm{rk}(HF^+_{red}(Y)) = -\delta(Y) - \lambda(Y)$, so $\delta_\infty^{(p)}(Y) = m + \delta(Y) \leq -\lambda(Y)$.

If $m = 0$, then we still have that $\widehat{U} E_\infty(Y)_{red} \subseteq E_\infty(Y)_{red}$, for if this were not the case then we would have some $x \in E_\infty(Y)^{a,d(Y)-1}_{red}$ such that $U_{(1,1)} x \notin E_\infty(Y)_{red}$. Further, since $HF^+_{red}(Y)$ is concentrated in degrees at most $d(Y)-1$, it follows that $U_{(0,2)} x = 0$ and that $U_{(1,1)} x = c\theta$ for some non-zero element $c \in H^{a+1}_{\mathbb{Z}_p}$. Then since $U_{(j,2-j)} x \in \mathcal{F}_{a+2}(Y)$ for $j \geq 2$, it follows that

$$\widehat{U} x = c\theta \pmod{\mathcal{F}_{a+2}(Y)}.$$

Hence $\delta_{\mathbb{Z}_p,c}(Y) < \delta(Y)$. On the other hand, we have already shown that $\delta_{\mathbb{Z}_p,1} = \delta_0^{(p)}(-Y) = -\delta(Y)$. Using this and [5, Theorem 4.4], we have

$$0 \leq \delta_{\mathbb{Z}_p,c}(Y) + \delta_{\mathbb{Z}_p,1}(-Y) < \delta(Y) - \delta(Y) = 0,$$

a contradiction. So even in the $m = 0$ case we still have that $\widehat{U} E_\infty(Y)_{red} \subseteq E_\infty(Y)_{red}$ and so we again conclude that $\delta_j^{(p)}(Y) = \delta_\infty^{(p)}(Y)$ for all $j \geq 0$ and that $\delta_\infty^{(p)}(Y) \leq -\lambda(Y)$. This proves (2).

We have proven everything except the inequality $\delta_j^{(p)}(-Y) \geq \lambda(Y)$. But from $\delta_0^{(p)}(Y) \leq -\lambda(Y)$ and [5, Theorem 4.4], it follows that

$$0 \leq \delta_j^{(p)}(-Y) + \delta_0^{(p)}(Y) \leq \delta_j^{(p)}(-Y) - \lambda(Y),$$

for all $j \geq 0$. Hence $\delta_j^{(p)}(-Y) \geq \lambda(Y)$. □

4. Computation of $\theta^{(c)}(T_{a,b})$

Let $a, b, c > 1$ be coprime integers and suppose also that c is prime. In this section we will prove that $\theta^{(c)}(T_{a,b}) = (a-1)(b-1)/2$, where $T_{a,b}$ is the (a,b)-torus knot. Since $\Sigma_c(T_{a,b}) = \Sigma(a,b,c)$, we will be interested in computing the invariant $j^{(c)}(\Sigma(a,b,c))$ of the Brieskorn sphere $\Sigma(a,b,c)$. Consequently we are interested in studying the structure of the Floer homology of $\Sigma(a,b,c)$. Combined with the results of Section 3, we will be able to carry out the computation of $\theta^{(c)}(T_{a,b})$.

4.1. **Floer homology of $-\Sigma(a,b,c)$.** Let $1 < a < b < c$ be pairwise coprime integers and let $\Sigma(a,b,c)$ denote the Brieskorn homology 3-sphere oriented so that it is the link of the singularity $x^a + y^b + z^c = 0$ in $\mathbb{C}^3$. The graded roots algorithm of Némethi can be used to compute the Floer homology $HF^+(-\Sigma(a,b,c))$ [31]. In this case the algorithm produces a τ function $\tau\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ from which a graded root (R,χ) may be constructed [31, Example 3.4 (3)]. The graded root (R,χ) gives rise to an associated $\mathbb{F}[U]$-module $\mathbb{H}(R,\chi)$, which up to a grading shift coincides with $HF^+(-\Sigma(a,b,c))$.

For a Brieskorn sphere $\Sigma(a,b,c)$, the τ function $\tau(i)$ is given as follows [31, §11], [6]. Let e_0, p_1, p_2, p_3 be the unique integers with $0 < p_1 < a$, $0 < p_2 < b$, $0 < p_3 < c$ and

$$abce_0 + p_1bc + ap_2c + abp_3 = -1.$$

Then

$$\tau(i) = \sum_{n=0}^{i-1} \Delta(n), \tag{4.1}$$

where

$$\Delta(n) = 1 - e_0 n - \left\lceil \frac{p_1 n}{a} \right\rceil - \left\lceil \frac{p_2 n}{b} \right\rceil - \left\lceil \frac{p_3 n}{c} \right\rceil. \tag{4.2}$$

By [6, Theorem 1.3], we have $\tau(n+1) \geq \tau(n)$ for $n > N$, where

$$N = abc - bc - ac - ab.$$

Note that $N > 0$ except when $(a,b,c) = (2,3,5)$. We will exclude this case from the discussion unless stated otherwise. For the purpose of computing the Floer homology, it suffices to consider the τ function up until the point where it is increasing. So we only need the restricted τ function $\tau\colon [0, N+1] \to \mathbb{Z}$, or equivalently, it suffices to determine the Δ-function $\Delta\colon [0,N] \to \mathbb{Z}$. Furthermore, the function Δ is completely determined on $[0,N]$ as follows [6, Theorem 1.3]. Let

$$G = \{g \in \mathbb{Z}_{\geq 0} \mid g = bci + acj + abk \text{ for some } i,j,k \in \mathbb{Z}_{\geq 0}\}$$

be the additive semigroup generated by bc, ac, ab. We have that $\Delta(n) \in \{-1, 0, 1\}$ for all $n \in [0,N]$. Moreover, for any $n \in [0,N]$, we have that

$$\Delta(n) = \begin{cases} 1 & n \in G, \\ -1 & N - n \in G, \\ 0 & \text{otherwise.} \end{cases}$$

It follow that for $n \in [0,N]$, we have

$$\tau(n+1) = |G \cap [0,n]| - |G \cap [N-n, N]| \tag{4.3}$$

where $|S|$ denotes the cardinality of a finite set S. From this expression it is clear that τ satisfies $\tau(N+1-n) = \tau(n)$ for all $n \in [0,N]$.

Remark 4.1. Note that since $0 < N < abc$, it follows from the Chinese remainder theorem that any $n \in G \cap [0,N]$ can be written as $abc(i/a + j/b + k/c)$ for *uniquely determined* integers i,j,k satisfying $0 < i < a$, $0 < j < b$, $0 < k < c$.

We briefly explain how to obtain $HF^+(-\Sigma(a,b,c))$ from the τ function. First we recall the definition of a graded root:

Definition 4.2 ([31])**.** Let R be an infinite tree with vertices $\mathcal{V}$ and edges $\mathcal{E}$. Denote by $[u,v]$ the edge joining u and v. Edges are unordered, so $[u,v] = [v,u]$. We say that R is a *graded root* with grading $\chi\colon \mathcal{V} \to \mathbb{Z}$ if

(a) $\chi(u) - \chi(v) = \pm 1$ for all $[u,v] \in \mathcal{E}$,
(b) $\chi(u) > \min\{\chi(v), \chi(w)\}$ if $[u,v], [u,w] \in \mathcal{E}$ and $v \neq w$,

(c) χ is bounded from below, $\chi^{-1}(k)$ is finite for any $k \in \mathbb{Z}$ and $|\chi^{-1}(k)| = 1$ for all sufficiently large k.

Define a partial relation $\succeq$ on (R, χ) by declaring $v \succeq w$ if there exists a sequence of vertices $w = w_0, w_1, \ldots, w_k = v$ such that $[w_{i-1}, w_i] \in \mathcal{E}$ and $\chi(w_i) - \chi(w_{i-1}) = 1$ for $i = 1, \ldots, k$. From the axioms (a)-(c), it is easily seen that for each $u \in \mathcal{V}$ there is a unique vertex $v \in \mathcal{V}$ with $[u, v] \in \mathcal{E}$ and $\chi(v) = \chi(u) + 1$. From this and axiom (c), it is seen that for any two vertices $u, v \in \mathcal{V}$ there is a unique $\succeq$-minimal element $w \in \mathcal{V}$ with $w \succeq v$ and $w \succeq u$. We denote this element by $\sup(u, v)$.

For $v \in \mathcal{V}$, let δ_v be the number of edges which have v as an endpoint. Clearly $\delta_v \geq 1$ and $\delta_v = 1$ if and only if v is $\succeq$-minimal.

Given a graded root (R, χ) define $\mathbb{H}(R, \chi)$ to be the graded $\mathbb{F}[U]$-module constructed as follows. A basis for $\mathbb{H}(R, \chi)$ is given by $\{e_v \mid v \in \mathcal{V}\}$ where $deg(e_v) = 2\chi(v)$ and $Ue_v = e_u$, where u is the unique $u \in \mathcal{V}$ such that $[u, v] \in \mathcal{E}$ and $\chi(u) = \chi(v) + 1$. Note that our definition of $\mathbb{H}(R, \chi)$ differs slightly from the one in [31]. This is because we are using Floer cohomology rather than homology. The underlying graded abelian group of $\mathbb{H}(R, \chi)$ is the same as that in [31], but our $\mathbb{F}[U]$-module action is the transpose of the one in [31].

For a positive integer n let $\mathcal{T}^+(n)$ denote the $\mathbb{F}[U]$-module $\mathbb{F}[U]/(U^n)$. For any integer d and any $\mathbb{F}[U]$-module M, let M_d be the degree-shift of M defined by $(M_d)^j = M^{j-d}$. In particular, for any positive integer n and any integer d, we have the $\mathbb{F}[U]$-module $\mathcal{T}^+_d(n) = (\mathcal{T}^+(n))_d$. We refer to $\mathcal{T}^+_d(n)$ as a *tower of length* n. The lowest degree in $\mathcal{T}^+_d(n)$ is d and the highest degree is $d + 2n - 2$.

