Holomorphic Bundles on Foliations

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Outline

- Background and motivation from gauge theory
- Transverse Hermitian–Einstein metrics and holomorphic bundles
- A transverse Hitchin–Kobayashi correspondence
- Applications

Joint work with David Baraglia

The Donaldson–Hitchin-Kobayashi–Uhlenbeck-Yau correspondence

Let E be a holomorphic vector bundle on a compact Hermitian manifold.

A Hermitian metric on E is **Hermitian–Einstein** if its associated Chern connection A satisfies

$$i\Lambda F_A = \gamma_A id_E.$$

Theorem

E admits a Hermitian–Einstein metric if and only if E is **polystable***.*

Brief history

1965 Narasimhan and Seshadri for Riemann surfaces,

1982 Kobayashi proved partial results for Kähler manifolds,

1983 Donaldson for Riemann surfaces,

1985 Donaldson for algebraic surfaces,

1986 Uhlenbeck and Yau for Kähler manifolds,

1988 Buchdahl for Hermitian surfaces,

1987 Li and Yau for Hermitian manifolds.

Motivation from gauge theory

Instantons

Let E be a unitary vector bundle on a compact Riemannian 4-manifold X.

A unitary connection A on E is an **anti-self-dual instanton** if

$$* F_A = -F_A.$$

On a Hermitian 4-manifold X, the anti-self-duality equations become

$$F_A^{0,2} = 0, \qquad \Lambda F_A = 0.$$

Theorem

$$\left\{ \begin{array}{c} \textit{Moduli space of instantons} \\ \textit{on } X \textit{ of rank } r \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Moduli space of polystable} \\ \textit{holomorphic vector bundles} \\ \textit{on } X \textit{ of degree } 0 \textit{ and rank } r \end{array} \right\}$$

Instantons in higher dimensions

Ω-instantons (Donaldson–Thomas, Tian)

The anti-self-dual **Ω-instanton equations** on a Hermitian *n*-manifold,

$$*F_A = -\Omega \wedge F_A, \qquad \text{where } \Omega = \frac{\omega^{n-2}}{(n-2)!},$$

are equivalent to the Hermitian-Einstein equations with $\gamma_A = 0$.

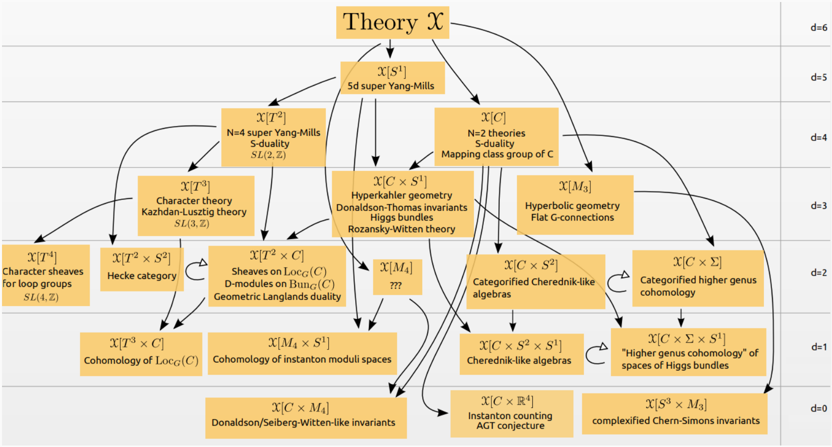
Contact instantons (Zabzine et al, Baraglia–H)

Let (X, η) be a contact 5-manifold. The anti-self-dual **contact instanton equations** are

$$*F_A = -\eta \wedge F_A.$$

If X is Sasakian, then contact instantons are *basic* connections that are Hermitian-Einstein *transverse* to the Reeb foliation.

Twisted 5D supersymmetric Yang–Mills theory



General problem

Instantons on Sasaki manifolds

If X is Sasakian of dimension $2n+1$ with $n \geq 2$, then

$$*F_A = -\Omega \wedge F_A, \qquad \text{where } \Omega = \eta \wedge \frac{(d\eta)^{n-2}}{(n-2)!}.$$

Equivalently an Ω-instanton A is a basic connection that is Hermitian-Einstein transverse to the Reeb foliation.

