A Poincaré-Hopf Theorem for Orbiline Fields

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Abstract

A line field on an orbifold $\mathcal{O}$ is field of locally invariant tangent lines, that is, a section of its projectivised tangent bundle $PT\mathcal{O}$. Generalising the work done by Crowley and Grant in ([CG17], 2017), we show that for a line field with a finite set of singularities, the orbifold Euler-Satake characteristic $\chi^{\mathrm{orb}}(\mathcal{O}) \in \mathbb{Q}$ can be computed by means of local data (the so called projectivised index) about each singularity. The result can be viewed therefore as a Poincaré-Hopf Theorem for line fields on an orbifold. We take a classical approach, where $\mathcal{O}$ is effective, viewed as a suitable topological space with an atlas consisting of charts, following Satake.

In passage towards this result, we first recall the classical Poincaré-Hopf Index Theorem by means of intersection theory, and secondly the result of Crowley and Grant in [CG17] regarding line fields on a smooth manifold. Following this, we recall some basic orbifold theory, in particular, Satake's Poincaré-Hopf Theorem for vector fields on an orbifold. We then contrast those intersection theoretic results outlined in the first chapter, concluding with brief remarks regarding intersection theory in the category of orbifolds. We conclude this thesis with a generalisation of Crowley and Grant's result in [CG17] to the setting of orbifolds.

Acknowledgements

I wish to first thank my advisor, Pedram, for introducing me to orbifolds, pointing out the potential to generalise the work in [CG17], and for providing a platform on which I was able to encounter a vast array of mathematics. From discussions regarding the Atiyah-Singer Index Theorem, to the Moduli seminar, to the conference on Poisson Geometry and Quantisation, it has all been the genesis of a mathematical journey.

I am indebted to my partner and my family, both for enduring my ramblings about differential geometry and orbifolds, and for the ability to spend hours absent, somewhere in front of a whiteboard. Finally, I wish to acknowledge my cohort and office mates, Elliott, Pankaj, Ben, Chris and Oliver, thank you for your input to this thesis, and for the much (and not so much) needed distraction around the office.

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Introduction

The Poincaré-Hopf Index Theorem is a remarkable result which relates the behaviour of a vector field on a smooth manifold to a topological invariant of the manifold. To be more precise, given a vector field v on a closed smooth n-manifold M with isolated (and thus a finite number of) zeroes, we assign an integer to each zero p, called the (local) index of v at p, denoted $\mathrm{ind}_v(p) \in \mathbb{Z}$. This integer is obtained by restricting v to a small oriented sphere S centered at p, on which there are no other zeroes, normalising via an auxillary metric so as to obtain a map $v|_S : S \to STM|_S$ into the associated sphere bundle, and finally, by choosing S sufficiently small we may compose with a local trivialisation to obtain a self map of oriented $(n-1)$-spheres. The index $\mathrm{ind}_v(p)$ is the Brouwer degree of this resultant composition, that is, it is the degree of the composition, $S \to STM|_S \to S \times S^{n-1} \to S^{n-1}$. The index is independent of all choices made. The Poincaré-Hopf Index Theorem relates the indices of the vector field v to the Euler characteristic, $\chi(M)$, of M, a classical topological invariant. Precisely,

$$\chi(M) = \sum_{\substack{p \in M \\ v(p)=0}} \mathrm{ind}_v(p)$$

Instead of a vector field, consider a line field ξ on M. Intuitively, a line field assigns to each point a tangent line, and these lines vary smoothly along M. Formally, let PTM be the fiberwise projectivisation of TM. A line field is a smooth section $\xi : M \to PTM$. We say that the line field ξ has singularities $\{x_1, \ldots, x_k\} \subset M$ if it is defined only on the submanifold $M - \{x_1, \ldots, x_k\}$. Another way to think about ξ is as a line subbundle of TM, and singularities as points where the rank drops to zero. In the spirit of the Poincaré-Hopf Theorem, we can ask whether a similar index calculation about each singularity $x \in M$ of ξ called the projective index, denoted $\mathrm{p\,ind}_\xi(x) \in \mathbb{Z}$, can be used to compute the Euler characteristic. This question was affirmed for surfaces by Hopf in the 1950's [Hop83], with a slightly different definition of index, which is now called the Hopf index. In [Mar55] Lawrence Markus published a paper in the Annals of Mathematics, which contained a 'generalisation' of Hopf's Theorem to all dimensions. Unfortunately, his result was shown by Crowley and Grant in [CG17] to be invalid for surfaces and in odd dimensions. Put simply, the subtleties are as follows. First, for surfaces, singularities can be 'non-orientable' (this means that ξ is not locally generated by a vector field in a neighbourhood of the singularity). Second, in odd dimensions, Markus gave a slightly more complex definition of the index, which ultimately caused issues. In [CG17], Crowley and Grant corrected the statement of Markus, and provided a generalisation as follows. For closed M (if

$\partial M \neq \varnothing$, they require all singularities to be in the interior, and the line field normal to ∂M), one has

$$2\chi(M) = \sum_{i=1}^{k} \mathrm{p}\,\mathrm{ind}_{\xi}(x_i)$$

where equality is congruence modulo 2 in odd dimensions.

Rather than a manifold, we may consider vector and line fields on more singular spaces. For us, these are *orbifolds.* An orbifold (of dimension n) is a space in which, locally about any given point, one has a neighbourhood of the form $\mathbb{R}^n/G$ for a finite group G, where the groups may vary over the space. Orbifolds form a class of singular spaces; the singular points arising as fixed points of a finite group action. Motivation initially draws to this object simply because it is quite easy to encounter an orbifold in 'nature'. For example, when one takes the quotient of a smooth manifold by a group action, we don't in general get a manifold, but more often an orbifold. It becomes of interest to know if we can still perform any differential geometry on the resulting quotient, and this has been confirmed a long time ago, by Ichirō Satake who (formally) introduced the concept under the title of a 'V-manifold' (see [Sat56]). Satake simultaneously provided natural generalisations of many standard tools and theorems from the differential geometry of manifolds, such as de Rham cohomology and the Gauss-Bonnet Theorem ([Sat56], [SAT57]).

To explain the term that stuck, 'orbifold' was famously introduced via a democratic procedure in a course of William Thurston[1] in 1976-77, and evidently represents a contraction of the two words, 'orbit' and 'manifold'. In modern times, there are two takes on orbifold theory, varying in popularity. The first is a simple generalisation of the definition of a manifold, using charts and atlases. This definition has to its advantage simplicity, however, hidden in details is subtlety. For example, defining the 'correct' notion of a morphism is not immediate. An alternative way to view an orbifold is as a special kind of differentiable stack, which we view as a Morita equivalence class of Lie groupoids. Whilst more technical, this perspective settles in clarity how to define an orbifold morphism, amongst other things. We shall briefly touch on both perspectives, but not give a full treatise on differentiable stacks.

Amongst the results that Satake established for orbifolds, is a Poincaré-Hopf Index Theorem. In particular, for a vector field v on a closed orbifold $\mathcal{O}$ (which we can think of as a locally invariant section of the tangent bundle) with zeroes $\{x_1, \ldots, x_k\} \subset \mathcal{O}$, we can formulate in the usual way a notion of index v about any given zero, written $\mathrm{ind}_v(x)$. Satake defines an orbifold index of v at x to be the usual index weighted by the order of the isotropy of x, denoted $|G_x|$. We define $\mathrm{orb}\,\mathrm{ind}_v(x) := \mathrm{ind}_v(x)/|G_x|$. Satake proved in [SAT57] that,

$$\chi^{\mathrm{orb}}(\mathcal{O}) = \sum_{i=1}^{k} \mathrm{orb}\,\mathrm{ind}_v(x_i) \in \mathbb{Q}$$

[1]Thurston used orbifolds in his geometrization program for 3-manifolds, and introduced the notion of the orbifold fundamental group.

where $\chi^{\mathrm{orb}}(\mathcal{O})$ is the orbifold Euler characteristic of $\mathcal{O}$, a rational number, which is an orbifold homotopy invariant, generalising the Euler characteristic to the setting of orbifolds (in the sense that we define it via a 'compatible' triangulation of our underlying space).

Our contribution is to generalise the main result of Crowley and Grant in [CG17] to the setting of orbifolds, providing a Poincaré-Hopf Index Theorem for line fields on an orbifold. We define a line field on an orbifold to be a section of its projectivised tangent orbibundle. For an orbifold $\mathcal{O}$, we say that a line field ξ has singularities $\{x_1, \ldots, x_k\} \subset \mathcal{O}$ if it is defined only on the suborbifold $\mathcal{O} - \{x_1, \ldots, x_k\}$. About each singularity x of ξ, we define a notion of orbifold projective index at x, given by weighting the usual projective index, something we compute locally, by $|G_x|$, and this is denoted $\mathrm{orb}\,\mathrm{p}\,\mathrm{ind}_\xi(x) \in \mathbb{Q}$. We show that if $\mathcal{O}$ is closed, then

$$2\chi^{\mathrm{orb}}(\mathcal{O}) = \sum_{i=1}^{k} \mathrm{orb}\,\mathrm{p}\,\mathrm{ind}_\xi(x_i)$$

where equality is congruence modulo 2 if $\dim \mathcal{O}$ is odd.

Outline of the Thesis

We record here an outline of the stucture of the thesis. In Chapter 1, we present a standard generalisation of the classical Poincaré-Hopf Index Theorem to the setting of an arbitrary oriented vector bundle over a compact base, by means of intersection theory. Throughout the Chapter, an auxillary goal is to demonstrate that almost all essential constructions can be traced back to Poincaré-duality. In Chapter 2, we recall generalities regarding line fields on smooth manifolds, and in particular, provide an outline of the techniques used by Crowley and Grant in [CG17] to prove a Poincaré-Hopf Theorem for line fields on smooth manifolds. In Chapter 3, we recall some basic theory regarding orbifolds, together with Satake's Poincaré-Hopf Theorem, then as a stepping stone, we discuss the incarnation of orbifolds as Lie groupoids, concluding the Chapter with intersection theory on orbifolds, from the perspective of differentiable stacks. In Chapter 4, we present a proof of a Poincaré-Hopf Theorem for line fields on orbifolds, discussed above.

Assumed Background and Notation

We assume the reader is familiar with standard modern constructions in differential geometry and differential topology, most importantly, the de Rham cohomology and its associated duality. We also assume the reader has at least some familiarity with orbifold theory (both perspectives, in particular, as a stack). Let us fix some notation. For two sets A and B, we write $A \subset B$ to mean A is a subset of B (not necessarily proper). All group actions are, unless otherwise stated, left actions. By smooth, we shall always mean C^∞, and by manifold (resp. orbifold), we always mean smooth manifold (resp. orbifold). For a smooth map of manifolds $f : M \to N$, we denote its differential at $x \in M$ by $df_x : T_xM \to T_{f(x)}N$, where T_xM denotes the tangent space at $x \in M$. We write $H^k_{dR}(M)$ to mean the k-th de Rham cohomology group of M. Unless clarity is needed, we shall simply write $H^k(M)$.

Chapter 1

The Poincaré-Hopf Index Theorem

The Poincaré-Hopf Theorem can be realised as a statement about the zero locus of sections of the tangent bundle of M. Precisely, given an n-manifold M, we embed M into TM by means of the zero section, and consider the intersection 'of M' (i.e. the image of the zero section) with the image of our vector field. Hence we are concerned with the intersection of two submanifolds in an ambient setting, leading us intersection theory. Clearly a natural generalisation is to ask for an entirely similar result, where rather than sections of the tangent bundle $TM \to M$, we consider sections of an arbitrary oriented vector bundle $E \to M$ over M of rank n. In this chapter, we'll outline a proof of such a generalisation. Rather than the Euler characteristic of our manifold, we consider a cohomological invariant of our vector bundle, called the Euler class $e(E) \in H^n_{dR}(M)$. We show that the Euler number, which is by definition $\int_M e(E)$, can be computed out of local degree calculations about the zeroes of any section $s : M \to E$ with isolated zeroes, thus generalising the Poincaré-Hopf Index Theorem.

As our motivation derives from vector fields, we have chosen to proceed within the realm of differential forms and de Rham cohomology, however, the theory we present has a formulation in terms of singular (co)homology, and we may pass between the two perspectives by means of de Rham's Isomorphism Theorem. Finally, it is not the intention of this chapter to present an entirely self contained exposition, as this would lead us astray. On the other hand, an auxillary goal of this chapter is to demonstrate that Poincaré duality of the cohomology groups encapsulates an incredible amount of geometric content. As a consequence, we obtain the Poincaré-Hopf Theorem (certainly a different approach to our forefathers).

Let us make a brief comment on our orientation conventions. Let M and N be oriented smooth manifolds, of dimension m and n respectively. We orient the product manifold $M \times N$ with the so called product orientation, namely, if $(v_1, \ldots, v_m)$ is a positively oriented basis for T_xM, and $(w_1, \ldots, w_n)$ is a positively oriented basis for T_yN, we say that $(v_1, \ldots, v_n, w_1, \ldots, w_m)$ is a positively oriented basis for $T_{(x,y)}(M \times N) \cong T_xM \times T_yN$. Let $E \to M$ be an oriented vector bundle over an oriented manifold M (that is, we have chosen a section of the sphere bundle $S(\det E)$.) Our convention shall be that the total space E is equipped with the local product orientation, that is, in a oriented trivialising cover, we declare each local trivialisation $E|_U \cong U \times \mathbb{R}^n$ to be an orientation preserving diffeomorphism. A direct sum of vector bundles $E_1 \oplus E_2$ for $E_1, E_2 \to M$ is oriented by declaring a positive basis of the fiber $(E_1)_x \oplus (E_2)_x$ to be a positive basis of $(E_1)_x$, followed by a positive basis of $(E_2)_x$. Let $S \subset M$ be a compact submanifold. Write ν_S^M for its normal bundle. For a choice of metric, one has a canonical decomposition $TM|_S \cong TS \oplus \nu_S^M$, and thus orientations of S and M canonically determine an orientation for the normal bundle, where we follow the base first convention. By submanifold, we mean regular submanifold (see [Tu]). We follow [Bot82], [Nic07], [Mad97], [RS18], [Ebe14] and [Hir76] closely. Unless otherwise specified, we assume all manifolds to be connected, and submanifolds compact.