Proposition 4.3 ([31], Proposition 3.5.2)**.** *Let (R, χ) be a graded root. Set $I = \{v \in \mathcal{V} \mid \delta_v = 1\}$. We choose an ordering on the set I as follows. The first element $v_1 \in I$ is chosen so that $\chi(v_1)$ is the minimal value of χ. If $v_1, \ldots, v_k \in I$ are already determined and $J = \{v_1, \ldots, v_k\} \neq I$, then v_{k+1} is chosen from $I \setminus J$ such that $\chi(v_{k+1}) = \min_{v \in I \setminus J} \chi(v)$. Let $w_{k+1} \in \mathcal{V}$ be the unique $\succeq$-minimal vertex of R which dominates both v_{k+1} and some element of J. Then we have an isomorphism of $\mathbb{F}[U]$-modules:*

$$\mathbb{H}(R, \chi) \cong \mathbb{F}[U]_{2\chi(v_1)} \oplus \bigoplus_{k \geq 2} \mathcal{T}^+_{2\chi(v_k)}(\chi(w_k) - \chi(v_k)).$$

In particular we have an isomorphism

$$\mathbb{H}_{red}(R, \chi) \cong \bigoplus_{k \geq 2} \mathcal{T}^+_{2\chi(v_k)}(\chi(w_k) - \chi(v_k)).$$

Let $\tau\colon \{0, 1, \ldots, l\} \to \mathbb{Z}$ be any function and suppose there is an l such that $\tau(i+1) \geq \tau(i)$ for all $i \geq l$. Following [31, Example 3.4], we construct a graded root (R, χ) from τ as follows. Start with the vertex set $\widetilde{\mathcal{V}} = \{v_i^k\}$ where $0 \leq i \leq l$ and $k \geq \tau(i)$, edge set $\widetilde{\mathcal{E}} = [v_i^k, v_i^{k+1}]$ and define $\widetilde{\chi}$ by $\widetilde{\chi}(v_i^k) = k$. Now define $\mathcal{V} = \widetilde{\mathcal{V}}/{\sim}$ and $\mathcal{E} = \widetilde{\mathcal{E}}/{\sim}$ where for $i \leq j$ we set $v_i^k \sim v_j^k$ and $[v_i^k, v_i^{k+1}] \sim [v_j^k, v_j^{k+1}]$ if $k \geq \max_{i \leq u \leq j} \tau(u)$. We define χ to be the function on $\mathcal{V}$ induced by $\widetilde{\chi}$. It is easily checked that $(\mathcal{V}, \mathcal{E}, \chi)$ is a graded root. We call it the *graded root associated to* τ.

Let $1 < a < b < c$ be pairwise coprime integers and let τ be the τ-function defined as in Equations (4.1)-(4.2). Let (R_τ, χ_τ) be the associated graded root. Then from [31] it follows that up to a grading shift $HF^+(-\Sigma(a,b,c))$ is isomorphic to $\mathbb{H}(R_\tau, \chi_\tau)$:

$$HF^+(-\Sigma(a,b,c)) \cong \mathbb{H}(R_\tau, \chi_\tau)[u].$$

The precise value of the grading shift u can be computed [31, Proposition 4.7], but we will not need this.

Observe that $\tau(0) = \tau(N+1) = 0$ and that $\tau(1) = \tau(N) = 1$. We will see in Proposition 4.5 that $\tau(n) \leq 1$ for all $n \in [0, N+1]$, hence 1 and N are global maxima of τ. Let $n \in [0, N+1]$ be a global maximum of τ, that is, $\tau(n) = 1$. We will say n is a *trivial maximum* if we either have that $\tau(i) = 1$ for all $1 \leq i \leq n$ or $\tau(i) = 1$ for all $n \leq i \leq N$.

Proposition 4.4. *Suppose that all global maxima of τ are trivial. Let $\ell^+(-\Sigma(a,b,c))$ denote the highest non-zero degree in $HF^+_{red}(-\Sigma(a,b,c))$. Any non-zero element in the image of $U\colon HF^+_{red}(-\Sigma(a,b,c)) \to HF^+_{red}(-\Sigma(a,b,c))$ has degree strictly less than $\ell^+(-\Sigma(a,b,c))$.*

Proof. Let (R_τ, χ_τ) be the graded root associated to τ. From the definition of (R_τ, χ_τ), it is clear that (R_τ, χ_τ) depends only on the values of the local minima and local maxima of τ. More precisely, let S^{min} be the set of $m \in [0, N+1]$ such that $m = 0$, $m = N+1$, or $\tau(m-1) > \tau(m)$ and there exists a $k \geq 0$ such that $\tau(i) = \tau(m)$ for $m \leq i \leq m+k$ and $\tau(m+k+1) > \tau(m)$. Similarly, let S^{max} be the set of $M \in [1, N]$ such that $\tau(M) > \tau(M-1)$ and there exists a $k \geq 0$ such that $\tau(i) = \tau(M)$ for $M \leq i \leq M+k$ and $\tau(M+k+1) < \tau(M)$. Write the elements of S^{min} and S^{max} in increasing order as $S^{min} = \{m_0, m_1, \ldots, m_r\}$, $S^{max} = \{M_1, M_2, \ldots, M_r\}$ where $0 = m_0 < m_1 < \cdots < m_r = N+1$ and $1 = M_1 < M_2 < \cdots < M_r$. The local minima and maxima must occur in alternating order, so we have

$$0 = m_0 < M_1 < m_1 < M_2 < \cdots < m_{r-1} < M_r < m_r = N+1.$$

Furthermore, M_1 and M_r correspond to the trivial global maxima. By assumption these are the only global maxima. Hence $\tau(M_j) \leq 0$ for $1 < j < r$.

Recall that (R_τ, χ_τ) is constructed as follows. Start with $\widetilde{\mathcal{V}} = \{v_i^k\}$ where $0 \leq i \leq N+1$ and $k \geq \tau(i)$, edge set $\widetilde{\mathcal{E}} = [v_i^k, v_i^{k+1}]$ and define $\widetilde{\chi}$ by $\widetilde{\chi}(v_i^k) = k$. Then (R_τ, χ_τ) is obtained by taking the quotient of this by the equivalence relation $\sim$, where for $i \leq j$ we set $v_i^k \sim v_j^k$ and $[v_i^k, v_i^{k+1}] \sim [v_j^k, v_j^{k+1}]$ if $k \geq \max_{i \leq u \leq j} \tau(u)$.

It is easily seen that the minima of $\succeq$ on (R_τ, χ_τ) are precisely the elements $I = \{v_{m_i}^{\tau(m_i)}\}_{0 \leq i \leq r}$. From Proposition 4.5 we have that $\tau(n) \leq 1$ for all $n \in [0, N+1]$ and hence $\tau(m_i) \leq 0$ for any local minimum. Hence we may choose a permutation $\sigma\colon \{0, 1, \ldots, r\} \to \{0, 1, \ldots, r\}$ such that $\tau(m_{\sigma(i)}) \leq \tau(m_{\sigma(i+1)})$ for $0 \leq i \leq r-1$ and $\sigma(r-1) = 0$, $\sigma(r) = r$. This gives an ordering of the set I as in the statement of Proposition 4.3, namely $I = \{v_0', v_1', v_2', \ldots, v_r'\}$ where $v_i' = v_{m_{\sigma(i)}}^{\tau(m_{\sigma(i)})}$. For each $i \in \{1, \ldots, r\}$ and each $j \in \{0, 1, \ldots, i-1\}$ one finds that $sup(v_i', v_j') = v_{m_{\sigma(i)}}^{K_{ij}}$, where K_{ij} is the maximum of $\tau(M_a)$ for all a such that M_a lies in the interval joining $m_{\sigma(i)}$ and $m_{\sigma(j)}$. Now let $w_1, w_2, \dots, w_r$ be defined as in the statement of Proposition 4.3. Then it follows that $w_i = v^{K_i}_{m_{\sigma(i)}}$, where $K_i = \min_{0 \le j \le i-i} K_{ij}$. Now suppose that $i \neq r-1, r$. Then for all $j \in \{0, \dots, i-1\}$ we have that $\sigma(i) \neq 0, r$ and $\sigma(j) \neq 0, r$. It follows that each maximum M_a in the interval joining $m_{\sigma(i)}$ and $m_{\sigma(j)}$ is not trivial and hence $\tau(M_a) \le 0$. Hence $K_{ij} \le 0$ for all $i \le r-2$ and $j < i$. This also implies that $\chi(w_i) = K_i \le 0$ for all $i \le r-2$. On the other hand since $\tau(M_1) = \tau(M_r) = 1$, it follows that $K_{r-1} = K_r = 1$ and thus $\chi(w_{r-1}) = \chi(w_r) = 1$.