General problem:

Does the Hitchin–Kobayashi correspondence extend to compact foliated manifolds with a transverse Hermitian structure?

Transverse Hermitian-Einstein metrics

Transverse Riemannian geometry

Let $(X, \mathcal{F})$ be a smooth oriented m-manifold with a codimension $2n$ Riemannian foliation.

Let χ denote the leafwise volume form and $H = TX/T\mathcal{F}$.

A foliation is said to be **taut** if X admits a metric such that every leaf of $\mathcal{F}$ is a minimal submanifold.

We have the space of **basic differential k-forms**, with differential d_B

$$\Omega_B^k(X) = \{\alpha \in \Omega^k(X) \mid i_\xi \alpha = 0, \;\; i_\xi d\alpha = 0 \;\; \forall \xi \in \Gamma(X, T\mathcal{F})\}.$$

Basic Stokes' theorem

Proposition

Let X be a closed oriented manifold with a taut Riemannian foliation of codimension $2n$, then

$$\int_X d_B\alpha \wedge \chi = 0$$

for all $\alpha \in \Omega_B^{2n-1}(X)$.

Transverse Hermitian geometry

A **transverse Hermitian structure** on $(X, \mathcal{F})$ consists of a bundle-like Riemannian metric g and a compatible transverse integrable complex structure $I\colon H \to H$. Let $\omega \in \Omega^2_B(X)$ denote the associated Hermitian 2-form.

We have the basic **Lefschetz operator**

$$L : \Omega^j_B(X) \to \Omega^{j+2}_B(X), \quad L(\alpha) = \omega \wedge \alpha$$

and the **contraction operator**

$$\Lambda : \Omega^j_B(X) \to \Omega^{j-2}_B(X), \quad \Lambda(\alpha) = L^*(\alpha).$$

The transverse complex structure I gives rise to the usual decomposition $\Omega^{p,q}_B(X, \mathbb{C})$ with $d_B = \partial + \overline{\partial}$.

Transverse holomorphic bundles

A **foliated principal G-bundle** corresponds to a basic Čech cocycle $\{g_{\alpha\beta}\colon U_{\alpha\beta} \to G\}$. A vector bundle E is foliated if its frame bundle is foliated.

A connection A on P is **basic** if $A \in \Omega^1_B(P, \mathfrak{g})$.

A **transverse Hermitian metric** on E is a Hermitian metric which is basic as a section of $E^* \otimes \overline{E}^*$.

A **transverse holomorphic structure** on E is a transverse semi-connection

$$\overline{\partial}_E : \Omega_B^{p,q}(X, E) \to \Omega_B^{p,q+1}(X, E)$$

that is *integrable*, i.e. $\overline{\partial}_E^2 = 0$.

Let E be a foliated Hermitian vector bundle.

Proposition (Transverse Chern correspondence)

Basic unitary connections on E with curvature of type $(1,1)$ are in bijective correspondence with transverse holomorphic structures.

A transverse Hermitian metric on X is called **Gauduchon** if $\partial\overline{\partial}(\omega^{n-1}) = 0$.

Theorem

Let (X, g) be a compact oriented, taut, transverse Hermitian foliated manifold. Then g can be conformally rescaled by a basic positive real valued smooth function such that the rescaled transverse metric g_0 is Gauduchon.

Transverse Hermitian–Einstein metrics

Let $E \to X$ be a foliated Hermitian vector bundle on a compact oriented, taut, transverse Gauduchon manifold.

A basic unitary connection A on E is called **transverse Hermitian-Einstein** if its curvature 2-form F_A is of type $(1,1)$ and satisfies

$$i\Lambda F_A = \gamma_A id_E.$$

A transverse Hermitian metric on E is Hermitian-Einstein if the associated Chern connection is transverse Hermitian-Einstein.

Stable transverse holomorphic bundles

Degree of transverse holomorphic bundles

Let E be a foliated holomorphic vector bundle of rank r which admits a transverse Hermitian metric h.

The **degree** of E is

$$\deg(E) = \frac{i}{2\pi} \int_X tr(F_E) \wedge \omega^{n-1} \wedge \chi,$$

where F_E is the curvature of the Chern connection associated to h.