1.1 Intersection Theory

We recall several aspects regarding differential topology. Let us emphasise that, upon introducing machinery, one of our cornerstones is the Poincaré-duality, whose statement we now recall. For a orientable n-manifold M of finite type, we have a bilinear pairing

$$\begin{aligned} \int : H^k(M) \times H_c^{n-k}(M) &\to \mathbb{R} \\ ([\omega], [\tau]) &\mapsto \int_M \omega \wedge \tau. \end{aligned}$$

Poincaré-duality asserts that this pairing is non-degenerate, or, what is the same thing, that $(H^k(M))^* \cong H_c^{n-k}(M)$. Observe that, for $k = 0$, the pairing is just integration of compactly supported (which may be dropped if M is compact) top forms over M, and this brings us to the first fundamental homotopy invariant, namely the degree of a smooth mapping.

The degree generalises the winding number from complex analysis to higher dimensional manifolds, and is an incredible topological-geometric tool introduced by Brouwer. Let $f : M \to N$ be a smooth map map of closed, oriented and connected n-manifolds. By means of the de Rham functor and Poincaré-duality, we have a commutative diagram

$$\begin{tikzcd} H^n(N) \arrow[r, "f^*"] \arrow[d, "\cong"'] & H^n(M) \arrow[d, "\cong"] \\ \mathbb{R} \arrow[r, "\deg f"'] & \mathbb{R} \end{tikzcd}$$

where the vertical isomorphisms are induced by integration over the respective manifold. The lower horizontal map $\mathbb{R} \to \mathbb{R}$, being linear, must act by scalar multiplication, and we call this scalar the degree of f, denoted $\deg f$. This is to say that, given $[\omega] \in H^n(N)$ we have

$$\deg f \int_N \omega = \int_M f^*\omega.$$

As smoothly (even continuously) homotopic maps induce the same map in cohomology, it follows that the degree is a homotopy invariant of f. A priori, the degree of f is just a real number, but it turns out, quite remarkably, that it is in fact an integer. Recall that $y \in N$ is called a regular value of f if $df_x : T_xM \to T_yN$ is surjective, for all $x \in f^{-1}(y)$, and by Sards Theorem (see [Mil65], Page 16), regular values of f are dense in N.

Theorem 1.1.1. *Let M, N and f be as above. Let $z \in N$ be a regular value for f (whose existence is guaranteed by Sard's Theorem). If $z \notin \operatorname{im} f$, then $\deg f = 0$. If $z \in \operatorname{im} f$, then $f^{-1}(z)$ is a finite set of points, and*

$$\deg f = \sum_{x \in f^{-1}(z)} \operatorname{sign}(df_x)$$

where $\operatorname{sign}(df_x) = +1$ or -1 according to whether f preserves or reverses orientation.

Proof. This is a standard result, we refer the reader to ([Mad97], Page 101). □

With this perspective we may have the following geometric interpretation of the degree of f. For generic $y \in N$, we may find a neighbourhood U of y whose preimage under f consists of a 'stack of records', (namely a finite collection of disjoint open sets, each of which is mapped diffeomorphically onto U) and the degree of f is the algebraic number of times f covers U.

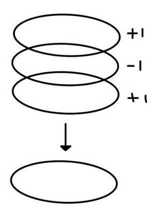


Figure 1.1.1: Visualising the degree as an algebraic covering number.

We record an easy observation that shall be used frequently, without explicit reference.

Proposition 1.1.1. *Suppose $f : M \to N$ is a diffeomorphism. For $\omega \in \Omega^n(N)$, we have*

$$\int_N \omega = \pm \int_M f^*\omega$$

according to whether f preserves or reverses orientation.

Let us now provide some standard examples.

Example 1.1.1. Given any integer k, there is a map whose degree is k. First, let us handle the case of a positive integer $k \in \mathbb{Z}$. Define a smooth map $f : S^1 \to S^1$ by $f(z) = z^k$. Let $\phi : \mathbb{R} \to S^1$ defined by $\phi(t) = (\cos t, \sin t)$ be orientation preserving. Let $\psi : \mathbb{R} \to \mathbb{R}$ be multiplication by k.

Now, $f \circ \phi = \phi \circ \psi$. We take the differential, and apply the chain rule to obtain a commutative diagram,

$$\begin{tikzcd}
T_z S^1 \arrow[r, "df_z"] & T_{f(z)} S^1 \\
\mathbb{R} \arrow[u, "d\phi_t"] \arrow[r, "\psi"'] & \mathbb{R} \arrow[u, "d\phi_{kt}"']
\end{tikzcd}$$

in which $z = \phi(t)$. The Jacobian of ϕ at t is given by scaling the vector $(-\sin t, \cos t)^T$. Now, a non-zero linear map between two vector spaces of dimension 1 is necessarily an isomorphism, and so we may identify $df_z : T_z S^1 \to T_{f(z)} S^1$ with multiplication by k, that is, identify with ψ via the vertical (orientation preserving) isomorphisms. The point $1 \in S^1$ is a regular value of f. Consider the preimage $f^{-1}(1)$; it consists of k-points, the k-th roots of unity. It is now an easy consequence of Theorem 1.1.1 that the map f has degree k (it is also easy directly, computing the pullback and integrating). For a negative integer we proceed as follows. With similar identifications outlined above, the differential of conjugation $z \to \overline{z}$ can be viewed as reflection, now the degree is multiplicative under composition, and so the map $z \mapsto \overline{z}^k$ has degree $-k$. We have shown that the map

$$\deg : [S^1, S^1] \to \mathbb{Z}$$

is in fact surjective, where $[S^1, S^1]$ denotes homotopy classes of self maps of the circle. In fact, this mapping is injective, that is, for two maps $f, g : S^1 \to S^1$ with $\deg f = \deg g$, we have that f is homotopic to g. The latter is a special case of Hopf's Degree Theorem, which says that homotopy classes of maps from a closed oriented n-manifold M to the n-sphere are classified by their degree.

Example 1.1.2. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be given by $x \mapsto Lx$ for $L \in \mathrm{GL}(n, \mathbb{R})$. There is an induced map f given by $f(x) = F(x)/\|F(x)\|$. With Theorem 1.1.1, we observe that the degree of f is the sign of the determinant of L. In particular, $\deg f = +1$ or -1 according to whether A preserves or reverses orienation.

Example 1.1.3. Let $\pi : E \to M$ be a smooth orientation preserving k-fold covering map, where both M and E are connected, orientable, compact, and of the *same* dimension. It is an easy consequence of Theorem 1.1.1 that π has degree k. In particular, if $\dim M = n$ and ω is an n-form on M, one has

$$\int_E \pi^* \omega = k \int_M \omega.$$

This special case foreshadows the so called *projection formula* (cf. (1.1.1)), which involves 'integration along the fiber coordinates', the fiber in this case being dimension 0.

We now discuss another consequence of Poincaré-duality. Let S be a closed oriented submanifold of M, of dimension k. Let $i : S \hookrightarrow M$ be inclusion. Define a linear functional on $H^k(M)$ by

$$\begin{aligned} H^k(M) &\to \mathbb{R} \\ [\omega] &\to \int_S i^* \omega. \end{aligned}$$

As S is without boundary, this is well-defined by Stokes' Theorem. By Poincaré-duality, this corresponds to a unique class $[\eta_S] \in H_c^{n-k}(M)$ which we call the (compact) Poincaré-dual of S. By definition, the class $[\eta_S]$ is characterised by the relation

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S$$

for all $\omega \in H^k(M)$.

Remark 1.1.1. The class above is titled the compact Poincaré-dual because the submanifold S is compact, so we can integrate any k-form over S. If the compactness assumption is dropped, one must, of course, integrate forms with compact support, and require that S is topologically closed, then one obtains a class in $H^{n-k}(M)$. Some authors title this the 'closed Poincaré-dual' ([Bot82], Page 51). We shall only work with compact submanifolds, and so by Poincaré-dual, we always mean compact Poincaré-dual.

Let us provide the simplest such example.

Example 1.1.4. Set $M = \mathbb{R}^n$ and $S = \{\mathrm{pt}\} \subset \mathbb{R}^n$. By definition, the Poincaré dual of S is (represented by) a compactly supported n-form η_S such that, for any constant $\lambda \in \mathbb{R} \cong H^0(M)$, we have

$$\lambda = \int_{\{\mathrm{pt}\}} \lambda = \lambda \int_{\mathbb{R}^n} \eta_S$$

where we recall that to wedge with a constant is just to multiply. Thus η_S is just a compactly supported n-form on $\mathbb{R}^n$ with total integral 1. Observe that we can localise the support of any representative as much as we'd like.

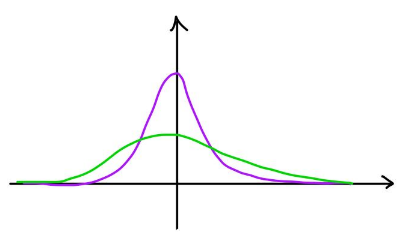


Figure 1.1.2: Inspired by [Nic07],Example 7.3.6; the dual of a point is Dirac's distribution.

Localising the support of representatives of certain cohomology classes shall be instrumental in what follows. For example, we have the following *localisation principle* ([Bot82], Page 53). Suppose $W \subset M$ is an open set whom contains S, and write $[\eta_{S,W}] \in H_c^{n-k}(W)$ for the Poincaré-dual of S in W. Extend $\eta_{S,W}$ by zero to obtain a compactly supported $n-k$ form η'_S defined on all of M. Then,

$$\int_S i^*\omega = \int_W \omega \wedge \eta_{S,W} = \int_M \omega \wedge \eta'_S$$

so that $[\eta'_S]$ is the Poincaré-dual of S in M. Equipped with localisation of supports, we may consider neighbourhoods whom admit significantly more structure. For a compact submanifold

$S \hookrightarrow M$ of codimension $n-k$, there is an especially important open neighbourhood of S, namely that of a tubular neighbourhood; a neighbourhood diffeomorphic to a vector bundle of rank $n-k$ over S, with S diffeomorphic to the (image of the) zero section.

Observe that there is a canonical inclusion $TS \hookrightarrow (TM)|_S$. The normal bundle of S in M, denoted ν_S^M (our notation follows that of ([Ebe14],Page 79)) is defined as the quotient bundle $(TM)|_S/TS$, so that we have an exact sequence

$$0 \to TS \to (TM)|_S \to \nu_S^M \to 0$$

of vector bundles over S (which, a priori, does not admit a *canonical* splitting). Recall that we orient the normal bundle of S in M by means of the base first convention, namely, if M is equipped with a Riemannian metric, then the fibres of the normal bundle may be identified with orthogonal complements,

$$(\nu_S^M)_x = T_xM/T_xS \cong (T_xS)^\perp.$$

In particular, the normal bundle of S in M may be identifed with the orthogonal complement $(TS)^\perp$, a subbundle of TM. We have a canonical splitting $(TM)|_S = TS \oplus TS^\perp$, and so an orientation of ν_S^M, in keeping with our base first convention. For submanifolds with nowhere vanishing outward normals, their normal bundle is trivial. For example,

Example 1.1.5. Let (x, y) be the usual global coordinates on $\mathbb{R}^2$ with the standard metric. Consider the 1-sphere $S^1 = \{(x, y) \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$. For each $p \in S^1$, we have $T_pS^1 \cong \mathrm{span}_{\mathbb{R}}(p)^\perp$ so that $v : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $v(p) = p$ restricted to S^1 yields a nowhere vanishing section of the normal bundle. Thus $\nu_{S^1}^{\mathbb{R}^2} \cong S^1 \times \mathbb{R}$. We can in fact be even lazier. There are only two real line bundles over the circle, and so if it can be oriented, we are done.

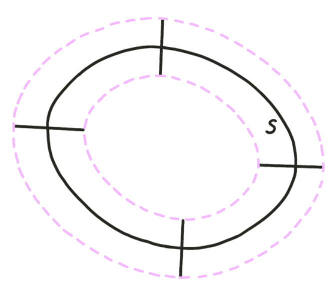


Figure 1.1.3: Normal bundle of the circle viewed as an embedded submanifold of $\mathbb{R}^2$.

Let us briefly formulate a general definition of normal bundle.

Definition 1.1.1. *Let $f : S \to M$ be an immersion of smooth manifolds. The normal bundle of the mapping f is by definition the vector bundle $\nu f := f^*TM/TS$. If f is inclusion, we write ν_S^M, called the normal bundle of S in M.*

We define formally our neighbourhoods of interest, but only in the special case of inclusion.

Definition 1.1.2. *Let M be a smooth manifold, and S a submanifold. A tubular neighbourhood of S in M is an open neighbourhood U of S in M that is diffeomorphic to the total space of the normal bundle of S in M, denoted ν_S^M, and for which the diffeomorphism carries $S \subseteq U$ to the zero section of ν_S^M. The data is often presented as a pair, the diffeomorphism is called a tubular map.*

For compact submanifolds, tubular neighbourhoods exist. The proof of this result is non-trivial, yet we shall need some aspects of the details later, and so we recall the easy details.

Theorem 1.1.2 (Tubular Neighbourhood Theorem)**.** *Let S be a compact submanifold of M. Then S admits a tubular neighbourhood.*

Proof. This is a classical result, see for example ([Spi70],Volume 1,Page 334). □

We recall some brief details. First, equip M with a Riemannian metric. Let us denote the resultant 'norm' on TM by $\|\cdot\|$ and the (honest) metric on M by $d(\ ,\ )$. For $\varepsilon > 0$, we consider the following open neighbourhoods,

$$N_\varepsilon = \{v \in (TS)^\perp \mid \|v\| < \varepsilon\}$$
$$U_\varepsilon = \{p \in M \mid d(p, S) < \varepsilon\}.$$

For sufficiently small $\varepsilon > 0$, one shows that the exponential map (from Riemannian geometry) $\exp : N_\varepsilon \to U_\varepsilon$ is well-defined (as S is compact) and further, is a diffeomorphism. Hence U_ε is a tubular neighbourhood of S.