From Proposition 4.3, we have an isomorphism

$$\mathbb{H}_{red}(R_\tau, \chi_\tau) \cong \bigoplus_{k=1}^{r} \mathcal{T}^+_{2\chi(v'_k)}(\chi(w_k) - \chi(v'_k)).$$

The highest degree in the tower $\mathcal{T}^+_{2\chi(v'_k)}(\chi(w_k) - \chi(v'_k))$ is $2\chi(w_k) - 2$. For $k \neq r-1, r$ we have $2\chi(w_k) - 2 \le -2$ whereas for $k = r-1, r$ we have $2\chi(w_k) - 2 = 0$. Thus the highest non-zero degree in $\mathbb{H}_{red}(R_\tau, \chi_\tau)$ is 0 and only the towers for $k = r-1, r$ attain this degree. Moreover the length of the tower $\mathcal{T}^+_{2\chi(v'_k)}(\chi(w_k) - \chi(v'_k))$ is $\chi(w_k) - \chi(v'_k)$ which for $k = r-1, r$ equals 1, since $v'_{r-1} = v^{\tau(m_{\sigma(r-1)})}_{\sigma(r-1)} = v^{\tau(0)}_0 = v^0_0$ and $v'_r = v^{\tau(m_{\sigma(r)})}_{\sigma(r)} = v^{\tau(m_r)}_r = v^0_r$, so $\chi(v'_{r-1}) = \chi(v'_r) = 0$. It follows that any non-zero element in the image of $U\colon \mathbb{H}_{red}(R_\tau, \chi_\tau) \to \mathbb{H}_{red}(R_\tau, \chi_\tau)$ has degree strictly less than 0. Now since $HF^+_{red}(-\Sigma(a,b,c))$ is isomorphic to a grading shift of $\mathbb{H}_{red}(R_\tau, \chi_\tau)$, it also follows that any non-zero element in the image of $U\colon HF^+_{red}(-\Sigma(a,b,c)) \to HF^+_{red}(-\Sigma(a,b,c))$ has degree strictly less than $\ell^+(-\Sigma(a,b,c))$. □

Proposition 4.5. *Let $1 < a < b < c$ be pairwise coprime integers and assume that $(a,b,c) \neq (2,3,5)$. Then $\tau(n+1) \le 1$ for all $n \in [0, N]$. Furthermore, all maxima of τ are trivial, except in the following cases: $(a,b,c) = (2,3,6n-1)$, $n \ge 2$ or $(a,b,c) = (2,3,6n+1)$, $n \ge 1$.*

Proof. For $n \in [0, N]$, let α_n be the largest element of G less than or equal to $N - n$. Suppose that $j \in G \cap [0, n] \setminus \{0\}$. We claim that $j + \alpha_n \in G \cap [N-n, N]$. Clearly $j + \alpha_n \in G$ because $j, \alpha_n \in G$. Since $j \le n$ and $\alpha_n \le N - n$, we also have $j + \alpha_n \le N$. Lastly, since $j > 0$, we have $j + \alpha_n \ge N - n$ for if not then α_n is not the largest element of G less than or equal to $N - n$. Therefore $j + \alpha_n \in G \cap [N-n, N]$. So we have constructed an injective map $\phi_n : G \cap [0, n] \setminus \{0\} \to G \cap [N-n, N]$, given by $\phi(j) = j + \alpha_n$. Therefore $|G \cap [0, n]| \le 1 + |G \cap [N-n, N]$. Comparing with Equation (4.3), we have shown that $\tau(n) \le 1$.

Now suppose that $n_0 \in [1, N]$ is a maximum of τ, so $\tau(n_0) = 1$. By the symmetry $\tau(N+1+i) = \tau(i)$ of the τ function, it suffices to consider only maxima n_0 such that $n_0 \ge (N+1)/2$. Since τ can only change by ± 1, there exists an $n \ge n_0$ such that $\tau(i) = 1$ for $n_0 \le i \le n$ and $\tau(n+1) = 0$. Since $\tau(N+1) = 0$, we see that $n \le N$. Also we have $n \ge n_0 \ge (N+1)/2$.

Since $\tau(n) = 1$ and $\tau(n+1) = 0$, we have that $\Delta(n) = \tau(n+1) - \tau(n) = -1$. Hence $N - n \in G$ and $n \notin G$. Since $N - n \in G$, it follows that $\alpha_n = N - n$. Consider the map $\phi\colon G \cap [0, n] \to G \cap [N-n, N]$ given by $\phi(j) = j + N - n$ (we already saw that $\phi(j) \in G \cap [N-n, N]$ for $j > 0$ and we also have $\phi(0) = N - n \in G \cap [N-n, N]$).

Since $\tau(n+1)=0$, it follows that ϕ is a bijection. Thus every $g \in G \cap [N-n, N]$ can be written uniquely as $g = (N-n)+g'$ for some $g \in G \cap [0,n]$. Since $N-n \in G$, we may write it in the form

$$N - n = abc\left(\frac{i_0}{a} + \frac{j_0}{b} + \frac{k_0}{c}\right)$$

for some $i_0, j_0, k_0 \geq 0$. It follows that any $g \in G \cap [N-n, N]$ has the form

$$g = abc\left(\frac{i}{a} + \frac{j}{b} + \frac{k}{c}\right) \tag{4.4}$$

where $i \geq i_0$, $j \geq j_0$, $k \geq k_0$. Furthermore, by Remark 4.1, this is a necessary condition. That is, if $g \in G \cap [N-n, N]$ is written as in (4.4) then we must have $i \geq i_0$, $j \geq j_0$, $k \geq k_0$.

Suppose that $i_0 > 0$. There exists an integer k such that $1/a$ lies in the interval $[(k-1)/c, k/c]$. Thus

$$\frac{1}{a} \leq \frac{k}{c} \leq \frac{1}{a} + \frac{1}{c}.$$

Consider

$$g = (N-n) + abc\left(-\frac{1}{a} + \frac{k}{c}\right) = abc\left(\frac{i_0 - 1}{a} + \frac{j_0}{b} + \frac{k_0 + k}{c}\right).$$

Then $g \in G$ and $g \geq N-n$ since $k/c - 1/a \geq 0$. But g is not of the form $abc(i'/a + j'/b + k'/c)$ with $i' \geq i_0$, $j' \geq j_0$, $k' \geq k_0$, hence $g \notin G \cap [N-n, N]$. The only way this can happen is that $g > N$. Thus

$$g = (N-n) + abc\left(-\frac{1}{a} + \frac{k}{c}\right) > N,$$

which implies that $n < abc(-1/a + k/c)$. But $k/c \leq 1/a + 1/c$, so $n < abc/c = ab$. Note that since $a < b < c$, ab is the smallest positive element of G. Thus $i \notin G$ for all $1 \leq i \leq n$. So $\Delta(i) \leq 0$ for $1 \leq i \leq n$. But we also have that

$$\tau(n) = 1 = \sum_{i=0}^{n-1} \Delta(i) = 1 + \sum_{i=1}^{n-1} \Delta(i).$$

Since $\Delta(i) \leq 0$ for $1 \leq i \leq n$, the only way we can have equality is that $\Delta(i) = 0$ for $1 \leq i \leq n$ and hence $\tau(i) = 1$ for $1 \leq i \leq n$. This means that n is a trivial maximum.

Next, suppose that $i_0 = 0$ and $j_0 > 0$. Then by a similar argument to the $i_0 > 0$ case, there exists a k such that $1/b \leq k/c \leq 1/b + 1/c$. Consider

$$g = (N-n) + abc\left(-\frac{1}{b} + \frac{k}{c}\right).$$

Arguing as in the $i_0 > 0$ case, we see that n is again a trivial maximum.

Now consider the case that $i_0 = j_0 = 0$. Hence $N - n = abc(k_0/c)$ for some $k_0 \geq 0$. We will assume that $k_0 > 0$ for if $k_0 = 0$, then $n = N$ is a trivial maximum. Furthermore, recall that we are assuming $n \geq (N+1)/2 > N/2$. Therefore

$$\frac{k_0}{c} < \frac{N}{2abc} = \frac{1}{2} - \frac{1}{2a} - \frac{1}{2b} - \frac{1}{2c}. \tag{4.5}$$

Recall that each element of $G \cap [N-n, N]$ has the form given by (4.4) with $i \geq 0$, $j \geq 0$, $k \geq k_0$. It follows that there does not exist a solution to

$$\frac{k_0}{c} \leq \frac{j}{b} + \frac{k_0 - 1}{c} \leq 1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} \tag{4.6}$$

with $j \geq 0$. For if such a j exists, we would have

$$N - n = abc\left(\frac{k_0}{c}\right) < abc\left(\frac{j}{b} + \frac{k_0 - 1}{c}\right) \leq N$$

and thus $g = abc(j/b + (k_0 - 1)/c)$ would be an element of $G \cap [N-n, N]$ not of the form $abc(i'/a + j'/b + k'/c)$ with $i' \geq i_0$, $j' \geq j_0$, $k' \geq k_0$. In particular, $j = 1$ is not a solution. But since $b < c$, we have $1/b + (k_0 - 1)/c > k_0/c$. So it must be the second inequality in (4.6) that is violated. That is, we must have

$$\frac{j}{b} + \frac{k_0 - 1}{c} > 1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}.$$

Rearranging and using (4.5), we find that

$$\frac{1}{a} + \frac{3}{b} - \frac{1}{c} > 1. \tag{4.7}$$

If this condition is not satisfied, then all maxima of τ are trivial.

If $a \geq 4$, then $b \geq 5$ and

$$\frac{1}{a} + \frac{3}{b} - \frac{1}{c} < \frac{1}{4} + \frac{3}{5} < 1.$$

So (4.7) implies $a < 4$, hence $a = 2$ or 3.

If $a = 3$, then since $1/3 + 3/5 < 1$, (4.7) implies that $b \leq 5$. So $b = 4$ and then (4.7) implies that $c > 12$.

If $a = 2$, then since $1/2 + 3/6 - 1/c < 1$, we must have $b < 6$. Hence $b = 3$ or 5.

If $a = 2, b = 5$, then (4.7) is satisfied for any $c > 10$.

If $a = 2, b = 3$, then (4.7) is satisfied for any $c > 3$. But we are excluding $(2, 3, 5)$ so $c > 5$.