Let $\mathcal{O}$ denote the sheaf of *basic* holomorphic functions on X.

A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ is called a **transverse coherent sheaf** if locally, $\mathcal{F}$ is the cokernel of a sheaf map $\mathcal{O}^p \to \mathcal{O}^q$ for some p, q.

Torsion-free transverse coherent sheaves are locally free outside a transverse analytic subvariety of codimension at least 2.

Transverse coherent subsheaves with torsion free quotient

Let s be a positive integer less than r and let $q : Gr(s, E) \to X$ be the foliated Grassmannian bundle associated to E.

Let $\mathbb{F} \to Gr(s, E)$ denote the tautological rank s transverse holomorphic vector bundle.

Proposition

$$\left\{ \begin{array}{c} \textit{Rank } s \textit{ transverse} \\ \textit{coherent subsheaves} \\ \textit{of } E \textit{ with torsion–} \\ \textit{free quotient} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Closed irreducible transverse} \\ \textit{subvarieties } Y \subset Gr(s, E) \\ \textit{that generically project} \\ \textit{isomorphically onto } X \end{array} \right\}$$

Transverse resolution of singularities

Let Z be a foliated manifold with transverse complex structure and let $\iota : Y \to Z$ be a transverse analytic subvariety.

Theorem

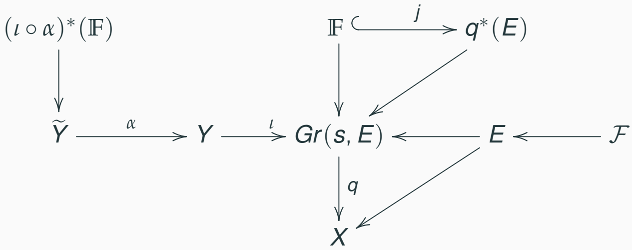
There exists a foliated manifold $\widetilde{Y}$ with transverse complex structure and a proper map $\alpha\colon \widetilde{Y} \to Y$ such that

(i) $\iota \circ \alpha\colon \widetilde{Y} \to Z$ *is smooth and transverse holomorphic,*

(ii) $\alpha\colon \widetilde{Y} \to Y$ *is an isomorphism over the smooth locus of* Y.

Degree of transverse coherent subsheaves

Let $\iota : Y \to Gr(s, E)$ be the irreducible transverse analytic subvariety of $Gr(s, E)$ corresponding to a rank s coherent subsheaf $\mathcal{F} \to E$ with torsion-free quotient $\mathcal{O}(E)/\mathcal{F}$. Fix a transverse resolution $\alpha : \widetilde{Y} \to Y$.



Degree of transverse coherent subsheaves

The bundle $(\iota \circ \alpha)^*(\mathbb{F}) \to \widetilde{Y}$ inherits a natural transverse Hermitian metric from $q^*(E)$ and has an associated Chern connection with curvature $\widetilde{F}_1$.

The **degree** of $\mathcal{F}$ is defined by

$$\deg(\mathcal{F}) = \frac{i}{2\pi} \int_{\widetilde{Y}} tr(\widetilde{F}_1) \wedge (q \circ \iota \circ \alpha)^*(\omega^{n-1} \wedge \chi).$$

This is independent of the choice of transverse resolution α.

Stability of transverse holomorphic bundles

Let E be a transverse holomorphic vector bundle which admits a transverse Hermitian metric. Let $\mu(E) = \deg(E)/rk(E)$ denote the **slope**.

- E is **stable** if for all transverse coherent subsheaf $\mathcal{F}$ of E with $0 < rk(\mathcal{F}) < rk(E)$ and torsion-free quotient $\mathcal{O}(E)/\mathcal{F}$,

$$\mu(\mathcal{F}) < \mu(E).$$

- E is **polystable** if it is the direct sum of stable transverse holomorphic bundles of the same slope.

A transverse holomorphic bundle E is called **simple** if

$$H^0_B(X, End(E)) = \mathbb{C}Id.$$

Main result

Let E be a transverse holomorphic vector bundle which admits transverse Hermitian metrics.