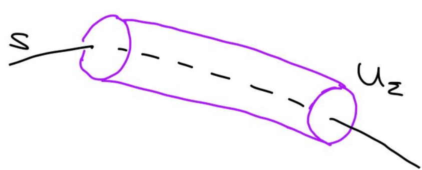


Figure 1.1.4: A tube about a nice curve S in $\mathbb{R}^3$.

Thus far we have discussed how to establish a relationship between bundles and open neighbourhoods of submanifolds. In what is to come, this relationship shall be used extensively. We now recall some theory related to cohomological invariants of vector bundles, in particular, the Thom Isomorphism. As we shall only be concerned with a compact manifold M, it is again a consequence of Poincaré-duality!

Given a oriented vector bundle $\pi : E \to M$ of rank k over a manifold M, let $\Omega^p_{cv}(E)$ be the space of all p-forms on E with *vertically compact support.* Recall that a form ω on E is said to have vertical compact support if the map $\pi : \mathrm{supp}(\omega) \to M$ is proper. Clearly, such forms admit compact support on each fiber E_x. For vertical compactly supported forms, we have a cochain complex, together with cohomology groups $H^*_{cv}(E)$, called the *cohomology of E* *with compact support in the vertical direction.* The Thom Isomorphism Theorem asserts that $H^*_{cv}(E) \cong H^{*-k}(M)$. If M is compact, we have $H^*_{cv}(E) = H^*_c(E)$ (as compact sets in Hausdorff spaces are closed) and obtaining such an isomorphism is a consequence of duality. Indeed, let $s : M \to E$ denote the zero section. Then π^* and s^* are inverse isomorphisms in cohomology as $s \circ \pi$ is homotopic to the identity. In particular, E is of finite type. Now by Poincaré-duality observe that

$$H^p_{cv}(E) = H^p_c(E) \cong (H^{n+k-p}(E))^* \cong (H^{n+k-p}(M))^* \cong H^{p-k}_c(M) = H^{p-k}(M).$$

Let the resultant isomorphism be denoted by π_*. Let $[\omega] \in H^p_{cv}(E)$. If we trace through the above isomorphisms, we see that the equality of two linear functionals is asserted. Namely, for each $[\tau] \in H^{n+k-p}(M)$, we have

$$\int_E \pi^*\tau \wedge \omega = \int_M \tau \wedge \pi_*\omega. \tag{1.1.1}$$

This identity is known as the *projection formula.* Set $p = k$, and choose $[\Phi_E] \in H^k_{cv}(E)$ for which $\pi_*[\Phi_E] = 1 \in H^0(M)$. We call $[\Phi_E]$ the *Thom Class* of the oriented rank k vector bundle $\pi : E \to M$. Representatives of the Thom class are called *Thom forms.* By the projection formula above, we may develop a unique characterisation of Thom forms as those forms whom restrict to the generator on each fiber, precisely,

Proposition 1.1.2. *The Thom class $[\Phi_E]$ on an oriented vector bundle $\pi : E \to M$ of rank k over a closed n-manifold M may be uniquely characterised as the cohomology class in $H^k_{cv}(E)$ which restricts to the generator on each fiber. That is, for each $x \in M$, we have*

$$\int_{E_x} \Phi_E = 1.$$

Proof. The proof given here is drawn from ([Ebe14], Page 84). The forwards direction is a consequence of the projection formula. Indeed, let Φ_E be a Thom form on E. Choose an oriented chart (U, x) on M and an n-form η on U with total integral 1, extended by zero to a form on M. In coordinates on U, we can write $\eta = a(x)dx^1 \wedge \cdots \wedge dx^n$ for a smooth compactly supported function $a : U \to \mathbb{R}$. We arrange U so that $E|_U \cong U \times \mathbb{R}^k$, and write ξ for our bundle coordinates $E|_U$. We may write Φ_E on $E|_U$ as a sum

$$b(x,\xi)d\xi^1 \wedge \cdots \wedge d\xi^k + \zeta$$

where ζ is a combination of forms where the terms involve at most $(n-1)$ $d\xi_i$'s, and therefore at least one dx_j. In $E|_U$, we then have,

$$\pi^*\eta \wedge \Phi_E = a(x)b(x,\xi)dx^1 \wedge \cdots \wedge dx^n \wedge d\xi^1 \wedge \cdots \wedge d\xi^k$$

where we've identified each x^i coordinate with its pullback. Therefore,

$$\begin{aligned}\int_E \pi^*\eta \wedge \Phi_E &= \int_{U\times\mathbb{R}^k} a(x)b(x)dx^1 \wedge \cdots \wedge dx^n \wedge d\xi^1 \wedge \cdots \wedge d\xi^k \\ &= \int_U \left(\int_{\mathbb{R}^k} a(x)b(x,\xi)d\xi\right) dx\end{aligned}$$

where we've used Fubini's Theorem and several identifications. On the other hand, by the projection formula,

$$\int_E \pi^*\eta \wedge \Phi_E = \int_U \eta = 1.$$

Let $c(x) = \int_{\mathbb{R}^k} b(x, \xi) d\xi$. We then have,

$$\int_U a(x)c(x)dx = \int_U a(x)dx$$

The equality above holds for each compactly supported smooth function $a : U \to \mathbb{R}$, and therefore $c(x) = 1$. This states precisely that,

$$\int_{E_x} \Phi_E = 1.$$

As $x \in M$ was arbitrary, the claim follows. For the converse direction, one uses the second part of the Theorem 1.1.3. □

Remark 1.1.2. The proof above different to the usual one, but it allows us to circumvent a complete diversion to integration along the fiber coordinates. Finally, for the conclusion that $c(x) = 1$ above, use linearity and that if a measurable function has Lebesgue integral zero, then it is zero almost everywhere.

Remark 1.1.3 (The General Case)**.** The Thom Isomorphism holds true even if M is only of finite type. One defines first a homomorphism called 'integration along the fibers', $\pi_* : \Omega^*_{cv}(E) \to \Omega^{*-k}(M)$ by the following classical procedure. We first make a definition of π_* within a local product coordinate system. Then, by means of a trivialising open cover of M, lifted to E via π, and a partition of unity subordinate to this open cover, we extend π_* linearly to an arbitrary $\omega \in \Omega^*_{cv}(E)$. One then checks that the resulting form on M is independent of all choices made. To this end, fix a oriented chart $(U, x^1, \dots, x^n)$ on M for which $E|_U = \pi^{-1}(U) \cong U \times \mathbb{R}^k$ is trivial. Let $(t^1, \dots, t^k)$ denote coordinates on $\mathbb{R}^k$. Coordinates on $\pi^{-1}(U)$ are given by $(\pi^*x^1, \dots, \pi^*x^n, t^1, \dots, t^k)$. Write $\overline{x}^j = \pi^*x^j$. A differential form on $\pi^{-1}(U)$ is a linear combination of two types of forms, those who do not contain $dt^1 \wedge \cdots \wedge dt^k$ (type I), and those who do (type II). For $\omega \in \Omega^*_{cv}(\pi^{-1}(U))$, we may uniquely decompose into a sum of type I and type II forms. We define π_* for forms on $\pi^{-1}(U)$ by mapping type I forms to zero, and for a type II form

$$\nu = \sum_I a_I(x,t) d\overline{x}^I \wedge dt^1 \wedge \cdots \wedge dt^k$$

where I runs over strictly ascending indices, and $a_I(x,t)$ is a smooth real valued function, whose factor in t has compact support in $\mathbb{R}^n$. We define

$$\pi_*(\nu) = \sum_I \left(\int_{\mathbb{R}^n} a_I(x,t) dt^1 \cdots dt^k \right) dx^I.$$

With the procedure outline above, it can be shown that this extends to a well defined homomorphism $\pi_* : \Omega^*_{cv}(E) \to \Omega^{*-k}(M)$. (Note, our convention in which we write the order of the forms is that of [Bot82], other authors may decompose with t first as above, and a sign difference appears in the projection formula.) Let us now summarise the above by recording the discussed results. For complete proofs, we refer the reader to [Bot82],[RS18].

Theorem 1.1.3 (Thom Isomorphism Theorem)**.** *Suppose $\pi : E \to M$ is a orientable rank k vector bundle over a manifold M of finite type. Then there exists a homomorphism $\pi_* : \Omega^*_{cv}(E) \to \Omega^{*-k}(M)$, called integration along the fiber, which satisfies*

1. *Denote by d the exterior derivative. Then $\pi_* \circ d = d \circ \pi_*$, so that there is an induced map $\pi_* : H^*_{cv}(E) \to H^{*-k}(M)$.*
2. *If $\tau \in \Omega^*(M)$ and $\omega \in \Omega^*_{cv}(E)$, then $\pi_*(\pi^*\tau \wedge \omega) = \tau \wedge \pi_*\omega$.*
3. *(Projection Formula) If M is orientable, then for $\omega \in \Omega^p_{cv}(E)$ and $\tau \in H^{n+k-p}_c(M)$, one has*
$$\int_E (\pi^*\tau) \wedge \omega = \int_M \tau \wedge \pi_*\omega.$$

Furthermore, the induced map in cohomology $\pi_* : H^*_{cv}(E) \to H^{*-k}(M)$ is an isomorphism.

We'll now return to our special case.

Lemma 1.1.4. *Let $\pi : E \to M$ be an oriented vector bundle of rank k over a closed oriented n-manifold M. Let Φ_E be a Thom form. Then given a closed form n-form $\sigma \in \Omega^n(E)$ we have,*

$$\int_E \sigma \wedge \Phi_E = \int_M \iota^*\sigma,$$

where $\iota : M \to E$ is the zero section.

Proof. This is part of ([RS18],Lemma 7.2.15, Page 194). As Φ_E is a Thom form, we have $\pi_*\Phi_E = 1$, and in particular, for an n-form η on M

$$\int_E \pi^*\eta \wedge \Phi_E = \int_M \eta.$$

Let $\sigma \in \Omega^n(E)$ be a closed n-form. Now $\iota \circ \pi : E \to E$ is homotopic to the identity, so $\sigma - \pi^*\iota^*\sigma \in \Omega^n(E)$ is an exact form. Thus,

$$\int_E \sigma \wedge \Phi_E = \int_E \pi^*\iota^*\sigma \wedge \Phi_E = \int_M \iota^*\sigma.$$

□

Remark 1.1.4 (Support of Thom Forms)**.** Let $\pi : E \to M$ be as above. Let $U \subset E$ be an open neighbourhood of the zero section. Then we may choose a representative of the Thom class of $E \to M$ whose support is contained in U. Intuitively, the Thom class is uniquely characterised as the form whose restriction to each fiber is the generator, and so this is clear. For a formal proof, we refer the reader to ([RS18], Page 199).

We shall now introduce arguably the most important notion: transversality.

Definition 1.1.3. *Let M, L and S be smooth manifolds. Let $f : L \to M$ and $g : S \to M$ be smooth maps. We say that f and g are transverse, denoted $f \pitchfork g$, if for all x, y with $f(x) = g(y) = z$, we have*

$$df_x(T_xL) + dg_y(T_yS) = T_zM.$$

If g is the inclusion of a submanifold, we write $f \pitchfork S$.

Example 1.1.6. Let us provide some intuitive examples of a transverse and non-transverse intersection. Fix $\mathbb{R}^2$ as our ambient space, and consider the following intersections.

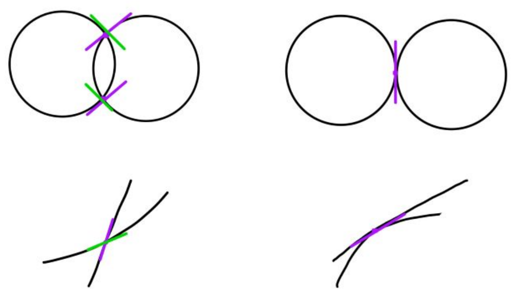


Figure 1.1.5: The left hand side consists of transverse intersections, the right hand side non-transverse.

Let us now state several major results regarding approximation and transversality.

Proposition 1.1.3. *Let $f : L \to M$ be an immersion, $S \subset M$. Suppose $f \pitchfork S$. Then $f^{-1}(S)$ is a submanifold of L, of dimension $\dim M - \dim S$. Moreover, the following relation holds true for the normal bundles,*

$$\nu^L_{f^{-1}(S)} \cong f^*\nu^M_S.$$

Proof. See ([Ebe12],Proposition 1.1.2) and ([Lee],Page 144). □

Theorem 1.1.5. *Let $f : L \to M$ and $g : S \to M$ be smooth maps. Let M be equipped with the data of a Riemannian metric, denote the resultant metric on M by $d(\ ,\ )$. Let $\varepsilon : M \to (0, \infty)$ a function. Then there exists a map $h : L \to M$ with $d(f(x), h(x)) < \varepsilon(x)$ such that $h \pitchfork g$, that is, we can approximate f with mappings transverse to g.*

Proof. We refer the reader to ([Ebe12], Theorem 1.1.3) and ([BJ82],Page 149). □

We provide an example of these results applied to sections of a vector bundle.

Example 1.1.7 (Perturbation of a section)**.** With the results above, we may perturb maps as needed, however, if we perturb a section, we must ensure that it is again a section. Indeed, for sufficiently small perturbation things are okay. Suppose $\pi : E \to M$ is a vector bundle over a compact manifold M. Let $s : M \to E$ be a smooth section, then we can approximate s by a smooth map transverse to the zero section, call this map $f : M \to E$. Then $g := \pi \circ f$ is a perturbation of $\pi \circ s = 1_M$. In particular, for f close to s, g will be a diffeomorphism, for it is close to the identity. Define $t(x) = f(g^{-1}(x))$. Then $(\pi \circ t)(x) = x$, so t is a section, and in particular, t is transverse to the zero section, and approximates s. (This example is from [Bot82], proof of Proposition 11.14).