To summarise, (4.7) is satisfied only in the following cases:

(1) $(a, b, c) = (3, 4, c)$, $c > 12$.
(2) $(a, b, c) = (2, 5, c)$, $c > 10$.
(3) $(a, b, c) = (2, 3, c)$, $c > 5$.

It remains to show in cases (1) and (2) we still have that all maxima of τ are trivial. We do this by a direct computation of the τ function.

In case (1) there are four subcases:

(1a) $(a, b, c) = (3, 4, 12k + 1)$, $k \geq 1$.
(1b) $(a, b, c) = (3, 4, 12k + 5)$, $k \geq 1$.
(1c) $(a, b, c) = (3, 4, 12k + 7)$, $k \geq 1$.
(1d) $(a, b, c) = (3, 4, 12k + 11)$, $k \geq 1$.

Similarly in case (2) there are four subcases:

(2a) $(a, b, c) = (2, 5, 10k + 1)$, $k \geq 1$.
(2b) $(a, b, c) = (2, 5, 10k + 3)$, $k \geq 1$.
(2c) $(a, b, c) = (2, 5, 10k + 7)$, $k \geq 1$.

(2d) $(a, b, c) = (2, 5, 10k + 9)$, $k \geq 1$.

Case (1a): $N = 60k - 7$, G is generated by $\{12, 36k + 3, 48k + 4\}$. By the symmetry of the τ function, it suffices to only look for maxima of $\tau(n + 1)$ with $n \in [0, N/2]$. We consider the intersection of G with $[0, N/2]$. Since $36k + 3 > N/2$, all elements of $G \cap [0, N]$ have the form $12j$ for some j. Thus

$$G \cap [0, N/2] = \{12j\}_{0 \leq j \leq \lfloor N/24 \rfloor}.$$

Now we consider the intersection of $N - G$ with $[0, N]$. Elements of $N - G$ in the range $[0, N]$ have the form $N - 12u$, $N - (36k + 3) - 12u$ or $N - (48k + 4) - 12u$ for some $u \geq 0$. In the first case, we have

$$N - 12u = 12(5k - 1 - u) + 5.$$

In the second case, we have

$$N - (36k + 3) - 12u = 12(2k - 1 - u) + 2$$

and in the third case, we have

$$N - (48 + 4) - 12u = 12(k - 1 - u) + 1.$$

Therefore

$$(N - G) \cap [0, N] = \{12j + 5\}_{0 \leq j \leq 5k-1} \cup \{12j + 2\}_{0 \leq j \leq 2k-1} \cup \{12j + 1\}_{0 \leq j \leq k-1}.$$

Partition $[0, N]$ into subintervals $I_j = [12j, 12j + 11]$, $0 \leq j \leq 5k - 2$ and $I_{5k-1} = [60k - 12, 60k - 7]$. We are only interested in the subintervals which intersect with $[0, N/2]$, so we can assume $j \leq \lfloor N/24 \rfloor$. Under this condition, we have $G \cap I_j \cap G = \{12j\}$ and

$$(N - G) \cap I_j = \begin{cases} 12j + 1, 12j + 2, 12j + 5 & 0 \leq j \leq k - 1, \\ 12j + 2, 12j + 5 & k \leq j \leq 2k - 1, \\ 12j + 5 & 2k \leq j. \end{cases}$$

It follows easily that τ has only trivial maxima.

Case (1b): $N = 60k + 13$, G is generated by $12, 36k + 15, 48k + 20$. Arguing similarly to case (1a), we find

$$G \cap [0, N/2] = \{12j\}_{0 \leq j \leq \lfloor N/24 \rfloor}$$

and

$$(N - G) \cap [0, N] = \{12j + 1\}_{0 \leq j \leq 5k+1} \cup \{12j + 10\}_{0 \leq j \leq 2k-1} \cup \{12j + 5\}_{0 \leq j \leq k-1}.$$

By a similar argument, we see that τ has only trivial maxima.

Case (1c): $N = 60k + 23$, G is generated by $12, 36k + 21, 48k + 28$. We find

$$G \cap [0, N/2] = \{12j\}_{0 \leq j \leq \lfloor N/24 \rfloor}$$

and

$$(N - G) \cap [0, N] = \{12j + 11\}_{0 \leq j \leq 5k+1} \cup \{12j + 2\}_{0 \leq j \leq 2k} \cup \{12j + 7\}_{0 \leq j \leq k-1}.$$

We then see that τ has only trivial maxima.

Case (1d): $N = 60k + 43$, G is generated by $12, 36k + 33, 48k + 44$. We find

$$G \cap [0, N/2] = \{12j\}_{0 \le j \le \lfloor N/24 \rfloor}$$

and

$$(N - G) \cap [0, N] = \{12j + 7\}_{0 \le j \le 5k+3} \cup \{12j + 10\}_{0 \le j \le 2k} \cup \{12j + 11\}_{0 \le j \le k-1}.$$

We then see that τ has only trivial maxima.

Case (2a): $N = 30k - 7$, G is generated by $10, 20k + 2, 50k + 5$. We find

$$G \cap [0, N/2] = \{10j\}_{0 \le j \le \lfloor N/20 \rfloor}$$

and

$$(N - G) \cap [0, N] = \{10j + 3\}_{0 \le j \le 3k-1} \cup \{10j + 1\}_{0 \le j \le k-1}$$

We then see that τ has only trivial maxima.

Case (2b): $N = 30k - 1$, G is generated by $10, 20k + 6, 50k + 15$. We find

$$G \cap [0, N/2] = \{10j\}_{0 \le j \le \lfloor N/20 \rfloor}$$

and

$$(N - G) \cap [0, N] = \{10j + 9\}_{0 \le j \le 3k-1} \cup \{10j + 3\}_{0 \le j \le k-1}$$

We then see that τ has only trivial maxima.

Case (2c): $N = 30k + 11$, G is generated by $10, 20k + 14, 50k + 35$. We find

$$G \cap [0, N/2] = \{10j\}_{0 \le j \le \lfloor N/20 \rfloor}$$

and

$$(N - G) \cap [0, N] = \{10j + 1\}_{0 \le j \le 3k+1} \cup \{10j + 7\}_{0 \le j \le k-1}$$

We then see that τ has only trivial maxima.

Case (2d): $N = 30k + 17$, G is generated by $10, 20k + 18, 50k + 45$. We find

$$G \cap [0, N/2] = \{10j\}_{0 \le j \le \lfloor N/20 \rfloor}$$

and

$$(N - G) \cap [0, N] = \{10j + 7\}_{0 \le j \le 3k+1} \cup \{10j + 9\}_{0 \le j \le k-1}$$

We then see that τ has only trivial maxima.

□

Proposition 4.6. *Let $1 < a < b < c$ be pairwise coprime integers and assume $(a, b, c) \neq (2, 3, 5)$. Let $Y = -\Sigma(a, b, c)$ and let $\ell^+(Y)$ denote the highest non-zero degree in $HF^+_{red}(Y)$. Then $\ell^+(Y) = 2\delta(Y) - 2\min\{\tau\}$. Moreover, any non-zero element in the image of $U\colon HF^+_{red}(Y) \to HF^+_{red}(Y)$ has degree strictly less than $\ell^+(Y)$.*

Proof. Recall that $HF^+(Y)$ is isomorphic to $\mathbb{H}(R_\tau, \chi_\tau)$ up to an overall grading shift. The lowest degree in $HF^+(-\Sigma(a, b, c))$ is $2\delta(Y)$ and the lowest degree in $\mathbb{H}(R_\tau, \chi_\tau)$ is $2\min\{\tau\}$, hence the grading shift is $2\delta(Y) - 2\min\{\tau\}$. The highest non-zero degree in $\mathbb{H}(R_\tau, \chi_\tau)$ is easily seen to be 0, hence $\ell^+(Y) = 2\delta(Y) - 2\min\{\tau\}$.

If $(a, b, c) \neq (2, 3, 6n \pm 1)$ for any n, then all global maxima of τ are trivial by Proposition 4.5. Then by Proposition 4.4 we have that any non-zero element in the image of $U\colon HF^+_{red}(Y) \to HF^+_{red}(Y)$ has degree strictly less than $\ell^+(Y)$.

If $(a,b,c) = (2,3,6n \pm 1)$ for some n, then it is easily seen that $HF^+_{red}(Y)$ is concentrated in a single degree and hence U acts trivially on $HF^+_{red}(Y)$.

□

4.2. **Computation of $j^{(c)}(T_{a,b})$ and $\theta^{(c)}(T_{a,b})$.** Following [6], define $\kappa(a,b,c)$ to be the cardinality of $G \cap [0,N]$.