Theorem (Transverse Hitchin–Kobayashi correspondence)

E admits a transverse Hermitian-Einstein metric if and only if E is polystable.

If E is simple, then the transverse Hermitian-Einstein metric is unique up to constant rescaling.

Sketch of the proof $(\Rightarrow)$

- Let $\mathcal{F}$ be a coherent subsheaf of E with torsion-free quotient. Let $F \subset E$ be the corresponding holomorphic subbundle on $X \backslash S$, where $S \subset X$ is some transverse analytic subvariety of complex codimension at least 2.

- Let $\pi \in \mathcal{C}_B^\infty(X \setminus S, End(E))$ be the orthogonal projection from E to F. Then

$$i\Lambda tr(F_1) = i\Lambda tr(\pi F_E \pi) - |\partial_{End(E)}\pi|^2.$$

- Integrating over X and using the Hermitian-Einstein equation, we get

$$2\pi n \deg(\mathcal{F}) = \pi n\, rk(\mathcal{F})\mu(E) - ||\partial_{End(E)}\pi||^2_{L^2}.$$

  So $\mu(\mathcal{F}) \leq \mu(E)$ and equality occurs if and only if $\bar{\partial}_{End(E)}\pi = 0$.

- Hartog's theorem implies that π extends to a basic holomorphic section on X.

- Hence F extends to a holomorphic subbundle on all of X and we obtain an orthogonal, holomorphic splitting $E = F \oplus F^\perp$, where $\mu(F) = \mu(F^\perp) = \mu(E)$. Iteration yields the result.

Sketch of the proof $(\Leftarrow)$

- Let h_0 be a fixed choice of a transverse Hermitian metric on E and F_0 the associated Chern connection.
- Any transverse Hermitian metric h on E is of the form $h(\cdot,\cdot) = h_0(f\cdot,\cdot)$ for a unique $f \in Herm_B^+(E, h_0)$.
- The Chern connection associated to h is $A_0 + f^{-1}\partial_0(f)$. Define

$$L_0(f) = i\Lambda F_0 - \gamma Id_E + i\Lambda\left(\bar{\partial}(f^{-1}\partial_0(f))\right).$$

- Then the Hermitian-Einstein equation for h becomes $L_0(f) = 0$.
- For a real number $\epsilon \in [0,1]$ consider the perturbed equation

$$L_\epsilon(f) = i\Lambda F_0 - \gamma Id_E + i\Lambda\left(\bar{\partial}(f^{-1}\partial_0(f))\right) + \epsilon \cdot \log(f) = 0.$$

Sketch of the proof $(\Leftarrow)$

- One shows easily that there exists a solution f_1 to $L_1(f_1) = 0$.
- Let $J \subset (0,1]$ be the subset of $\epsilon \in (0,1]$ for which there is a map $f : [\epsilon, 1] \to Herm_B^+(E, h_0)$ such that $f(1) = f_1$ and $L_{\epsilon'}(f_{\epsilon'}) = 0$ for all $\epsilon' \in [\epsilon, 1]$. Note that J is an interval containing 1.
- One shows that J is open and closed in $(0,1]$ and therefore $J = (0,1]$. Consequently, for every $\epsilon \in (0,1]$ we get a solution f_ϵ of $L_\epsilon(f_\epsilon) = 0$.
- Finally consider the limit $\lim_{\epsilon \to 0} f_\epsilon$:
  - (i) If $f_0 = \lim_{\epsilon \to 0} f_\epsilon$ exists, one shows that $h(\cdot,\cdot) = h_0(f_0\cdot,\cdot)$ is a transverse Hermitian-Einstein metric.
  - (ii) If the limit does not exist, one constructs a transverse coherent subsheaf violating the stability condition for E.

Applications

Transverse Narasimhan–Seshadri theorem

Let X be a compact oriented, taut, transverse Hermitian foliated of complex codimension **one**. Let $E \to X$ be a transverse holomorphic vector bundle which admits transverse Hermitian metrics. Define $vol(X) = \int_X \omega \wedge \chi$.