We shall now define the oriented intersection number of two submanifolds who intersect transversally, and are of complimentary dimension.

Definition 1.1.4. *Let $S \subset M$ be a submanifold of dimension k. Let $f : L \to M$ be a smooth map where L is a compact manifold of dimension $m - k$, so that $\dim L + \dim S = \dim M$. Suppose that $f \pitchfork S$, then $f^{-1}(S)$ consists of a finite collection of points. To each point $x \in f^{-1}(S)$, we assign an integer, denoted $\iota_x(S, f)$, defined as follows. Consider the direct sum $T_{f(x)}S \oplus df_x(T_xL) = T_{f(x)}M$. We set $\iota_x(S, f) = +1$ if the positive orientations of the summands pair up to give the positive orientation of $T_{f(x)}M$, else we set $\iota_x(S, f) = -1$. We define the oriented intersection number $I(S, f)$ to be the finite sum,*

$$I(S, f) = \sum_{x \in f^{-1}(S)} \iota_x(S, f).$$

The order in which the mapping and submanifold are written is important, and swapping the factors results in a (potentially trivial) sign contribution. In our definition here, we input the submanifold first. The reason for this shall become clear later. The submanifold of interest will be our base M embedded into the total space by the zero section, and the intersection number is counting the number of zeroes, with signs.

Example 1.1.8. Let $\mathbb{R}^2$ be equipped with global coordinates x and y, consider the x-axis $L_x = \mathbb{R} \times \{0\}$ and y-axis $L_y = \{0\} \times \mathbb{R}$ respectively. An orientaton of a 1-dimensional manifold is a choice of positive direction. Orient L_x by calling $e_1 = (1, 0)$ the positive direction, and L_y by calling $-e_2 = (0, -1)$ the positive direction (i.e, standard y-axis, but with orientation reversed. The frame $(e_1, -e_2)$ has opposite orientation to (e_1, e_2), so $I(L_x, L_y) = -1$, on the other hand the frame $(e_2, -e_1)$ has positive orientation (i.e. positive determinant), so $I(L_y, L_x) = +1$. Linked to our orientation is the corresponding top form, in this case one has top forms dx and dy for the standard orientations and in our case with the reversed orientation, dx and $d(-y) = -dy$. We'll show that by manipulating with suitable functions and integrating these we can pickup intersection numbers.

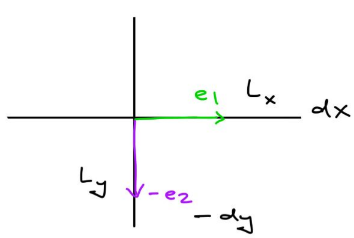


Figure 1.1.6: Intersection number of axes with opposite orientations.

We now come to a cornerstone of this chapter. We shall establish a relationship between the Poincaré-dual of a submanifold and the Thom class of its normal bundle. The ability to utilise Thom forms as representatives of duals is exceptionally useful, and is a genesis in our intersection theory, as we shall demonstrate. Fix an auxillary Riemannian metric on M, and identify the normal bundle of S with $TS^{\perp}$. By the Tubular Neighbourhood Theorem, there exists an $\varepsilon > 0$ for which the two neighbourhoods,

$$\begin{aligned} N_\varepsilon &= \{v \in (TS)^{\perp} \mid \|v\| < \varepsilon\} \\ U_\varepsilon &= \{p \in M \mid d(p, S) < \varepsilon\} \end{aligned}$$

are diffeomorphic via the exponential map $\exp : N_\varepsilon \to U_\varepsilon$. Choose a Thom form $\Phi_\varepsilon \in \Omega_c^{n-k}(TS^{\perp})$ for which $\operatorname{supp}(\Phi_\varepsilon) \subset N_\varepsilon$. Define a form Φ_S on M by $(\exp^{-1})^*\Phi_\varepsilon$ and extend by zero to all of M, precisely we have

$$\Phi_S = \begin{cases} (\exp^{-1})^*\Phi_\varepsilon & \text{on } U_\varepsilon \\ 0 & \text{on } M\backslash U_\varepsilon. \end{cases}$$

It is clear that Φ_S is a closed $(n-k)$-form on M with compact support. Let $\iota : S \to TS^{\perp}$ be the zero section. Let $\omega \in \Omega^k(M)$. Observe that $\exp \circ \iota$ is just inclusion $S \hookrightarrow M$. Then, noting that, with our conventions exp is orientation preserving, using Lemma 1.1.4

$$\int_M \omega \wedge \Phi_S = \int_{U_\varepsilon} \omega \wedge (\exp^{-1})^*\Phi_\varepsilon = \int_{N_\varepsilon} \exp^* \omega \wedge \Phi_\varepsilon = \int_S \iota^* \exp^* \omega = \int_S \omega.$$

Therefore $[\Phi_S] \in H_c^{n-k}(M)$ is the Poincaré dual of S.

Theorem 1.1.6. *Let $S \subset M$ be a closed oriented k-dimensional submanifold of an oriented n-manifold M. Let L be a closed oriented manifold of dimension $n-k$, and let $f : L \to M$ be a smooth map transverse to S. Let $[\eta_S] \in H_c^{n-k}(M)$ be the Poincaré dual of S. Then,*

$$I(S, f) = \int_L f^*\Phi.$$

The proof uses a common homotopy trick, namely that of 'dragging fibers' along a contractible neighbourhood. (We invite the reader to compare with Page 443 of [Spi70]).

Proof. We sketch a proof following ([RS18],Theorem 7.2.18, Page 198). As outlined above, let us work with the representative $\Phi_S \in \Omega_c^{n-k}(M)$ of the Poincaré-dual, obtained via extending (a representative of) the Thom class of S by zero. Futhermore, carry through the notation above. As $f \pitchfork S$, the preimage $f^{-1}(S)$ is a finite set of points (see 1.1.3). Write $f^{-1}(S) = \{x_1, \ldots, x_q\}$. As $\dim S + \dim L = \dim M$, f is an immersion at each point of $f^{-1}(S)$. In particular, about each $x_j \in f^{-1}(S)$ we have an open neighbourhood V_j about x_j for which $f|_{V_j}$ is a smooth embedding, whose image is transverse to S. We may without loss of generality assume that the sets $\{V_j\}$ are pairwise disjoint, and even further, we may choose $\varepsilon > 0$ sufficiently small so as to obtain $f^{-1}(U_\varepsilon) = \bigcup_{j=1}^q V_j$. Now $\operatorname{supp}(f^*\Phi_S) \subset \cup_{j=1}^q V_j$ and we have,

$$\int_L f^*\Phi_S = \sum_{j=1}^q \int_{V_j} f^*\Phi_S = \sum_{j=1}^q \int_{V_j} (\exp^{-1} \circ f)^*\Phi_\varepsilon.$$

In particular, it is sufficient to show that

$$\int_{V_j} (\exp^{-1} \circ f)^* \Phi_\varepsilon = \iota_{x_j}(S, f).$$

To this end, let us drop subscripts and fix $x \in f^{-1}(S)$ and the corresponding open neighbourhood V about x. Let W be a contractible neighbourhood of $f(x)$ for which $TS^\perp|_W$ is trivial. Let $\psi : TS^\perp \to W \times \mathbb{R}^{n-k}$ be a local trivialisation. We may assume that $N_\varepsilon|_W$ is mapped diffeomorphically under ψ to $W \times B_\varepsilon$ where B_ε denotes the open ball of radius ε in $\mathbb{R}^{n-k}$, centered at the origin. Define a form τ on $W \times B_\varepsilon$ by $\psi^*\tau = \Phi_\varepsilon|_W$. We observe that τ is a Thom form on $W \times B_\varepsilon$. Now,

$$\int_V (\exp^{-1} \circ f)^* \Phi_\varepsilon = \int_V (\psi \circ \exp^{-1} \circ f)^* \tau. \tag{1.1.2}$$

Let $\mathrm{pr}_1 : W \times B_\varepsilon \to B_\varepsilon$ and $\mathrm{pr}_2 : W \times B_\varepsilon \to B_\varepsilon$ denote projection onto the first and second factor respectively. Now, we may choose $\varepsilon > 0$ sufficiently small so that $f(V) \subset W$, and for which the following mapping,

$$g := \mathrm{pr}_2 \circ \psi \circ \exp^{-1} \circ f|_V : V \to B_\varepsilon$$

is a diffeomorphism. We observe that if g preserves orientation, then f is compatible with the local product orientation on the normal bundle, and in particular $\iota_x(S, f) = +1$. If g reverses orientation, then we have $\iota_x(S, f) = -1$. By assumption, W is contractible, so there is a homotopy $H : V \times [0, 1] \to W$ for which

$$\begin{aligned} &H(\cdot, 0) \equiv f(x) : V \to W \\ &H(\cdot, 1) = \mathrm{pr}_1 \circ \psi \circ \exp^{-1} \circ f|_V : V \to W. \end{aligned}$$

Let us write $h_t = H(\cdot, t)$. In particular, $h_1 \times g = \psi \circ \exp^{-1} \circ f|_V : V \to W \times B_\varepsilon$ is the map appearing in (1.1.2). The pullback of $\tau \in \Omega_c^{n-k}(W \times B_\varepsilon)$ by the collection $h_t \times g$ is compactly supported in $V \times [0, 1]$. Homotopic maps induce the same map in cohomology, and so

$$\int_V (\psi \circ \exp^{-1} \circ f)^* \tau = \int_V (h_1 \times g)^* \tau = \int_V (h_0 \times g)^* \tau = \iota_x(S, f) \int_{\{f(x)\} \times B_\varepsilon} \tau = \iota_x(S, f).$$

Hence the result follows. □

If we apply this result to two submanifolds, where f is realised as an inclusion map, we get the following corollary.

Corollary 1.1.1. *Let S and L be two transverse compact submanifolds of M, with complimentary dimension. Let η_S and η_L be forms representing the Poincaré-duals of S and L respectively. Then,*

$$I(S, L) = \int_M \eta_S \wedge \eta_L.$$

In particular, the left hand side was only defined for transverse intersections, but the right hand side makes sense without a transversality assumption. In particular, one has a way to generalise the intersection number.

Example 1.1.9 (Loops on the Torus)**.** Consider two loops on the Torus, S and L, given by taking horizontal and vertical line segments, in the representation of the Torus as a square with edges identified, oriented in the usual way. Write η_S and η_L for the respective Poincaré-duals, viewed as Thom forms of normal bundles of S and L respectively. Both S and L are copies of S^1, so have trivial normal bundle, and we can therefore write down global bundle coordinates, say $\eta_S = \rho(y)dy$ and $\eta_L = -\rho(x)dx$ for a suitable bump function ρ with total integral 1 (see [Bot82], Page 68). In particular, abusing notation, we write,

$$I(S,L) = \int_M \eta_S \wedge \eta_L = \int \rho(x)\rho(y)dxdy = +1.$$

where we can use Fubini to evaluate the final integral over the square, as expected.

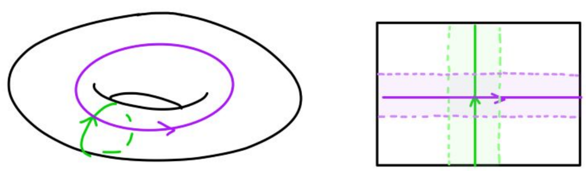


Figure 1.1.7: Loops on the Torus, and associated tubes.

At the moment, it might be temporarily unclear as to how the above result relates to sections of bundles. Let us make the connection now. Suppose $s : M \to E$ is a section of a oriented vector bundle $E \to M$ of rank k with a compact oriented base M. Suppose that s is transverse to the zero section, in which case it has a finite number of zeroes, each of which is non-degenerate (this means the vertical derivative (see Definition 1.1.6) of the section at the given zero is surjective). The zero section in E is a submanifold, and it has a Poincaré dual, whom we represent with a Thom form say Φ. Moreover, choose Φ with support contained in a small tube about the zero section, so that $s^*\Phi$ is supported in balls $\{B_1, \ldots, B_k\}$ about the finite number of zeroes. About each of these balls B_ℓ, take a local trivialisation $B_\ell \times \mathbb{R}^k$ of E over the ball, coordinates say (x,t). Then, on this trivialisation, Φ locally looks like $f(t)dt^1 \wedge \cdots \wedge dt^k$, where $\int_{\mathbb{R}^k} f(t)dt^1 \cdots dt^k = 1$. The section locally on B_ℓ looks like $x \to (x, t(x))$ for some mapping t. Then locally, due to the zeroes of our section being non-degenerate,

$$\int_{B_\ell} s^*(f(t)dt^1 \wedge \cdots \wedge dt^k) = \pm 1$$

according to whether or not the (vertical) derivative of s (i.e., differentiate the mapping $x \to t(x)$) preserves $(+1)$ or reverses (-1) orientation. In totality, the integral $\int_M s^*\Phi$ can be thought of as an oriented intersection number $I(M,s)$, identifying M with its image under the zero section.

The program, as it remains, is to make the discussion above slightly more precise.

We now define a cohomological invariant of a vector bundle, called the Euler class, and demonstrate that it is the primary obstruction to the existence of a nowhere vanishing section.

Intuitively, when our rank of our vector bundle lines up with the dimension of a manifold, we can pullback the Thom form and integrate it. This is pulling back the Poincaré-dual of the zero section and integrating it against the manifold, so as to 'detect' zeroes. Formally, let $E \to M$ be a vector bundle over M. View M as an embedded submanifold of E, via the zero section, identified with its image. Recall that there is an exact sequence of vector bundles over M,

$$0 \to TM \to TE|_M \to E \to 0,$$

where the rightmost map is the vertical derivative. In particular, the normal bundle of M in E is E itself. It follows that the Poincaré-dual of E is the Thom class of E.