Lemma 4.7. *Let $\tau_1(a,b,c)$ denote the number of integers x,y,z with $0 < x < a$, $0 < y < b$, $0 < z < c$ and $x/a + y/b + z/c < 1$. Then $\tau_1(a,b,c) = \kappa(a,b,c)$.*

Proof. By Remark 4.1, any $x \in G$ with $x \leq N$ has a unique representation $xbc + yac + zab$ with $0 \leq x < a$, $0 \leq y < b$, $0 \leq z < c$. Thus $\kappa(a,b,c)$ is the number of points $(x,y,z) \in \mathbb{Z}^3_{\geq 0}$ such that

$$xbc + yac + zab \leq abc - bc - ac - ab.$$

From [6, Theorem 1.3] it follows that $N \notin G$, so $\kappa(a,b,c)$ is also the number of $x,y,z \geq 0$ such that

$$xbc + yac + zab < abc - bc - ac - ab.$$

Dividing through by abc, this is equivalent to

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} < 1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}$$

which can be rewritten as

$$\frac{x+1}{a} + \frac{y+1}{b} + \frac{z+1}{q} < 1.$$

Setting $x' = x+1$, $y' = y+1$, $z' = z+1$, we see that $\kappa(a,b,c)$ is equal to $\tau_1(a,b,c)$. □

Recall that λ denotes the Casson invariant. From [38, §19], we have $8\lambda(\Sigma(a,b,c)) = -(a-1)(b-1)(c-1) + 4\tau_1(a,b,c)$. Thus Lemma 4.7 gives:

$$8\lambda(\Sigma(a,b,c)) = -(a-1)(b-1)(c-1) + 4\kappa(a,b,c). \tag{4.8}$$

From [15], [10], we also have that

$$8\lambda(\Sigma(a,b,c)) = \sum_{j=1}^{c-1} \sigma_{T_{b,c}}(j/c) = \sigma^{(c)}(T_{a,b}).$$

Theorem 4.8. *Let $a,b,c > 1$ be coprime integers and suppose that c is a prime number. Then*

$$j^{(c)}(-T_{a,b}) = \begin{cases} \kappa(a,b,c) & \text{if } c \text{ is odd,} \\ 2\kappa(a,b,c) & \text{if } c = 2 \end{cases}$$

and

$$\theta^{(c)}(T_{a,b}) = (a-1)(b-1)/2.$$

Proof. Let $Y = -\Sigma(a,b,c)$. If (a,b,c) is a permutation of $(2,3,5)$, then $HF^+_{red}(Y) = 0$ and $\kappa(2,3,5) = 0$. Then from [5, Proposition 3.16], it follows that $j^{(c)}(-T_{a,b}) = 0$ and $\theta^{(c)}(T_{a,b}) = -\sigma^{(c)}(T_{a,b})/(c-1)$. But since $\kappa(2,3,5) = 0$, Equation (4.8) gives $\sigma^{(c)}(T_{a,b}) = 8\lambda(\Sigma(a,b,c)) = -(a-1)(b-1)(c-1)$. Hence $\theta^{(c)}(T_{a,b}) = (a-1)(b-1)/2$.

Henceforth we assume that (a,b,c) is not a permutation of $(2,3,5)$ and hence $HF^+_{red}(Y) \neq 0$. Recall that $\Sigma(a,b,c)$ is the boundary of a negative definite plumbing [32] whose plumbing graph has only one bad vertex in the sense of [35]. Then from [35, Corollary 1.4], we have that $HF^+(Y)$ is concentrated in even degrees. As explained in [5, §7] we have $\Sigma(a,b,c) = \Sigma_c(T_{a,b})$ and the generator of the $\mathbb{Z}_c$-action on $\Sigma(a,b,c)$ is isotopic to the identity. Therefore in the spectral sequence $\{E_r^{p,q}, d_r\}$ for the equivariant Seiberg–Witten–Floer cohomology of Y, we have $E_2^{0,q} = H^0(\mathbb{Z}_c; HSW^q(Y)) \cong HSW^q(Y)$. Furthermore, the graded roots algorithm implies that $HF^+_{red}(Y)$ is concentrated in degrees $d(Y)$ and above. It follows that there can be no differentials in the spectral sequence and hence $\delta_{\mathbb{Z}_c,S^0}(Y) = \delta(Y)$. Recall that $\delta_j^{(c)}(K) = -\sigma^{(c)}(K)/2$ for all sufficiently large j. Then since $Y = -\Sigma(a,b,c) = \Sigma_c(-T_{a,b})$, it follows that

$$\delta_{\mathbb{Z}_c,S^j}(Y) = -\sigma^{(c)}(-T_{a,b})/8 = \lambda(\Sigma(a,b,c)) = -\lambda(Y)$$

for sufficiently large j. But since $HF^*_{red}(Y)$ is concentrated in even degrees, we also have $dim_{\mathbb{F}}(HF^*_{red}(Y)) = \delta(Y) + \lambda(Y)$. Hence for large enough j, we have $\delta_{\mathbb{Z}_c,S^0}(Y) - \delta_{\mathbb{Z}_c,S^j}(Y) = \delta(Y) + \lambda(Y) = dim_{\mathbb{F}}(HF^*_{red}(Y))$. From Proposition 4.6, it follows that any element in the image of $U\colon HF^+_{red}(Y,\mathfrak{s}) \to HF^+_{red}(Y,\mathfrak{s})$ has degree strictly less than $\ell^+(Y,\mathfrak{s})$. Thus conditions (1)-(4) of Proposition 3.5 are met. Therefore (since c is an odd prime and Y is an integral homology sphere) we have

$$j'(Y) = \ell^+(Y)/2 - \delta_{\mathbb{Z}_c,S^0}(Y) + \delta(Y) + \lambda(Y) = \ell^+(Y)/2 + \lambda(Y),$$

where $j'(Y) = j^{(c)}(Y)$ if c is odd and $j'(Y) = j^{(2)}(Y)/2$ if $c = 2$. Moreover, Proposition 4.6 also gives $\ell^+(Y) = 2(\delta(Y) - \min(\tau))$. From [6, §5], we also have that $-\min(\tau) + \delta(Y) = \kappa(a,b,c) - \lambda(Y)$. Thus $\ell^+(Y) = 2(\kappa - \lambda(Y))$. Hence

$$j'(-T_{a,b}) = j^{(c)}(Y) = \ell^+(Y)/2 + \lambda(Y) = \kappa(a,b,c),$$

which gives $j^{(c)}(-T_{a,b}) = \kappa(a,b,c)$ if c is odd and $j^{(2)}(-T_{a,b}) = 2\kappa(a,b,c)$ if $c = 2$. We also have $\sigma^{(c)}(T_{a,b}) = -8\lambda(Y) = 8\lambda(\Sigma(a,b,c))$ and thus we have

$$\begin{aligned}
\theta^{(c)}(T_{a,b}) &= \max\left\{0, \frac{2j'(Y)}{(c-1)} - \frac{\sigma^{(c)}(T_{a,b})}{2(c-1)}\right\} \\
&= \max\left\{0, \frac{2\kappa(a,b,c) - 4\lambda(\Sigma(a,b,c))}{(c-1)}\right\} \\
&= \frac{1}{2}(a-1)(b-1)
\end{aligned}$$

where the last line follows from Equation (4.8). □

5. Branched covers

Let $Y = \Sigma(a_1', a_2', \dots, a_r')$ be a Brieskorn homology sphere and p a prime such that p divides $a_1'a_2'\cdots a_r'$. Without loss of generality we may assume p divides a_1'. We set $a_1 = a_1'/p$ and $a_j = a_j'$ for $j > 1$. So $Y = \Sigma(pa_1, a_2, \dots, a_r)$. Then $\mathbb{Z}_p$ acts on Y with quotient space $Y_0 = Y/\mathbb{Z}_p = \Sigma(a_1, \dots, a_r)$ and the quotient map $Y \to Y_0$ is a p-fold cyclic branched cover.

Theorem 5.1. *We have that*

$$\delta(-Y) - \delta_\infty^{(p)}(-Y) \geq \mathrm{rk}(HF^+_{red}(Y)) - p\,\mathrm{rk}(HF^+_{red}(Y_0)).$$

Proof. Let W_0 be the negative definite plumbing bounded by Y_0. Since Y_0 is an integral homology sphere it follows that $H_1(W_0;\mathbb{Z}) = 0$. Let $k \subset Y_0$ denote the branch locus of $Y \to Y_0$. Then k is a knot in Y_0. Let $\Sigma \subset W_0$ be the pushoff of a Seifert surface for k, so Σ is a properly embedded surface in W_0 which meets $\partial W_0 = Y_0$ in k. Let $W \to W_0$ be the p-fold cyclic cover of W_0 branched over Σ. Then W has boundary Y and the $\mathbb{Z}_p$-action on Y extends to W. By [4, Proposition 2.5], for any characteristic $c \in H^2(W_0;\mathbb{Z})$ there exists a $\mathbb{Z}_p$-invariant spin^c-structure $\mathfrak{s}$ on W such that $c_1(\mathfrak{s}) = \pi^*(c)$ in $H^2(W;\mathbb{Q})$. Now we apply the equivariant Frøyshov inequality [5, Theorem 5.3] to W giving $\delta_\infty(-Y) + \delta(W,\mathfrak{s}) \leq 0$, where

$$\begin{aligned}\delta(W,\mathfrak{s}) &= \frac{c_1(\mathfrak{s})^2 - \sigma(W)}{8}\\ &= \frac{pc^2 - \sigma(W)}{8}\\ &= p\left(\frac{c^2 - \sigma(W_0)}{8}\right) + \frac{p\sigma(W_0) - \sigma(W)}{8}.\end{aligned}$$

The maximum of $(c^2 - \sigma(W_0))/8$ over all characteristics of $H^2(W_0;\mathbb{Z})$ equals $\delta(Y_0)$ [31, Theorem 8.3]. Hence we obtain

$$\delta_\infty^{(p)}(-Y) + p\delta(Y_0) + \frac{p\sigma(W_0) - \sigma(W)}{8} \leq 0,$$

which we may rewrite as

$$\delta(-Y) - \delta_\infty^{(p)}(-Y) \geq -(\delta(Y) - p\delta(Y_0)) + \frac{p\sigma(W_0) - \sigma(W)}{8}.$$

Next, we claim that $(p\sigma(W_0) - \sigma(W))/8 = p\lambda(Y_0) - \lambda(Y)$, where $\lambda(Y_0), \lambda(Y)$ are the Casson invariants of Y_0 and Y. Assuming this claim for the moment, our inquality becomes

$$\delta(-Y) - \delta_\infty^{(p)}(-Y) \geq -(\delta(Y) + \lambda(Y)) + p(\delta(Y_0) + \lambda(Y_0)).$$