Corollary

E admits a transverse Hermitian metric with Chern connection A satisfying

$$F_A = -2\pi i \frac{\mu(E)}{Vol(X)} \omega \otimes Id_E$$

if and only if E is polystable.

Transverse Narasimhan–Seshadri theorem: degree zero case

When E has degree zero, we get:

$$Rep(\pi_1(X), U(r)) \leftrightarrow \left\{ \begin{array}{c} \text{Moduli space of polystable rank } r\text{, degree} \\ \text{0 transverse holomorphic vector bundles} \\ \text{admitting transverse Hermitian metrics} \end{array} \right\}$$

Failure of the weak Uhlenbeck compactness theorem

Let $E \to X$ be a stable transverse holomorphic bundle of degree zero.

If the weak Uhlenbeck compactness theorem were true, it would follow that the moduli space of flat basic unitary connections on a vector bundle with a **fixed transverse structure** would be compact.

Counterexample.

Consider T^3 with a one-dimensional foliation $\langle \xi^1 \frac{\partial}{\partial x^1} + \xi^2 \frac{\partial}{\partial x^2} + \xi^3 \frac{\partial}{\partial x^3} \rangle$, where ξ_i are rationally independent. Then the moduli space of flat unitary line bundles with fixed transverse structure is $\mathbb{R}^2$.

An equivariant Hitchin-Kobayashi correspondence

Let M be a compact Hermitian manifold equipped with Gauduchon metric.

Let G be a finitely presented group acting on M by automorphisms of the Hermitian structure.

By a classical construction of Kervaire, G is the fundamental group of a closed oriented smooth 4-manifold Y:

- Let $Z = (S^1 \times S^3)\# \dots \#(S^1 \times S^3)$,
- A relation in G is a loop in Z with a tubular neighbourhood $U_j \cong S^1 \times D^3$,
- For each relation, apply the surgery $(Z \setminus int(U_i)) \cup_{S^1 \times S^2} (D^2 \times S^2)$,
- Y is the resulting closed oriented smooth 4-manifold with $\pi_1(Y) = G$.

An equivariant Hitchin-Kobayashi correspondence

Let $\widetilde{Y} \xrightarrow{G} Y$ be the universal cover and define $X = M \times_G \widetilde{Y}$. Then X is compact, oriented, taut, transverse Hermitian foliated and we have

$$\left\{ \begin{array}{c} \text{Transverse holomorphic} \\ \text{bundles on } X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} G\text{-equivariant holomorphic} \\ \text{bundles on } M \end{array} \right\}$$

Corollary (Equivariant Hitchin-Kobayashi correspondence)

Let E be a G-equivariant holomorphic vector bundle on M which admits G-invariant Hermitian metrics. Then E admits a G-invariant Hermitian-Einstein metric if and only if E is polystable.

Orbifold Narasimhan–Seshadri theorem

Let M be a compact Riemann surface and Γ a finite group acting on M by automorphisms of the Hermitian structure.

The quotient $\Sigma_g = M/\Gamma$ is an orbifold Riemann surface and by a result due to Selberg, every orbifold Riemann surface is of this form.

The equivariant Hitchin-Kobayashi correspondence implies:

Corollary (Mehta-Seshadri)

$$\bigsqcup_{\mathcal{C} \text{ admissible}} Rep_{\mathcal{C}}(\pi_1^{orb}(\Sigma_g), U(r)) \leftrightarrow \left\{ \begin{array}{c} \textit{Moduli space of parabolic} \\ \textit{bundles of rank } r\textit{, degree} \\ \textit{0 with admissible weights} \end{array} \right\}$$

Instantons on Sasaki manifolds

Recall that on a $2n+1$-dimensional Sasakian manifold X (with $n \geq 2$):

$$\left\{ \begin{array}{c} \text{Anti-self-dual } SU(r) \\ \text{contact instantons} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Rank } r \text{ transverse Hermitian-Einstein} \\ \text{connections with trivial determinant} \end{array} \right\}$$

When X is compact, we have

Corollary

Anti-self-dual $SU(r)$ higher contact instantons on X correspond to rank r polystable transverse holomorphic bundles on X with trivial determinant.

Biswas–Schumacher proved this for **quasi-regular** Sasaki manifolds.

Thank You!