Definition 1.1.5. *Suppose that $E \to M$ is an oriented vector bundle of rank k over a compact oriented manifold M. The Euler class of E, $e(E) \in H^k(M)$, is defined to be the pullback $s^*\Phi_E$ where $[\Phi_E] \in H^k_c(M)$ is the Thom class of E, and $s : M \to E$ is a smooth section.*

As smooth sections induce the same map in cohomology (because they are all are homotopic to the zero section), we have two immediate results.

Proposition 1.1.4. *With $E \to M$ as above, if there exists a nowhere vanishing section $s : M \to E$, then $e(E) = 0$.*

Proof. This proof follows ([Nic11],Theorem 4.4). Equip our bundle $E \to M$ with a metric. Let $\varepsilon > 0$, and denote by $D_\varepsilon(E)$ the set of all vectors in E of length less than ε; an open subset. As s is both nowhere vanishing and M is assumed compact, choose ε so that for each $x \in M$, we have $\|s(x)\| > \varepsilon$. Now, choose a representative $\Phi_\varepsilon \in \Omega^k_c(E)$ of the Thom class of E so that $\mathrm{supp}(\Phi_\varepsilon) \subset D_\varepsilon(E)$. Then $s^*\Phi_\varepsilon = 0$, for the support of Φ_ε is missed by s. As all sections induce the same map in cohomology, we deduce that $e(E) = 0$. □

Therefore the Euler class is an obstruction to the existence of a nowhere vanishing section.

Theorem 1.1.7. *Let $E \to M$ be an orientable vector bundle of rank k over a compact orientable n-manifold M. Let $s : M \to E$ be a section transverse to the zero section. Then we have,*

$$I(M, s) = \int_M e(E).$$

Proof. This is simply employing the definition of the Euler class. □

We have an alternative perspective in the language of intersection theory. The Euler class measures whether we can unlink the image of a given section and the zero section, and the net count is given by integration over M.

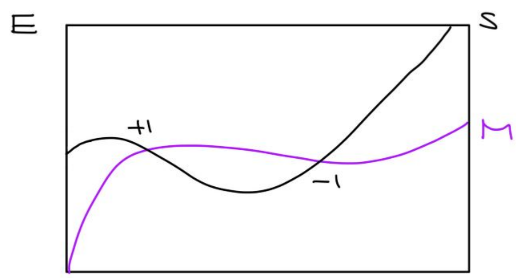


Figure 1.1.8: Manifolds M and $s(M)$ in E, with $e(E) = 0$ implying they can be seperated.

We now work towards dropping the transversality assumption on our section. The idea is to first obtain local information about a zero, where in the special case of a transverse intersection, we recover the oriented interesection number. If we use the local triviality of our bundle, and normalise our section, we can produce a mapping of spheres, for which we then recall the notion of degree. We shall need the following definition, which one can think of as differentiating a vector field on Euclidean space,

Definition 1.1.6. *Let $s : M \to E$ be a section of a vector bundle $\pi : E \to M$. Let $x \in M$ be a zero of s. The vertical derivative of s at x is a linear map $Ds_x : T_xM \to E_x$ constructed as follows. Choose a local trivialisation $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, where U is an open neighbourhood of x. There is an isomorphism of vector spaces $\Phi_x = pr_2 \circ \psi|_{\pi^{-1}(x)} : E_x \to \mathbb{R}^k$, and a map $s_U = pr_2 \circ \psi \circ s|_U : U \to \mathbb{R}^k$. Let $v \in T_xM$. Define,*

$$Ds_x(v) := \Phi_x^{-1}(ds_U)_x v \in E_x.$$

This linear map is independent of choices made.

For more details on the vertical derivative, we refer the reader to ([RS18],Page 201). We now define the local index about a zero. We let $k = \dim M$, and $s : M \to E$ be a section with an isolated zero at $x \in M$. Choose a coordinate disk D centered at $x \in M$, containing no other zeroes, by means of a chart. Let $S = \partial D$, oriented as such. Choose an auxillary metric on E. Choose D small enough so that $E|_S \cong S \times \mathbb{R}^k$. The restriction of s to our small sphere S and subsequent normalisation results in a compostion of sequence of mappings,

$$S \to S(E)|_S \cong S \times S^{k-1} \to S^{k-1}.$$

We define the local degree of s at x to be the degree of this composition, denoted $\mathrm{ind}_s(x) \in \mathbb{Z}$. It is independent of the choice of disk and Riemannian structure, for the degree is a homotopy invariant. We now have two results.

Proposition 1.1.5. *Let $E \to M$ be as above, with $k = \dim M$. Let $s : M \to E$ be a section transverse to the zero section, and let $x \in M$ be a zero of s. Then the local intersection number of s at x agrees with the local degree, that is, $\iota_x(s, M) = \mathrm{ind}_s(x) \in \mathbb{Z}$.*

Proof. Consider the vertical derivative of s at x, $Ds_x : T_xM \to E_x$. As s is transverse to M at x, we have, for $\iota : M \to E$ the zero section,

$$T_{0_x}s(M) \oplus T_{0_x}\iota(M) = T_{0_x}E \cong T_xM \oplus E_x.$$

In particular, the image of T_xM under ds_x is not in the kernel of vertical projection $T_{0_x}E \to E_x$. On the other hand, s is a section, so ds_x is an injection, and therefore the vertical derivative of s at x, Ds_x, is surjective. By dimensional reasoning, Ds_x is an isomorphism. Examining Definition 1.1.6, we see that the local degree of s at x is ± 1 (a local diffeomorphism), according to whether the vertical derivative preserves or reverses orientation. To say that $\iota_x(M, s) = +1$ (resp. -1) is to say that the orientation of $T_{0_x}s(M)$ agrees (resp. disagrees) with E_x, which is exactly that Ds_x preserves (resp. reverses) orientation. □

Before we give our ultimate result, we need some easy lemmata.

Lemma 1.1.8. *Let $F : W^{n+1} \to M^n$ be a smooth map from an oriented manifold W to a closed, oriented and connected manifold M. Let $X \subseteq W$ be a compact submanifold with boundary N, and suppose N is the disjoint union of submanifolds $N_1, \ldots, N_k$. Let $f_i := F|_{N_i}$. Then,*

$$\sum_{i=1}^{k} \deg(f_i) = 0.$$

Proof. This proof is drawn from ([Mad97],Page 102). Let $f = F|_N$, then $\deg(f) = \sum_{i=1}^{k} \deg(f_i)$. Let $\omega \in \Omega^n(M)$ be a closed bump n-form, i.e., we have $\int_M \omega = 1$. Then, by Stokes' Theorem,

$$\deg(f) = \int_N f^*\omega = \int_X dF^*(\omega) = \int_X F^*(d\omega) = 0.$$

□

We make use of the following local result, which should be understood as relating an index sum of a vector field (here as a section of the tangent bundle) to the degree of an associated Gauss map.

Lemma 1.1.9. *Let U be an open subset of $\mathbb{R}^n$. Let $F : U \to \mathbb{R}^n$ be a smooth function with isolated zeroes. At each zero z of F, the local degree $\mathrm{ind}_F(z)$ is defined. Let $R \subseteq U$ be a compact domain with smooth boundary, ∂R, and assume that F does not vanish on ∂R. Define $f : \partial R \to \mathbb{S}^{n-1}$ by $f(x) = F(x)/\|F(x)\|$. Then,*

$$\deg(f) = \sum_{z \in R, F(z)=0} \mathrm{ind}_F(z).$$

Proof. This proof is drawn from ([Mad97],Page 110). Write $p_1, \ldots, p_k$ for the zeroes of F in the interior of R. Choose a collection $D_j \subset \mathrm{int}(R)$ of pairwise disjoint closed disks each of which is centered at p_j. Define $f_j : \partial D_j \to S^{n-1}$ by $f_j(x) = F(x)/\|F(x)\|$. Define $X := R - \sqcup_j \mathrm{int}(D_j)$.

The boundary of X is the disjoint union of ∂R and ∂D_j for each j, where each ∂D_j has the opposite orientation when viewed as a boundary component of X. Thus, by the Lemma 1.1.8,

$$\deg(f) = \sum_{j=1}^{k} \deg(f_j)$$

and the claim follows. □

We now use a standard approximation trick to obtain our general result.

Theorem 1.1.10. *Let $\pi : E \to M$ be an oriented vector bundle of rank n over an oriented compact n-manifold M. Let $s : M \to E$ be a section with isolated zeroes $x_1, \ldots, x_q \in M$. Then,*

$$\int_M e(E) = \sum_{i=1}^{q} \mathrm{ind}_s(x_i).$$

Proof. (Sketch) The proof technique here is drawn from ([Ben19], Page 239-240, Theorem 14.4). For each zero x_j of s, choose a coordinate disk D_j about x_j, $S_j := \partial D_j$, so that the collection of such disks is pairwise disjoint, and say $\mathrm{ind}_s(x_j) = \deg(f_j : S_j \to S^{n-1})$ for a suitable mapping f_j. Let $\tilde{s} : M \to E$ be a section transverse to the zero section, and suitably close to s, so that the zeroes of $\tilde{s}$ are each distributed in the interior of the disks D_j. Fix a zero $x = x_j$ of s, with corresponding disk $D = D_j$. Write $z_1, \ldots, z_{r_j}$ for the corresponding zeroes of $\tilde{s}$ in $\mathrm{int}(D)$. By construction, the sections s and $\tilde{s}$ are homotopic along ∂D, so we may compute $\mathrm{ind}_s(x_j)$ with $\tilde{s}$ instead. However, by the Lemma above, we have that,

$$\mathrm{ind}_s(x_j) = \sum_i \mathrm{ind}_{\tilde{s}}(z_i).$$

where each z_i is a non-degenerate zero of $\tilde{s}$. In particular,

$$\int_M e(E) = I(M, \tilde{s}) = \sum_{z, \tilde{s}(z)=0} \mathrm{ind}_{\tilde{s}}(z) = \sum_{j=1}^{q} \mathrm{ind}_s(x_j).$$

□

We shall conclude this chapter with the Poincaré-Hopf Index Theorem.

1.2 Vector Fields on Manifolds

Let M be a closed oriented manifold. Let $v : M \to TM$ be a vector field on M with isolated zeroes $\{x_1, \ldots, x_q\}$. In light of previous work, we have shown that the Euler number of M is equal to the index sum of v about its zeroes, that is,

$$\int_M e(TM) = \sum_{i=1}^{q} \mathrm{ind}_v(x_i).$$

To prove the Poincaré-Hopf Index Theorem, it therefore remains to prove that for a compact, oriented manifold M, we have

$$\chi(M) = \int_M e(TM)$$

We take here, as a definition for the Euler characteristic $\chi(M)$,

$$\chi(M) = \sum_i (-1)^i \dim H^i_{dR}(M).$$

We follow the approach outlined in ([Bot82], Pages 126-129). In particular, we recognise the integral above as the self intersection number of the diagonal in $M \times M$. We assume the reader is familiar with the Künneth formula, as stated in ([Bot82], Page 47). Let M be a closed, oriented n-manifold. Let $\{\omega_i\}$ be a basis for $H^*(M)$, and $\{\tau_j\}$ the dual basis under Poincaré-duality, that is, we have,

$$\int_M \omega_i \wedge \tau_j = \delta_{ij}.$$

Let $\pi, p : M \times M \to M$ be projections onto the first and second factor respectively. The Künneth formula yields that,

$$H^*(M \times M) \cong \bigoplus_{q+k=n} H^q(M) \otimes H^k(M).$$

In particular, $\{\pi^*\omega_i \wedge p^*\tau_j\}$ is a basis for $H^*(M \times M)$. Write $\Delta = \{(x,x) \,|\, x \in M\}$ for the diagonal in $M \times M$, a closed oriented (via the diagonal map) submanifold. The Poincaré-dual η_Δ is of Δ in $M \times M$ is therefore defined, and can be written as a linear combination of our basis elements,

$$\eta_\Delta = \sum_{i,j} c_{ij} \pi^*\omega_i \wedge p^*\tau_j$$

for some coefficients c_{ij}. Using the definition of the Poincaré-dual, together with a pullback along the diagonal map $\iota : M \to M \times M$, we can determine our coefficients.