But since $HF^+_{red}(Y)$ and $HF^+_{red}(Y_0)$ are concentrated in odd degrees, [36, Theorem 1.3] gives

$$\mathrm{rk}(HF^+_{red}(Y)) = -\delta(Y) - \lambda(Y), \quad \mathrm{rk}(HF^+_{red}(Y_0)) = -\delta(Y_0) - \lambda(Y_0),$$

and hence we obtain

$$\delta(-Y) - \delta_\infty^{(p)}(-Y) \geq \mathrm{rk}(HF_{red}(-Y)) - p\,\mathrm{rk}(HF_{red}(-Y_0)).$$

It remains to prove the claim that $(p\sigma(W_0) - \sigma(W))/8 = p\lambda(Y_0) - \lambda(Y)$. From [11, Theorem 2], we have that

$$\lambda^{\mathbb{Z}/p}(Y) - p\lambda(Y_0) = \frac{\sigma(W) - p\sigma(W_0)}{8}$$

where $\lambda^{\mathbb{Z}/p}(Y)$ is the equivariant Casson invariant of Y with respect to the $\mathbb{Z}_p$-action [11]. Furthermore [11, Theorem 3] implies that $\lambda^{\mathbb{Z}/p}(Y) = \lambda(Y)$, because $k \subset Y_0 = \Sigma(a_1,\ldots,a_r)$ is a fibre of the Seifert fibration on Y_0, so it is a graph knot in the terminology of [11, §5]. This proves the claim that $\lambda(Y) - p\lambda(Y_0) = (\sigma(W) - p\sigma(W_0))/8$. □

Let Y, Y_0 be as in Theorem 5.1. Then by [17, Theorem 1.1], we have an inequality $\mathrm{rk}(HF^+_{red}(Y)) \geq p\,\mathrm{rk}(HF^+_{red}(Y_0))$. Combined with Theorem 5.1, this gives:

$$p\,\mathrm{rk}(HF^+_{red}(Y_0)) \leq \mathrm{rk}(HF^+_{red}(Y)) \leq p\,\mathrm{rk}(HF^+_{red}(Y_0)) + (\delta(-Y) - \delta^{(p)}_\infty(-Y)).$$

In particular, the equality $\delta^{(p)}_\infty(-Y) = \delta(-Y)$ can only happen if $\mathrm{rk}(HF^+_{red}(Y)) = p\,\mathrm{rk}(HF^+_{red}(Y_0))$. Hence we obtain:

Corollary 5.2. *Let Y, Y_0 be as in Theorem 5.1. If $\mathrm{rk}(HF^+_{red}(Y)) > p\,\mathrm{rk}(HF^+_{red}(Y_0))$ then $\delta^{(p)}_\infty(-Y) < \delta(-Y)$.*

6. Free actions

Suppose p does not divide $a_1 a_2 \cdots a_r$. Then the restriction of the S^1-action on $Y = \Sigma(a_1, \dots, a_r)$ acts freely on Y. Let $Y_0 = Y/\mathbb{Z}_p$ be the quotient. Then Y is a rational homology sphere. In fact if $Y = M(e_0, (a_1, b_1), \dots, (a_r, b_r))$, then it is easily seen that $Y_0 = M(pe_0, (a_1, pb_1), \dots, (a_r, pb_r))$. It is easy to see that $H_1(Y_0; \mathbb{Z}) \cong \mathbb{Z}_p$ and thus Y_0 has exactly p spin^c-structures. Further, the pullback to Y of any spin^c-structure on Y_0 must coincide with the unique spin^c-structure on Y.

Theorem 6.1. *For any spin^c-structure $\mathfrak{s}_0$ on Y_0, we have*

$$\delta^{(p)}_\infty(Y) - \delta(Y) = \mathrm{rk}(HF^+_{red}(Y)) - \mathrm{rk}(HF^+_{red}(Y_0, \mathfrak{s}_0)).$$

Proof. Set $G = \mathbb{Z}_p$ and $H^*_G = H^*_G(pt; \mathbb{F})$. By choosing a G-invariant metric on Y, we may construct a $S^1 \times G$-equivariant Conley index for $(Y, \mathfrak{s})$, which we denote by $I(Y, \mathfrak{s})$, here $\mathfrak{s}$ denotes the unique spin^c-structure on Y. As shown in [24, §3], the Conley index $I(Y_0, \mathfrak{s}_0)$ for $(Y_0, \mathfrak{s}_0)$ can be identified with the $\mathbb{Z}_p$-fixed point set $I(Y, \mathfrak{s})^{\mathbb{Z}_p}$ of $I(Y, \mathfrak{s})$. Thus we have isomorphisms

$$\begin{aligned} \widetilde{H}^*_{S^1 \times G}(I(Y, \mathfrak{s})^{\mathbb{Z}_p}; \mathbb{F}) &\cong \widetilde{H}^*_{S^1 \times G}(I(Y_0, \mathfrak{s}_0); \mathbb{F}) \\ &\cong \widetilde{H}^*_{S^1}(I(Y_0, \mathfrak{s}_0), \mathbb{F}) \otimes_{\mathbb{F}} H^*_G \\ &\cong HSW^*(Y_0, \mathfrak{s}_0) \otimes_{\mathbb{F}} H^*_G. \end{aligned}$$

It should be noted that the above isomorphisms only preserve relative gradings. Recall that $H^*_G \cong \mathbb{F}[Q]$, $deg(Q) = 1$ if $p = 2$ and $H^*_G \cong \mathbb{F}[R, S]/(R^2)$, $deg(R) = 1$, $deg(S) = 2$ if p is odd. In the case $p = 2$, define $S = Q^2$. Let $\mathcal{S} = \{1, S, S^2, \dots\}$. Then $\mathcal{S}$ is a multiplicative subset of $H^*_{S^1 \times G} = H^*_{S^1 \times G}(pt; \mathbb{F}) \cong H^*_G[U]$. The localisation theorem in equivariant cohomology [12, III, Theorem 3.8] applied to the pair $(I(Y, \mathfrak{s}), I(Y, \mathfrak{s})^{\mathbb{Z}_p})$ and multiplicative set $\mathcal{S}$ implies that the inclusion $I(Y, \mathfrak{s})^{\mathbb{Z}_p} \to I(Y, \mathfrak{s})$ induces an isomorphism

$$\mathcal{S}^{-1}\widetilde{H}^*_{S^1 \times G}(I(Y, \mathfrak{s}); \mathbb{F}) \to \mathcal{S}^{-1}\widetilde{H}^*_{S^1 \times G}(I(Y, \mathfrak{s})^{\mathbb{Z}_p}; \mathbb{F}). \tag{6.1}$$

Since $\widetilde{H}^*_{S^1 \times G}(I(Y, \mathfrak{s}); \mathbb{F}) \cong HSW^*_{\mathbb{Z}_p}$ and as we have shown above, $\widetilde{H}^*_{S^1 \times G}(I(Y, \mathfrak{s})^{\mathbb{Z}_p}; \mathbb{F}) \cong HSW^*(Y_0, \mathfrak{s}_0) \otimes_{\mathbb{F}} H^*_G$, we get an isomorphism

$$\mathcal{S}^{-1}HSW^*_{\mathbb{Z}_p}(Y, \mathfrak{s}) \cong HSW^*(Y_0, \mathfrak{s}_0) \otimes_{\mathbb{F}} \mathcal{S}^{-1}H^*_G.$$

This is an isomorphism of relatively graded $H^*_{S^1\times G}$-modules.

For the rest of the proof we restrict to the case that p is odd. The proof in the case that $p = 2$ is similar. We have

$$HSW^*(Y_0, \mathfrak{s}_0) \cong HF^+_*(Y_0, \mathfrak{s}_0) \cong \mathbb{F}[U]\theta_0 \oplus HF^+_{red}(Y_0, \mathfrak{s}_0)$$

for some θ_0. Combined with (6.1), we have an isomorphism

$$S^{-1}HSW^*_{\mathbb{Z}_p}(Y, \mathfrak{s}) \cong \frac{\mathbb{F}[U, R, S, S^{-1}]}{(R^2)}\theta_0 \oplus \frac{HF^+_{red}(Y_0, \mathfrak{s}_0)[R, S, S^{-1}]}{(R^2)}. \tag{6.2}$$

On the other hand, the proof of Proposition 3.6 gives an isomorphism

$$HSW^*_{\mathbb{Z}_p}(Y, \mathfrak{s}) \cong E_\infty(Y) \cong \frac{\mathbb{F}[U, R, S]}{(R^2)}U^m\theta \oplus E_\infty(Y)_{red}$$

under which the $\mathbb{F}[U]$-module structure is given by an endomorphism of the form $\widehat{U} = U_{(0,2)} + U_{(1,1)} + \cdots$ with $U_{(0,2)} = U$. Localising with respect to S gives an isomorphism

$$S^{-1}HSW^*_{\mathbb{Z}_p}(Y, \mathfrak{s}) \cong E_\infty(Y) \cong \frac{\mathbb{F}[U, R, S, S^{-1}]}{(R^2)}U^m\theta \oplus S^{-1}E_\infty(Y)_{red}.$$

Define $W \subseteq S^{-1}HSW^*_{\mathbb{Z}_p}(Y, \mathfrak{s}))$ to be the set of $x \in S^{-1}HSW^*_{\mathbb{Z}_p}(Y, \mathfrak{s}))$ such that for each $j \geq 0$, there exists a $k \geq 0$ for which $U^k x \in \mathcal{F}_j(Y)$. The proof of Proposition 3.6 demonstrates that $W \cong S^{-1}E_\infty(Y)_{red}$. On the other hand, the isomorphism (6.2) clearly shows that $W \cong HF^+_{red}(Y_0, \mathfrak{s}_0)[R, S, S^{-1}]/(R^2)$. Combining these, we have an isomorphism

$$S^{-1}E_\infty(Y)_{red} \cong HF^+_{red}(Y_0, \mathfrak{s}_0)[R, S, S^{-1}]/(R^2).$$

In any fixed degree j, the rank of $(HF^+_{red}(Y_0, \mathfrak{s}_0)[R, S, S^{-1}]/(R^2))^j$ is equal to $\mathrm{rk}(HF^+_{red}(Y_0, \mathfrak{s}_0))$, hence the same is true of $S^{-1}E_\infty(Y)_{red}$. From [5, Lemma 5.7], we have that $S\colon E_r^{p,q}(Y) \to E_r^{p+2,q}(Y)$ is an isomorphism for all large enough p. Hence for large enough p, $E_r^{2p,*}(Y)$ is independent of p and we denote the resulting group by M_r^*. We similarly define M^*_∞. Clearly the rank of $(S^{-1}E_\infty(Y)_{red})^j$ for any j equals $\mathrm{rk}(M_\infty)_{red}$. So we have proven that

$$\mathrm{rk}(HF^+_{red}(Y_0, \mathfrak{s}_0)) = \mathrm{rk}(M_\infty)_{red}.$$

Set $s_r = \mathrm{rk}(M_r)_{red}$. It follows from [5, Lemma 5.8] that $(M_{r+1})_{red}$ is a subquotient of $(M_r)_{red}$. Hence the sequence $s_2, s_3, \ldots$ is decreasing and equals $\mathrm{rk}(M_\infty)_{red}$ for sufficiently large r. Furthermore, $M_2 \cong HF^+(Y)$, so $s_2 = \mathrm{rk}(HF^+_{red}(Y))$.

Recall from the proof of Proposition 3.6 that

$$E_r(Y) \cong \frac{\mathbb{F}[U, R, S]}{(R^2)}U^{m_r}\theta \oplus E_r(Y)_{red}$$

for some increasing sequence $0 = m_0 \leq m_1 \leq \ldots$. Suppose $m_{r+1} > m_r$. Then $d_r(U^{m_r+j}\theta) \neq 0$ for $0 \leq j \leq m_{r+1} - m_r$. Notice that $U^{m_r+j}\theta$ has bi-degree $(0, a)$ where $a = 2m_r + 2j + d(Y)$ is even, hence $d_r(U^{m_r+j}\theta)$ has bi-degree $(r, a + (1-r))$. But all elements of $E_r(Y)_{red}$ have bi-degree (u, v) with v odd (because $E_r(Y)_{red}$ is a subquotient of $HF^+_{red}(Y)[R, S]/(R^2)$ and $HF^+_{red}(Y)$ is concentrated in odd degrees). Hence $a + (1-r)$ must be odd, which means that r is even. So we can regard d_r

as mapping into $(M_r)_{red}$. Hence the rank of $(M_{r+1})_{red}$ is at least $m_{r+1} - m_r$ less than the rank of $(M_r)_{red}$, that is,

$$s_r - s_{r+1} \geq m_{r+1} - m_r. \tag{6.3}$$

We claim that this inequality is actually an equality. Equivalently M_{r+1} is the quotient of M_r by the span of $\{d_r(U^{m_r+j})\}$, $0 \leq j \leq m_{r+1} - m_r$. This is also equivalent to saying that $d_r(x) = 0$ for all $r \geq 2$ and all $x \in E_r(Y)_{red}$. Consider a non-zero homogeneous element $x \in E_r(Y)^{a,b}_{red}$. Since $E_r(Y)_{red}$ is a subquotient of $HF^+_{red}(Y)[R,S]/(R^2)$ and $HF^+_{red}(Y)$ is concentrated in odd degrees, we have that b is odd. Then $d_r(x)$ has bi-degree $(a+r, b+1-r)$. If $d_r(x) \neq 0$, then $b+1-r$ must be odd and so r must be odd. Thus in order to prove the claim, it is sufficient to show that $d_r = 0$ for all odd r.

Consider the equivariant Seiberg–Witten–Floer cohomology of Y with integral coefficients $HSW^*_{\mathbb{Z}_p}(Y;\mathbb{Z})$. This is a module over $H^*_{\mathbb{Z}_p}(pt;\mathbb{Z}) \cong \mathbb{Z}[S]/(pS)$ where $deg(S) = 2$. The key point to observe here is that $H^*_{\mathbb{Z}_p}(pt;\mathbb{Z})$ is concentrated in even degrees. There is a spectral sequence $(E^{p,q}_r(Y;\mathbb{Z}), d_r)$ and a filtration $\{\mathcal{F}_j(Y;\mathbb{Z})\}$ such that $E_\infty(Y;\mathbb{Z})$ is the associated graded module of the filtration. The mod p reduction map $\mathbb{Z} \to \mathbb{Z}_p = \mathbb{F}$ induces a morphism of $HSW^*(Y;\mathbb{Z}) \to HSW^*(Y;\mathbb{F})$. There is also a reduction map of the equivariant Floer cohomologies, the filtrations and the spectral sequences. The results of [35] also hold with integer coefficients so

$$HF^+(Y;\mathbb{Z}) \cong \mathbb{Z}[U]_{d(Y)} \oplus HF^+_{red}(Y;\mathbb{Z})$$

and $HF^+_{red}(Y;\mathbb{Z})$ is concentrated in odd degrees. By the universal coefficient theorem $HF^+_{red}(Y;\mathbb{F}) \cong HF^+_{red}(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}$. Furthermore, we have that

$$\begin{aligned} E_2(Y;\mathbb{Z}) &\cong H^*(\mathbb{Z}_p; HSW^*(Y;\mathbb{Z})) \\ &\cong \frac{\mathbb{Z}[U,S]}{(pS)}\theta \oplus \frac{HF^+_{red}(Y;\mathbb{Z})[S]}{(pS)}. \end{aligned}$$

Clearly every homogeneous element in $\mathbb{Z}[U,S]/(pS)\theta$ has bi-degree (a,b) where a and b are even and every homogeneous element in $HF^+_{red}(Y;\mathbb{Z})[S]/(pS)$ has bi-degree (a,b) where a is even and b is odd. From this and the fact that the image of $d_r \colon E_r(Y;\mathbb{Z}) \to E_r(Y;\mathbb{Z})$ is contained in $E_r(Y;\mathbb{Z})_{red}$ it follows easily that $d_r \colon E_r(Y;\mathbb{Z}) \to E_r(Y;\mathbb{Z})$ is zero for odd r.

Now using induction on r one shows the following properties hold: (1) any $x \in E^{a,b}_r(Y)$ with a even is in the image of the reduction map $E^{a,b}_r(Y;\mathbb{Z}) \to E^{a,b}_r(Y)$, (2) every $x \in E^{a,b}_r(Y)$ with a odd is of the form $x = Ry$ where y is in the image of the reduction map $E^{a,b}_r(Y;\mathbb{Z}) \to E^{a,b}_r(Y)$ and (3) $d_r \colon E_r(Y) \to E_r(Y)$ is zero for even r. In particular, this proves the claim that $d_r \colon E_r(Y) \to E_r(Y)$ is zero for all odd r and hence the inequality (6.3) is actually an equality:

$$s_r - s_{r+1} = m_{r+1} - m_r. \tag{6.4}$$

Let $s = \mathrm{rk}((M_\infty)_{red}) = \mathrm{rk}(HF^+_{red}(Y_0, \mathfrak{s}_0))$ and let $m = \lim_{r\to\infty} m_r$. Summing (6.4) from $r = 2$ to infinity gives $s_2 - s = m - m_2$. But $m_2 = 0$ and $s_2 = \mathrm{rk}(HF^+_{red}(Y))$, so we get $m = \mathrm{rk}(HF^+_{red}(Y)) - \mathrm{rk}(HF^+_{red}(Y_0, \mathfrak{s}_0))$. But recall from the proof of

Proposition 3.6 that $\delta_\infty^{(p)}(Y) = \delta(Y) + m$, hence we get

$$\delta_\infty^{(p)}(Y) = \delta(Y) + \mathrm{rk}(HF^+_{red}(Y)) - \mathrm{rk}(HF^+_{red}(Y_0, \mathfrak{s}_0)).$$

□

We will take $\mathfrak{s}_0$ to be the restriction to Y_0 of the canonical spin^c-structure on the negative definite plumbing which Y_0 bounds. With this choice of spin^c-structure, the computation of $HF^+(-Y_0, \mathfrak{s}_0)$ is easily obtained through the graded roots algorithm [31]. Let $\Delta_p(n)$ denote the delta function for $(Y_0, \mathfrak{s}_0)$. Then (since $e_0 < 0$) we have

$$\Delta_p(n) = 1 + npe_0 - \sum_{j=1}^{r} \left\lceil \frac{npb_j}{a_j} \right\rceil \tag{6.5}$$

Proposition 6.2. *We have that $\Delta_p(n) \geq 0$ for $n > N/p$. In particular, if $p > N$, then $HF^+_{red}(-Y_0, \mathfrak{s}_0) = 0$.*

Proof. Following [6, §3], we let $f(x) = \lceil x \rceil - x$. Then from Equations (3.3) and (6.5), we see that

$$\Delta_p(n) = 1 + \frac{np}{a_1 \cdots a_r} - \sum_{j=1}^{r} f\left(\frac{npb_j}{a_j}\right).$$

For any integer m we have $f(m/a_j) \leq 1 - 1/a_j$, hence

$$\Delta_p(n) \geq 1 + \frac{np}{a_1 \cdots a_r} - r + \sum_{j=1}^{r} \frac{1}{a_j}. \tag{6.6}$$

Now suppose that $pn > N$. Thus

$$pn > a_1 \cdots a_n \left((r-2) - \sum_{j=1}^{r} \frac{1}{a_j} \right).$$

Re-arranging, we get

$$(1 - r) + \frac{np}{a_1 \cdots a_r} + \sum_{j=1}^{r} \frac{1}{a_j} > -1.$$

Combined with (6.6), we get $\Delta_p(n) > -1$. But Δ_p is integer-valued, so $\Delta_p(n) \geq 0$. □

Lemma 6.3. *We have that $\mathrm{rk}(HF^+_{red}(-Y)) > \mathrm{rk}(HF^+_{red}(-Y_0, \mathfrak{s}_0))$ unless $r = 3$ and up to reordering (a_1, a_2, a_3) is one of the following:*

$$(2,3,5), (2,3,7), (2,3,11), (2,3,13), (2,3,17), (2,5,7), (2,5,9), (3,4,5), (3,4,7).$$

Proof. We assume that Y is not $\Sigma(2,3,5)$ and hence $N > 0$. Let $N_p = \lfloor N/p \rfloor$. By Proposition 6.2 we only need to consider $\Delta_p(x)$ for $x \in [0, N_p]$.