Lemma 1.2.1. *The Poincaré-dual η_Δ of the diagonal in $M \times M$ is given by*

$$\eta_\Delta = \sum_i (-1)^{\deg \omega_i} \pi^*\omega_i \wedge \rho^*\tau_i.$$

Proof. The trick is to consider the integral $\int_\Delta \pi^*\tau_k \wedge \rho^* w_\ell$, and its pullback via $\iota : M \to M \times M$, namely,

$$\int_\Delta \pi^*\tau_k \wedge \rho^* w_\ell = \int_M \iota^*\pi^*\tau_k \wedge \iota^* p^* w_\ell = \int_M \tau_k \wedge w_\ell = (-1)^{(\deg\, w_\ell)(\deg\, \tau_k)} \delta_{k\ell}.$$

By definition,

$$\begin{aligned}\int_\Delta \pi^*\tau_k \wedge p^* w_\ell &= \int_{M\times M} \pi^*\tau_k \wedge p^* w_\ell \wedge \eta_\Delta \\ &= \sum_{i,j} c_{ij} \int_{M\times M} \pi^*\tau_k \wedge p^* w_\ell \wedge pi^*\omega_i \wedge p^*\tau_j \\ &= \sum_{i,j} c_{ij}(-1)^{(\deg\,\tau_k + \deg\, w_\ell)\deg\, w_i} \int_{M\times M} \pi^*(w^i \wedge \tau_k) \wedge p^*(w_\ell \wedge \tau_j) \\ &= \sum_{i,j} c_{ij}(-1)^{(\deg\,\tau_k + \deg\, w_\ell)\deg\, w_i} \left(\int_M \omega_i \wedge \tau_k\right)\left(\int_M w_\ell \wedge \tau_j\right) \\ &= (-1)^{(\deg\,\tau_k + \deg\, w_\ell)\deg\, w_k} c_{kl}.\end{aligned}$$

We thus have, for $k = \ell$,

$$(-1)^{(\deg\,\tau_k + \deg\, w_\ell)\deg\, w_k} c_{kl} = (-1)^{(\deg\, w_\ell)(\deg\,\tau_k)} \delta_{k\ell}$$

and in particular, it follows, that

$$c_{kl} = \begin{cases} 0 & \text{if } k \neq \ell, \\ (-1)^{\deg \omega_k} & \text{if } k = \ell. \end{cases}$$

□

Lemma 1.2.2. *The normal bundle of the diagonal Δ in $M \times M$, denoted by N_Δ, is canonically orientation preservingly isomorphic to the tangent bundle $T\Delta$.*

Proof. We only provide a sketch. Consider the following short exact sequences,

$$\begin{tikzcd} 0 \arrow[r] & TM \arrow[r, "v\mapsto(v,v)"] \arrow[d, "\cong"] & TM \oplus TM \arrow[r, "(v_0,v_1)\mapsto\frac{v_1-v_0}{2}"] \arrow[d, "\cong"] & TM \arrow[r] & 0 \\ 0 \arrow[r] & T\Delta \arrow[r] & T(M\times M)|_\Delta \arrow[r] & N_\Delta \arrow[r] & 0. \end{tikzcd}$$

Via the diagonal map we identify the lower sequence over Δ with the upper one over M. It follows that $N_\Delta \cong TM \cong T\Delta$, and it can be checked that this isomorphism preserves orientation. For a detailed proof, we refer the reader to ([Nic11], Page 17, Lemma 4.11). □

As proved in the previous section, the Poincaré-dual of a submanifold is one and the same thing (or more precisely, can be represented by the same form) as the Thom class of its normal bundle, which is diffeomorphic to a tubular neighbourhood. Let $\Phi(N_\Delta)$ be a form representing the Thom class of N_Δ, and identify Δ with its image under the zero section of N_Δ. Then, omitting restriction maps, together with the fact that the Euler class of a bundle is the pullback of its Thom class by a section,

$$\int_\Delta \eta_\Delta = \int_\Delta \Phi(N_\Delta) = \int_\Delta e(N_\Delta) = \int_\Delta e(T_\Delta) = \int_M e(TM)$$

where we've parsed through orientation preserving diffeomorphisms in the final equalities. In the language of intersection theory, we have shown that the self-intersection number of the diagonal in $M \times M$ is the Euler number, that is,

$$I(\Delta, \Delta) = \int_{M\times M} \eta_\Delta \wedge \eta_\Delta = \int_\Delta \eta_\Delta = \int_M e(TM).$$

Finally, we have,

$$\begin{aligned} I(\Delta, \Delta) = \int_\Delta \eta_\Delta = \sum_i (-1)^{\deg w_i} \int_\Delta \pi^* w_i \wedge \rho^* \tau_i &= \sum_i (-1)^{\deg w_i} \int_M \iota^* \pi^* w_i \wedge \iota^* p^* \tau_i \\ &= \sum_i (-1)^{\deg w_i} \int_M w_i \wedge \tau_i \\ &= \sum_i (-1)^{\deg w_i} \\ &= \sum_q (-1)^q \dim H^q(M) = \chi(M). \end{aligned}$$

This realises the Euler characteristic as the self interesection of Δ in M, as was desired. We now present the Poincaré-Hopf Index Theorem.

Theorem 1.2.3 (Poincaré-Hopf)**.** *Let v be a vector field on a closed manifold M, with isolated zeroes $x_1, \dots, x_q \in M$. Then,*

$$\chi(M) = \sum_{i=1}^q \operatorname{ind}_v(x_i).$$

Proof. If M is orientable we are done, and if M is non-orientable, the claim follows by passing to the orientation double cover of M, a two sheeted smooth covering $\widetilde{M} \to M$, and using the fact that $\chi(\widetilde{M}) = 2\chi(M)$. □

As it stands, there are two main directions in which the Theorem above can be generalised. Namely, one can suppose that M is a compact manifold with non-empty boundary. If the vector field v points outward along ∂M, the result again holds true with an identical formula. Somewhat lesser known is that if all zeroes lie within the interior, but the vector field may have points on the boundary where it is not normal, one has the so called Morse-Index Formula. This formula has a new boundary contribution term. We shall not concern ourselves with these generalisations here, but we recommned the paper [Jub09] for a summary.

We shall now briefly consider some easy consequences of the Poincaré-Hopf Theorem, together with some famous applications.

Example 1.2.1. Let G be a compact Lie group. Then G admits a nowhere vanishing vector field given by left translation of a fixed vector in T_eG (in fact, TG is easily seen to be trivial). Therefore $\chi(G) = 0$. In particular, spheres of even dimensions are not Lie groups, for they have Euler characteristic 2. Let us furnish an example of a vector field on S^2 vanishing at the poles. As usual, view the 2-sphere S^2 as an embedded submanifold of $\mathbb{R}^3$, where the standard coordinates on $\mathbb{R}^3$ are given by (x, y, z). We define a vector field $v : S^2 \to TS^2$ by

$$v = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

This vector field has two isolated zeroes; the poles N, S, both of which are non-degenerate (i.e. the differential of v at each pole is non-singular). The index of v at each pole is given by the determinant of the Jacobian matrix of v,

$$\det\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = +1.$$

And indeed, $\chi(S^2) = 2 = 1 + 1 = \mathrm{ind}_v(N) + \mathrm{ind}_v(S)$.

We'll conclude this chapter with one or two additional results that we use later on, and are of independent interest.

Theorem 1.2.4. *Suppose that M is a smooth manifold with $\chi(M) = 0$. Then M admits a nowhere vanishing vector field. In particular, compact odd-dimensional smooth manifolds admit nowhere vanishing vector fields.*

Proof. We refer the reader to ([Hir76],Page 137). □

If one allows boundary, we can double our manifold, that is, glue two copies along the identity map of the boundary (and obtain smooth charts via collars), and 'push' the zeroes to the other copy of our manifold.

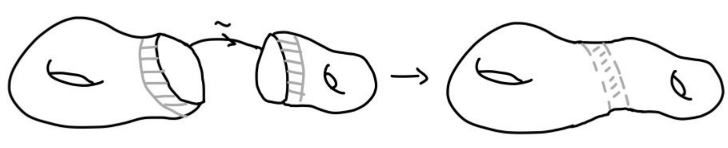


Figure 1.2.1: Doubling a 'handle' along its boundary.

Theorem 1.2.5. *Let M be a compact, connected manifold with boundary. Then M admits a nowhere vanishing vector field.*

Proof. We refer the reader to ([Hir76], Page 136). □

Let us make a technical remark that we shall make use of in Chapters 2 and 4. Namely, we, amongst other things, concern ourselves with the *radial* extension of a vector field over a disk in construction arguments.

Example 1.2.2. This example is drawn from ([Hir76],Page 155). Suppose that n is even. Consider $\mathbb{R}^n$ equipped with its standard Euclidean norm $\|\cdot\|$, and $S^{n-1} \subset \mathbb{R}^n$ the $(n-1)$-sphere. Let us consider a nowhere vanishing vector field defined on S^{n-1}, which we, after normalisation, view as a smooth mapping $f : S^{n-1} \to S^{n-1}$. Let $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. We may extend f radially to obtain a vector field F on D^n with an isolated zero at the origin. Precisely, define $F : D^n \to D^n$ by

$$F(x) = \begin{cases} \|x\| f\left(x/\|\|x\|\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

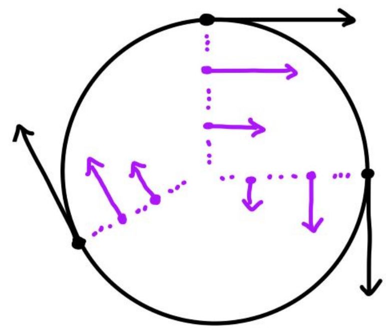


Figure 1.2.2: The radial extension of a vector field over a disk.

Suppose that f is an $O(n)$-equivariant mapping, i.e. we have an $O(n)$-invariant vector field. Let $g \in O(n)$. For $x \neq 0$, we have $F(g \cdot x) = \|g \cdot x\| f(g \cdot x/\|g \cdot x\|) = \|x\| f(g(x/\|x\|)) = g \cdot F(x)$, so that this extension procedures preserves invariance under an orthogonal action.

Chapter 2

Line Fields on Manifolds

A line field on a smooth manifold assigns to each point a tangent line, and these tangent lines vary smoothly with respect to the prescribed differentiable structure (see Figure 2.0.1). In the spirit of Chapter 1, one can ask whether the behaviour of a line field on a smooth manifold is captured by its topology. The answer is affirmative, and the purpose of this Chapter is to demonstrate this, namely, we recall a Poincaré-Hopf Theorem for line fields on a manifold, that is, given a line field defined on an open submanifold whose complement consists of a finite set of points, the Euler characteristic, a global invariant, can be captured by means of local behaviour of the line field near each singularity. For closed orientable surfaces, such a result can be traced back to Hopf, outlined in ([Hop83]). In particular, Hopf relates the Euler characteristic to an index sum of a line field, whose summands, the local indices, are now called Hopf Indices. These differ from the indices we use in this chapter, in the sense that they need not be integers, but lie within $\frac{1}{2}\mathbb{Z}$. The local indices whom are not integers convey additional information, namely that the line field is not locally generated by a vector field in a neighbourhood of the singularity. Such singularities are called *non-orientable*, and play an important role in this chapter. In 1955, Lawrence Markus published a paper [Mar55] in the Annals of Mathematics which contained a generalisation of Hopf's result for line fields to arbitrary dimensions. Unfortunately, this result was incorrect for surfaces, and odd dimensions, where counterexamples, and a new proof of the result, were given by Crowley and Grant in ([CG17],2017). In this Chapter, we present the generalisation to arbitrary compact manifolds, given by Crowley and Grant in ([CG17],2017). Finally, let us mention that there are various results in the literature regarding line fields, with varying degrees of generality amongst the definitions. We do not attempt here to give a full and complete overview, and refer the reader instead to page 2 of [CG17].

Let us define line fields in the precise sense.

Definition 2.0.1. *Let $\pi : E \to M$ be a vector bundle over a manifold M. A line field ξ on $E \to M$ is a smooth section $\xi : M \to P(E)$, where $P(E)$ denotes the fiberwise projectivisation of E. In particular, a line field ξ can be realised as a line subbundle of E. We say that the line field ξ has singularities $x_1, \dots, x_q \in M$ if the corresponding section is defined on the complement, that is, $\xi : M - \{x_1, \dots, x_q\} \to P(E)|_{M-\{x_1,\dots,x_q\}}$.*

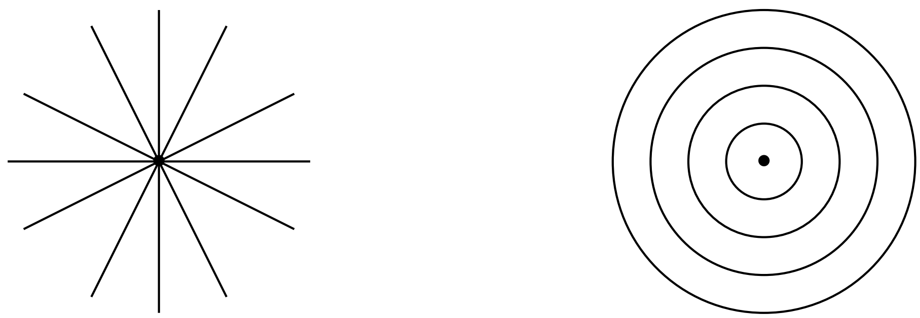


Figure 2.0.1: The integral curves of some line fields.

We shall momentarily specialise to $E = TM$, and return to the general case later.

Example 2.0.1. Let v be a vector field on M with a finite number of isolated zeroes, say $x_1, \ldots, x_q \in M$. By taking the span of each nonzero tangent vector, we obtain a line field ξ with singularities $x_1, \ldots, x_q$. By an abuse of notation, we write $\xi = \langle v \rangle$. To be more precise the line field ξ is obtained via post-composing v (when v is nonzero) with the quotient map $TM - \{0\} \to PTM$. In particular, note that nowhere vanishing vector fields generate globally defined line fields. We shall see that the converse is in fact true, that is, a globally defined line field yields a nowhere vanishing vector field.

The example above readily produces an uncountable number of examples, all of whom are generated by vector fields. In such cases, we can easily visually depict the line field by drawing integral curves without direction. However, not all line fields arise in this fashion. One key obstruction is non-orientability. To see this, consider a line field $\xi : M \to PTM$ on M. View ξ as a subbundle of TM; a line bundle over M. That ξ is generated by a vector field is equivalent to the existence of a nowhere vanishing section $v : M \to \xi$. It is an elementary fact that line bundles whom admit nowhere vanishing sections are trivial (see [Tu17], Page 235), and so in particular orientable. Note that line bundles admit *local* sections, and so a given line field (with no singularities!) is always locally generated by a vector field. We may summarise this observation by means of a cohomological obstruction.

Proposition 2.0.1. *A line field $\xi \to M$ is generated by a vector field if and only if the first Stiefel-Whitney class $w_1(\xi) \in H^1(M; \mathbb{Z}/2)$ (singular cohomology with $\mathbb{Z}/2$ coefficients) vanishes.*

Proof. The first Stiefel-Whitney class is zero if and only if the bundle is orientable, and a line bundle is orientable if and only if it is trivial. □

For background on Stiefel-Whitney classes, we refer the reader to [Mil74]. Let us now furnish examples of line fields whom are not globally generated by a vector field.

Example 2.0.2. Consider the Klein Bottle K, a closed non-orientable surface with $\chi(K) = 0$. There is thus a nowhere vanishing vector field v (see Theorem 1.2.4) and so a line field $\xi = \langle v \rangle$, which we view as a subbundle $\xi \subseteq TK$. We split $TK = \xi \oplus \nu$, and view ν as a line field on M.

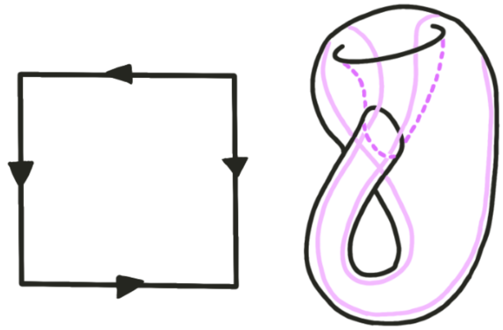


Figure 2.0.2: The Klein Bottle, viewed as a quotient.

If ν is generated by a vector field, TK decomposes into a direct sum of trivial line bundles, and in particular, K is orientable, and thus ν is an example of a line field not generated by a vector field.

Example 2.0.3. We exhibit a line field on a compact orientable 4-manifold which cannot be lifted to a vector field. This example is drawn from ([Gre81], Example 14). Let $M = S^1 \times \mathrm{SO}(3)$. Now M is parallelizable, so $TM \cong M \times \mathbb{R}^4$ and $PTM \cong M \times \mathbb{R}P^3$. Therefore, vector fields on M can be identified with maps $M \to \mathbb{R}^4$, and line fields identified with maps $M \to \mathbb{R}P^3$. We recall that $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{R}P^3$. Let $\varphi : \mathrm{SO}(3) \to \mathbb{R}P^3$ be a diffeomorphism. Define a line field ξ on M by $\xi(x, y) = \varphi(y)$ for $x \in S^1, y \in \mathbb{R}P^3$. There is an induced map $\xi_* : \pi_1(M) \to \pi_1(\mathbb{R}P^3)$. Let $\alpha \in \pi_1(S^1)$ and $\beta \in \pi_1(\mathrm{SO}(3))$ be generators, then

$$\xi_*(\alpha, \beta) = \varphi_*(\beta) \neq 0.$$

Suppose that $\xi = \langle v \rangle$. We may suppose that v is normalised, and therefore a mapping $M \to S^3$. Then $\xi = p \circ v$ where $p : S^3 \to \mathbb{R}P^3$ is the canonical double cover. In particular, $\xi_* = p_* \circ v_*$, but $v_*(\alpha, \beta) \in \pi_1(\mathbb{S}^3) = 0$, so $\xi_*(\alpha, \beta) = 0$, which is a contradiction. Therefore ξ is a not globally generated by a vector field.

Example 2.0.4. Consider the Möbius band as a bundle over the circle (see Figure 2.0.3). Take the subbundle tangent to the fibers, it is a line field, whom is not globally generated by a vector field. (For details, we refer the reader to [Kos93], Page 76).

Let us now introduce a construction fundamental to this chapter. Let $\xi : M \to PTM$ be a line field on a manifold M. Choose a Riemannian metric on M. View ξ as a subbundle of the tangent bundle, and consider the associated sphere bundle $S\xi \to M$, a two sheeted covering of M (an S^0-bundle). An element of $S\xi$ may be viewed (by an abuse of notation) as a pair (x, v) where $x \in M$ and $v \in \xi(x)$ has unit length, with respect to the metric. We define a vector field v_ξ on $S\xi$ by $v_\xi(x, w) = w \in T_{(x,w)}(S\xi) \cong T_x M$, where we have used the fact that the fiber is 0-dimensional. Utilisation of this associated vector field and covering forms the heart of this chapter, and shall be demonstrated in what is to come.

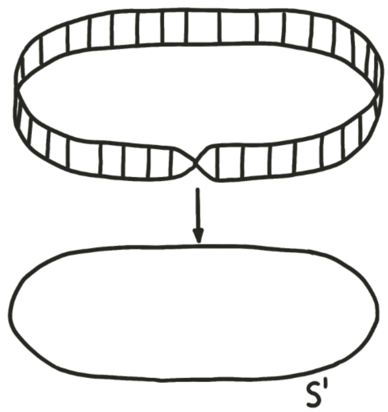


Figure 2.0.3: The Möbius band viewed as a bundle over the circle.

Returning to matters at hand, in Example 2.0.1, we claimed that the existence of a line field implies that of a nowhere vanishing vector field. Let us prove this in generality (the proof given is a trivial adaptation of the argument given in ([CG17], Remark 2.4) and for the benefit of the reader, we adapt similar notation).

Proposition 2.0.2. *Let $\pi : E \to M$ be a rank n vector bundle over a closed manifold M of dimension n. The bundle $\pi : E \to M$ admits a line field if, and only if, it admits a nowhere vanishing section.*

Proof. We only need prove the forward direction. Let $\xi : M \to P(E)$ be a smooth section, and view ξ as a subbundle of E. We may split $E \cong \xi \oplus F$ for some bundle F over M of rank $n-1$ (say, by means of an auxillary metric). Let $\nu : M \to \xi$ be transverse to the zero section. As M is compact, we conclude that the zeroes of ν form a finite set, say $\{x_1, \ldots, x_q\} \subseteq M$. Consider the zero section $M \to F$, by means of suitable bump functions, we may manipulate the zero section to construct a section $t : M \to F$ such that in a neighbourhood of each zero x_j of ν, the section t is non-zero. Define $s : M \to E$ by $s(x) = (\nu(x), t(x)) \in \xi \oplus F \cong E$, and one obtains the desired nowhere vanishing section □

In particular, if $E \to M$ is an oriented rank n vector bundle over a compact oriented manifold M, which admits a line field, the Euler number of the bundle $\pi : E \to M$ vanishes.

Corollary 2.0.1. *A closed manifold admits a line field if and only if it admits a nowhere vanishing vector field.*

Proof. This follows from Proposition 2.0.2. □

Note also, that by Theorem 1.2.5, if M is compact and has non-trivial boundary, it admits a nowhere vanishing vector field, and thus a line field. We shall now introduce singularities into the mix. Suppose that a line field ξ on M has a singularity at $x \in M$. By means of a coordinate chart, choose a disk D centered at x, and denote its boundary by S, oriented as such. Formulate the associated two sheeted covering ρ of $M - \{x\}$ induced by ξ, as above. Restrict ρ to S, $\rho|_S : \rho^{-1}(S) \to S$. This is a two sheeted covering of an $(n-1)$-sphere.

Definition 2.0.2. *Carry through notation from above. If $\rho|_S$ is trivial, we say that x is orientable, otherwise we say x is non-orientable.*

If $n \geq 3$, then S^{n-1} is simply connected, and so any double cover is necessarily trivial. In particular, for $n \geq 3$ all singularities are orientable. For $n = 2$, we are concerned with double covers of the circle, of which there are (up to identification) only two. Indeed, the connected double covers of the circle are in correspondence with homomorphisms of the form $\mathbb{Z} \cong \pi_1(S^1) \to \mathrm{Sym}(2) \cong \mathbb{Z}_2$ (see [Hat02], Page 70). In particular, if $\rho^{-1}(S)$ is connected, the covering is necessarily of the form $S^1 \to S^1$ given by $z \mapsto z^2$, where we've identified the circle as a subset of $\mathbb{C}$. If $\rho^{-1}(S)$ is disconnected, it is simply the trivial double cover. Distinguishing between non-orientable and orientable singularities in the case of surfaces is critical, and in doing so Crowley and Grant were able to give a unified proof of a Poincaré-Hopf Theorem for line fields. We shall now introduce the notion of projective index. The tool which we use to define it is of course the Brouwer degree. The alert reader may foresee a technicality; even dimensional real projective spaces are not orientable, and so if M has odd dimension, we resort to working modulo 2. Let ξ, x and S be as above. Suppose (by shrinking if necessary), that $PTM|_S$ is trivial, and $\Psi : PTM|_S \to S \times \mathbb{R}P^{n-1}$ is a local trivialisation. Consider the following composition $f : S \to \mathbb{R}P^{n-1}$ defined by,

$$f : S \xrightarrow{\xi_S} PTM|_S \xrightarrow{\Psi} S \times \mathbb{R}P^{n-1} \xrightarrow{\pi_2} \mathbb{R}P^{n-1}.$$

Definition 2.0.3. *With the data above, we define the projective index of ξ at x, denoted $\mathrm{p\,ind}_\xi(x)$, to be*

$$\mathrm{p\,ind}_\xi(x) = \begin{cases} \deg(f) \in \mathbb{Z}, & \text{if } n \text{ is even} \\ \deg_2(f) \in \mathbb{Z}, & \text{if } n \text{ is odd.} \end{cases}$$

Remark 2.0.1 (Hopf Indices)**.** We follow [BSS16]. Let (M, g) be a closed oriented Riemannian 2-manifold. Let ξ be a line field on M with singularity at $x \in M$. Let us recall how Hopf defined the index of a line field ξ about a singularity $x \in M$. Let U be a simply connected open coordinate neighbourhood of x, and Z a nowhere vanishing vector field on U. Let $C : [0,1] \to U$ be a simple closed curve which encircles x counterclockwise. We may define a map $F : [0,1] \to TM$ so that $\xi(C(t))$ is the span of $F(t)$ for each $t \in [0,1]$. Let $\angle[Z, F]_{C(t)}$ be the angle between $Z(C(t))$ and $F(t)$ with respect to the metric g. Let $\delta_C \angle[Z, F]$ be the total change (with sign) of this angle on $[0,1]$ by traversing C once counterclockwise. Define $j \in \mathbb{R}$ by

$$2\pi j = \delta_C \angle[Z, F].$$

It is clear that $j \in \frac{1}{2}\mathbb{Z}$. We call j the Hopf index of ξ at $x \in M$, denoted $j = \mathrm{h\,ind}_\xi(x)$. It is shown in ([Hop83], Chapter 3, Theorems 1.3,1.4) that this half integer is independent of C, the vector field Z, and the metric g. It can be shown that the projective index defined above, is exactly twice the Hopf index (see, [CG17], Remark 3.6). We refer the reader to Figure 2.0.4 for examples, both of which shall reappear later in Examples 2.0.9, 2.0.10. Let us conclude this diversion by stating Hopf's Theorem for line fields/elements.

(a) Hopf Index $+1/2$. (b) Hopf Index $+1$.

Figure 2.0.4: Two line fields, with singularities at the origin. The left hand side is a non-orientable singularity.

Theorem 2.0.1 (Hopf, [Hop83], pg 113, Theorem 2.2)**.** *For a line field ξ on a closed orientable surface M of genus g, with a finite number of singularities $x_1, \ldots, x_q$ in M, one has*

$$\sum_{k=1}^{q} \operatorname{h\,ind}_\xi(x_k) = \chi(M) = 2 - 2g.$$

Retuning to the projective index, let us consider a special case.

Example 2.0.5. Let M be a closed oriented smooth manifold of even dimension $n \geq 2$. Let v be a vector field on M with a isolated zeros $x_1, \ldots, x_k$. The vector field v defines a line field ξ on the open submanifold $M - \{x_1, \ldots, x_k\}$ by $\xi = \langle v \rangle$. Let $x \in \{x_1, \ldots, x_k\}$. Choose, by the means of a chart, a small coordinate disk D about x which contains no other zeros of v. Write $\partial D = S \cong S^{n-1}$. Restricting v to S and normalising yields a section $v : S \to STM|_S$. The restriction of ξ to S yields $\xi : S \to PTM|_S$. We may choose D sufficiently small so as to have the trivialisations $\Phi : STM|_S \xrightarrow{\sim} S \times S^{n-1}$ and $\Psi : PTM|_S \xrightarrow{\sim} S \times \mathbb{R}P^{n-1}$. There is a canonical map $\zeta : STM|_S \to PTM|_S$ given by taking the disjoint union of the standard 2-sheeted covering on map on each fibre. We now consider the following diagram,

$$\begin{tikzcd}
S \arrow[r, "v"] \arrow[dr, "\xi"'] & STM|_S \arrow[r, "\Phi"] \arrow[d, "\zeta"] & S \times S^{n-1} \arrow[r, "\pi_2"] & S^{n-1} \arrow[d, "p"] \\
& PTM|_S \arrow[r, "\Psi"'] & S \times \mathbb{R}P^{n-1} \arrow[r, "\pi_2"'] & \mathbb{R}P^{n-1}
\end{tikzcd}$$

where π_2 is projection onto the second factor and $p : S^{n-1} \to \mathbb{R}P^{n-1}$ is the standard 2-sheeted covering map. The degree of p is $+2$ or -2 according to whether p is orientation preserving or not; we orient $\mathbb{R}P^{n-1}$ (i.e. choose fundamental class) so that the map has degree $+2$. Let $f = \pi_2 \circ \Phi \circ v|_S$ and $g = \pi_2 \circ \Psi \circ \xi|_S$. We note that $\xi = \langle v \rangle$ on S is precisely $g = p \circ f$. The degree is multiplicative under composition, and so,

$$\operatorname{p\,ind}_\xi(x) = \deg(g) = \deg(p \circ f) = 2\deg(f) = 2\operatorname{ind}_v(x).$$

In particular, by the Poincaré-Hopf Index theorem, $2\chi(M) = 2\sum_{i=1}^{k} \operatorname{ind}_v(x_i) = \sum_{i=1}^{k} \operatorname{p\,ind}_\xi(x_i)$.

Remark 2.0.2. This calculation demonstrates another interesting property in odd dimensions. Given a vector field v on M, certainly $-v$ generates the same line field. For the above to hold, we'd need to have, at an isolated zero x of v, $\mathrm{ind}_v(x) = \mathrm{ind}_{-v}(x) = (-1)^{\dim M}\mathrm{ind}_v(x)$.

This example demonstrates that given a line field on an even dimensional manifold with a singularity, if it is generated by a vector field about the singularity, then the projective index is a multiple of 2. The contrapositive is useful. If the projective index is odd, then the line field is not locally generated by a vector field near the singularity.

Remark 2.0.3. Whilst we have specialised the example and definition above to $TM \to M$, precisely the same definition of projective index works for line fields on a vector bundle $E \to M$. The example given above also follows through, and for the final conclusion, one uses a core result from Chapter 1, namely Theorem 1.1.10.