In the proof we make use of abstract delta sequences [17, §3]. Let $P_p = G \cap [0, N_p]$. Consider the map $\phi\colon P_p \to [0, N]$ given by $\phi(n) = np$. From Equation (6.5) it is immediately clear that $\Delta(np) = \Delta_p(n)$, hence ϕ identifies the delta sequence (P_p, Δ_p) with a subsequence of the delta sequence (P, Δ). This gives the inequality

$\mathrm{rk}(HF^+_{red}(-Y)) \geq \mathrm{rk}(HF^+_{red}(-Y_0, \mathfrak{s}_0))$ by [17, Proposition 3.5]. To refine this to a strict inequality we will show that (except for the cases listed in the statement of the lemma) there exists an $x \in P \setminus \phi(P_p)$ such that $2x \leq N$. In this case $\{x, N-x\}$ defines a delta subsequence of (P, Δ) disjoint from $\phi(P_p)$ and then by [17, Corollary 3.12] we obtain a strict inequality $\mathrm{rk}(HF^+_{red}(-Y)) > \mathrm{rk}(HF^+_{red}(-Y_0, \mathfrak{s}_0))$, as the module $\mathbb{H}_{red}$ corresponding to $\{x, N-x\}$ has rank 1.

We seek an $x \in P \setminus \phi(P_p)$ such that $2x \leq N$. Equivalently $x \in P$, p does not divide x and $2x \leq N$. Reorder $a_1, \dots, a_r$ such that $a_1 > a_2 > \cdots > a_r$. Consider $x = a_2 a_3 \cdots a_r$. It follows that p does not divide x as p is coprime to $a_2, \dots, a_r$ and we also have that $x \in P$. Hence we obtain a strict rank inequality provided that $2x \leq N$. Suppose on the contrary that $2x > N$, that is,

$$2a_2 \cdots a_r > a_1 \cdots a_r \left((r-2) - \sum_{j=1}^{r} \frac{1}{a_j} \right).$$

Re-arranging, this is equivalent to

$$\frac{3}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_r} \geq (r-2). \tag{6.7}$$

Since $a_j \geq 2$ for all j and at most one can be equal to 2, we get that

$$\frac{3}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_r} < \frac{3}{a_1} + \frac{(r-1)}{2},$$

hence

$$\frac{3}{a_1} + \frac{(r-1)}{2} > (r-2).$$

From $a_1 > a_2 > \cdots > a_r \geq 2$, we see that $a_r \geq r+1$, hence

$$\frac{3}{r+1} + \frac{(r-1)}{2} > (r-2)$$

which simplifies to $(r+1)(r-3) < 6$. If $r \geq 5$, then $(r+1)(r-3) \geq 12$, so this leaves only the cases $r = 3, 4$.

If $r = 4$, then since $a_4 \geq 2$, $a_3 \geq 3$, $a_2 \geq 4$, $a_1 \geq 5$, we have

$$\frac{3}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \leq \frac{3}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = \frac{101}{60} < 2.$$

Hence (6.7) can not be satisfied with $r = 4$.

Now suppose that $r = 3$. If $a_3 \geq 4$, then $a_2 \geq 5$, $a_1 \geq 6$ and so

$$\frac{3}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \frac{3}{6} + \frac{1}{5} + \frac{1}{4} = \frac{19}{20} < 1$$

and so (6.7) can only be satisfied if $a_3 = 2$ or 3.

If $a_3 = 3$ and $a_4 \geq 5$, then $a_1 \geq 7$ (since a_1 must be coprime to $a_3 = 3$) and so

$$\frac{3}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \frac{3}{7} + \frac{1}{5} + \frac{1}{3} = \frac{101}{105} < 1.$$

So (6.7) implies that $a_4 = 4$. Then since $3/8 + 1/4 + 1/3 = 23/24 < 1$, it follows that $a_1 < 8$. Hence if $a_3 = 3$, then $(a_1, a_2, a_3) = (5, 4, 3)$ or $(7, 4, 3)$.

Lastly, suppose that $a_3 = 2$. If $a_2 > 5$, then $a_2 \geq 7$ and $a_1 \geq 9$ (since a_1, a_2 must be coprime to a_3). So

$$\frac{3}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \frac{3}{9} + \frac{1}{7} + \frac{1}{2} = \frac{41}{42} < 1.$$

So $a_2 \leq 5$, hence $a_2 = 3$ or 5.

If $a_3 = 2$ and $a_2 = 5$, then (6.7) implies $a_1 \leq 10$, hence $(a_1, a_2, a_3) = (7, 5, 2)$ or $(9, 5, 2)$.

If $a_3 = 2$ and $a_2 = 3$, then (6.7) implies $a_1 \leq 18$, hence (a_1, a_2, a_3) is one of $(5, 3, 2), (7, 3, 2), (11, 3, 2), (13, 3, 2), (17, 3, 2)$. □

Theorem 6.4. *We have that* $\mathrm{rk}(HF^+_{red}(-Y)) > \mathrm{rk}(HF^+_{red}(-Y_0, \mathfrak{s}_0))$ *except in the following cases:*

(1) $Y = \Sigma(2, 3, 5)$ *and* p *is any prime.*
(2) $Y = \Sigma(2, 3, 11)$ *and* $p = 5$.

In case (1) we have $\mathrm{rk}(HF^+_{red}(-Y)) = \mathrm{rk}(HF^+_{red}(-Y_0, \mathfrak{s}_0)) = 0$ *and in case (2) we have* $\mathrm{rk}(HF^+_{red}(-Y)) = \mathrm{rk}(HF^+_{red}(-Y_0, \mathfrak{s}_0)) = 1$.

Proof. By Lemma 6.3, we only need to consider the case that $r = 3$ and up to reordering (a_1, a_2, a_3) is one of:

$$(2, 3, 5), (2, 3, 7), (2, 3, 11), (2, 3, 13), (2, 3, 17), (2, 5, 7), (2, 5, 9), (3, 4, 5), (3, 4, 7).$$

In the case $Y = \Sigma(2, 3, 5)$ we have that $\mathrm{rk}(HF^+_{red}(-Y)) = 0$ and $N < 0$. Hence we also have $\mathrm{rk}(HF^+_{red}(-Y_0)) = 0$ by Proposition 6.2.

In the remaining cases we have $\mathrm{rk}(HF^+_{red}(-Y)) > 0$ and $N > 0$. Hence if $p > N$ then $\mathrm{rk}(HF^+_{red}(-Y)) > \mathrm{rk}(HF^+_{red}(-Y_0)) = 0$, by Proposition 6.2. So we only need to consider primes such that $p \leq N$ and coprime to a_1, a_2, a_3. This leaves only finitely many cases of 4-tuples $\{(a_1, a_2, a_3, p)\}$ to consider. We check each of these cases by directly computing the ranks of $HF^+_{red}(-Y)$ and $HF^+_{red}(-Y_0)$. The results are shown in Table 1. By inspection we see that $\mathrm{rk}(HF^+_{red}(-Y)) > \mathrm{rk}(HF^+_{red}(-Y_0, \mathfrak{s}_0))$ in all cases except for $Y = \Sigma(2, 3, 11)$ and $p = 5$.

□

Theorem 6.5. *Let* $Y = \Sigma(a_1, a_2, \ldots, a_r)$ *be a Brieskorn homology sphere and let* p *be a prime not dividing* $a_1 \cdots a_r$. *Then* $\delta^{(p)}_\infty(Y) > \delta(Y)$ *except in the following cases:*

(1) $Y = \Sigma(2, 3, 5)$ *and* p *is any prime.*
(2) $Y = \Sigma(2, 3, 11)$ *and* $p = 5$.

In both cases we have $\delta^{(p)}_\infty(Y) = \delta(Y) = 1$.

Proof. Theorems 6.1 and 6.4 imply that $\delta^{(p)}_\infty(Y) \geq \delta(Y)$ with equality only in the cases listed. In these cases, we have $\delta^{(p)}_\infty(Y) = \delta(Y)$ and $\delta(Y) = 1$ [36, §8]. □

(a_1, a_2, a_3)	N	$\mathrm{rk}(HF^+_{red}(-Y))$	p	$\mathrm{rk}(HF^+_{red}(-Y_0))$
$(2,3,7)$	1	1	none	
$(2,3,11)$	5	1	5	1
$(2,3,13)$	7	2	5	0
			7	1
$(2,3,17)$	11	2	5	1
			7	0
			11	1
$(2,5,7)$	11	2	3	0
			11	1
$(2,5,9)$	17	2	7	1
			11	0
			13	0
			17	1
$(3,4,5)$	13	2	2	0
			7	0
			11	0
			13	1
$(3,4,7)$	23	2	2	1
			5	0
			11	1
			13	0
			17	0
			19	0
			23	1

Table 1

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