Before we state the main result of Crowley and Grant's paper, we are going to introduce a fundamental technical tool regarding indices, namely that of the normal index. The normal index is used to compare the indices of a certain vector field with the projective indices of our line field, and ultimately leads to the proof of the Poincaré-Hopf Theorem for line fields given by Crowley and Grant. In order to motivate the definition, we will first retrieve the definition of the usual index via means of an oriented intersection number. Suppose $v : M \to TM$ is a vector field on M, with an isolated zero at $x \in M$. In the usual way, let S be a sphere centered at x, above which $STM|_S$ is trivial. Let $\Phi : STM|_S \to S \times S^{n-1}$ be a local trivialisation (an orientation preserving diffeomorphism, for we orient bundles with the local product orientation). Define $f : S \overset{v|_S}{\to} STM|_S \overset{\Phi}{\to} S \times S^{n-1} \overset{\pi_2}{\to} S^{n-1}$, where by an abuse of notation, $v|_S$ denotes the restriction of v to S, together with normalisation. By definition, $\mathrm{ind}_v(x) = \deg(f)$. Fix $a \in S^{n-1}$. Define $\sigma : S \to STM|_S$ by $\sigma(z) = \Phi^{-1}(z, a)$. Note $(\Phi \circ \sigma)(S) = S \times \{a\}$. For generic choice of $a \in S^{n-1}$ the embeddings σ and v intersect transversely i.e., transversality is a generic property. In such a case, we may consider the oriented intersection number $\sigma(S) \pitchfork v(S) \in \mathbb{Z}$. Let us now establish the desired result.

Proposition 2.0.3. *We have* $\mathrm{ind}_v(x) = \sigma(S) \pitchfork v(S)$.

Proof. As Φ is an orientation preserving diffeomorphism, it suffices to compute our prescribed intersection number in $S \times S^{n-1}$. We are thus concerned with $(S \times \{a\}) \pitchfork (\Phi \circ v)(S)$. Recall, for generic $a \in S^{n-1}$, we have, $\deg(f) = \sum_{q \in f^{-1}(a)} \mathrm{sign}(df_q)$. It is clear there is a bijective correspondence between the points $f^{-1}(a)$ and $(S \times \{a\}) \cap (\Phi \circ v)(S)$, for we may consult the following diagram,

$$\begin{tikzcd}
S \arrow[r, "v|_S"] & STM|_S \arrow[r, "\Phi"] \arrow[drr, "\pi"] & S \times S^{n-1} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & S^{n-1} \\
& S \arrow[u, "\sigma"] \arrow[rr, "\mathrm{id}"'] & & S
\end{tikzcd}$$

Given $q \in f^{-1}(a)$, consider the corresponding point $\Phi(v(q))$. To say that the local intersection number at $\Phi(v(q))$ is $+1$ is to say that the orientation of $v(S)$ (induced from v, which is an immersion, being a section) is the same as that of the fiber, which means exactly that $\text{sign}(df_q) = +1$. Similarly, a local intersection number of -1 at $\Phi(v(q))$ means exactly that the orientation of $v(S)$ is opposite to that of the fiber, and so $\text{sign}(df_q) = -1$. The result follows. $\square$

Carrying through notation as above, our auxillary metric determines an outward unit normal to the sphere S, which can be viewed as an embedding $\eta : S \to STM|_S$. We now define the normal index.

Definition 2.0.4. *The normal index of v at x, denoted $\text{ind}_v^{\perp}(x)$, is by definition the oriented intersection number,*

$$\text{ind}_v^{\perp}(x) = \eta(S) \pitchfork v(S) \in \mathbb{Z}.$$

Intuitively, this counts the number of times our vector field points outwards along the sphere S, with signs.

Example 2.0.6. Figure 2.0.5 describes a vector field (on a surface) along a circle centered at a zero whom nets a normal index of $+1$. Notice that the vector field wraps around the circle twice. Therefore,

$$\text{ind}_v^{\perp}(x) = 1 = 2 - 1 = \text{ind}_v(x) - 1.$$

This phenomena is in fact general, as the incoming results will state.

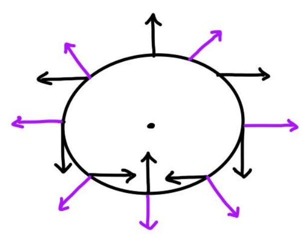


Figure 2.0.5: Here purple denotes the outward normal, and black the vector field.

Establishing a relationship between $\text{ind}_v(x)$ and $\text{ind}_v^{\perp}(x)$ is a natural first order of business, and is done so in [CG17] by comparing intersection numbers in the product $S \times S^{n-1}$, using homological techniques, similar to those outlined in the first chapter, but parsed through into the language of singular (co)homology. We have,

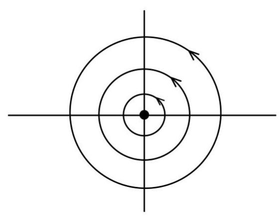
Lemma 2.0.2. *With the data above, one has, for $n \geq 2$,*

$$\text{ind}_v^{\perp}(x) = \text{ind}_v(x) + (-1)^{n-1}.$$

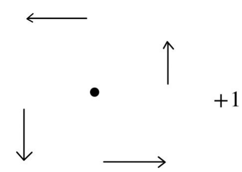
Proof. We refer the reader to Lemma 3.4 in [CG17]. $\square$

Example 2.0.7. Here we extend Example 1.2.1 from Chapter 1. Consider the 2-sphere S^2 as an embedded submanifold of $\mathbb{R}^3$, where the standard coordinates on $\mathbb{R}^3$ are given by (x, y, z). We recall that we defined a vector field $v : S^2 \to TS^2$ by $v = -y\partial_x + x\partial_y$ (where we use the standard contraction of partial operators). This vector field has non-degenerate zeroes at the poles. The index of v at each pole is given by the determinant of the Jacobian matrix of v,

$$\det \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = +1.$$



(a) Local behaviour about a pole.



(b) The Jacobian is rotation by $\pi/2$.

In particular, the normal index of v at either pole is zero (as intuitively expected, considering the Gauss normal to S^2 along a circle about a pole).

Example 2.0.8. Let $n \geq 3$ be an odd integer. Define a vector field $v : \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$ by $v(x) = Ax$, where A is an $n \times n$ matrix whom preserves the standard orietation on $\mathbb{R}^n$. Then v has an isolated zero at the origin, with $\operatorname{ind}_v(0) = \operatorname{sgn}\det(A) = +1$. Thus the normal index of v at the origin is $\operatorname{ind}_v^{\perp}(0) = 2$.

We now model our definition of the projective normal index. Let ξ be a line field with a singularity at $x \in M$. By means of a chart, choose a sphere S centered at x, and let $\eta : S \to PTM|_S$ be the projectivised outward unit normal to S, i.e., a *normal line* to S.

Definition 2.0.5. *We define the projective normal index of ξ at x, denoted $\operatorname{p\,ind}_\xi^{\perp}(x)$, to be the oriented intersection number, $\operatorname{p\,ind}_\xi^{\perp}(x) = \eta(S) \pitchfork \xi(S)$ in even dimensions, and the mod 2 intersection number $\eta(S) \pitchfork_2 \xi(S)$ in odd dimensions.*

Consider the example given in Figure 2.0.5, but with lines, rather than vectors. We see that the bottom vector whom pointed inwards now counts, and one nets a $+2$ normal projective index! In a similar way to Lemma 2.0.2, one obtains the following relationship between the projective indices.

Lemma 2.0.3. *For even n, $\operatorname{p\,ind}_\xi^{\perp}(x) = \operatorname{p\,ind}_\xi(x) - 2$. For $n \geq 3$ odd, we have $\operatorname{p\,ind}_\xi(x) = \operatorname{p\,ind}_\xi^{\perp}(x) = 0 \in \mathbb{Z}/2$.*

Proof. This is the content of Lemmata 3.8 and 3.9 in [CG17]. $\square$

In particular, one has the following immediate consequence of the previous lemmata.

Proposition 2.0.4. *Let M be a compact manifold of even dimension, and let v be a vector field on M with isolated zeroes $x_1, \ldots, x_k$. Let $\xi = \langle v \rangle$ on $M - \{x_1, \ldots, x_k\}$. Then, as for regular indices in Example 2.0.5, we have the following relationship between the normal indices,*

$$\operatorname{p\,ind}_\xi^{\perp}(x) = 2\operatorname{ind}_v^{\perp}(x).$$

We shall now state Crowley and Grants Poincaré-Hopf Theorem for line fields on a compact manifolds, and then provide the underlying strategy.

Theorem 2.0.4. *Let M be a compact manifold of dimension $n \geq 2$, and let ξ be a line field on M with finitely many singularities $x_1, \ldots, x_q$. If $\partial M \neq \varnothing$, we assume additionally that the singularities lie in the interior of M, and that the line field is normal to ∂M. Then,*

$$2\chi(M) = \sum_{i=1}^{q} \mathrm{p}\,\mathrm{ind}_{\xi}(x_i)$$

where the equality is interpreted as congruence modulo 2 if n is odd.

Remark 2.0.4. Notice that in even dimensions, this has the following consequence. A line field on a closed manifold cannot have only a single singularity, that of which is non-orientable.

Rather than prove this result, we shall provide an overview of the techniques used. The statement is first proved for a closed manifold M, for the general case follows by doubling M along its boundary to obtain a closed manifold, and applying the case that of which has been established. Thus let us assume that M is closed. If n is odd, it is simply the content of Lemma 2.0.3, and so we are concerned only with even dimensions at least 2. Here is a quick overview. First, excise the interiors of disks centered at each singularity, this yields a manifold with boundary on which the line field is globally defined. We pass to the associated cover, and consider the associated vector field. By gluing in disks along the boundary components of this associated covering, we obtain a closed manifold. We extend our vector field over these glued in disks to obtain a vector field with isolated zeroes. We now apply the classical Poincaré-Hopf Theorem together with the Riemann-Hurwitz formula and several established lemmata to obtain the result. We'll now proceed through the details a bit more carefully.

Construction 2.0.1 (Excise and Radially Extend)**.** For each singularity x_i of ξ, choose a coordinate disk D_i centered at x_i. Organise our disks so that the family $\{D_i\}_{i=1}^{q}$ is pairwise disjoint. Define,

$$N := M - \bigsqcup_{i=1}^{q} \mathrm{int}(D_i).$$

Then N is a compact manifold with boundary, and the restriction $\xi|_N$ yields a globally defined line field on N, and thus (by means of a metric) an associated double cover $p : \widetilde{N} \to N$, together with a vector field $v_{\xi|_N}$ on $\tilde{N}$. The restriction of this double cover to a boundary component on N yields a two sheeted covering of an $(n-1)$-sphere, which is trivial if and only if the corresponding singularity is orientable. We have $\partial\widetilde{N} = p^{-1}(\partial N)$. By gluing in disks along the boundary components of $\widetilde{N}$, we obtain a closed manifold $\widetilde{M}$, together with a map $\pi : \widetilde{M} \to M$ which extends p, and is a branched two sheeted covering, with branch points corresponding to the non-orientable singularities. Fix $i \in \{1\ \ldots, q\}$, then $\pi^{-1}(x_i)$ consists of either a single point (if x_i is non-orientable), or two points (if x_i is orientable). We now radially extend the vector field $v_{\xi|_N}$ over the glued in disks to obtain a vector field v on $\widetilde{M}$ with a finite number of isolated zeroes. See Example 1.2.2 for radial extension over a disk. In particular, if k denotes the number of non-orientable singularities, then the vector field v has $2q - k$ zeroes, corresponding to the number of disks glued along the boundary components of $\widetilde{N}$. In ([CG17]), Crowley and Grant establish the following.

Lemma 2.0.5. *For n even, with the data of ξ and v as above, we have*

$$\mathrm{p\,ind}_\xi^\perp(x) = \sum_{y \in \pi^{-1}(x)} \mathrm{ind}_v^\perp(y).$$

Proof. See Lemma 4.1 in [CG17]. We notify the reader that the proof uses a so called 'push-pull' formula. For details on this, we refer the reader to here. $\square$

Equipped with this result, and the Riemann Hurwitz formula, which yields $\chi(\widetilde{M}) = 2\chi(M) - k$, we use the Poincaré-Hopf Theorem together with several established lemmata to move as follows.

$$\begin{aligned}
2\chi(M) &= k + \chi(\widetilde{M}) \\
&= k + \sum_{i=1}^{q} \sum_{y \in \pi^{-1}(x_i)} \mathrm{ind}_v(y) \\
&= k + \sum_{i=1}^{q} \sum_{y \in \pi^{-1}(x_i)} (\mathrm{ind}_v^\perp(y) + 1) \\
&= k + (2q - k) + \sum_{i=1}^{q} \sum_{y \in \pi^{-1}(x_i)} \mathrm{ind}_v^\perp(y) \\
&= 2q + \sum_{i=1}^{q} \mathrm{p\,ind}_\xi^\perp(x_i) \\
&= 2q + \sum_{i=1}^{q} (\mathrm{p\,ind}_\xi - 2) \\
&= \sum_{i=1}^{q} \mathrm{p\,ind}_\xi(x_i)
\end{aligned}$$

which yields the desired result

We shall now furnish examples.

Example 2.0.9 (A line field on the 2-sphere)**.** Consider the vector field defined on S^2 as in Example 2.0.7. By taking the span of each non-zero tangent vector, we obtain a line field ξ on $S^2 - \{N, S\}$ where N and S denote the north and south poles respectively. About each pole, the line field looks as in 2.0.4, with projective index +2 (so Hopf index +1). In particular,

$$2\chi(S^2) = 4 = 2 + 2 = \mathrm{p\,ind}_\xi(N) + \mathrm{p\,ind}_\xi(S).$$

It is also interesting to compare the normal indices in the case where v has generated ξ, which is the content of Proposition 2.0.4.

In order to construct non-trivial examples, where one can still easily compute, we use the following construction.

Construction 2.0.2 (Proto-line fields)**.** Here we recall brielfy a construction given in [BSS16]. Let (M, g) be a closed oriented Riemannian 2-manifold. Let X and Y be vector fields on M, with zero sets z_X and z_Y respectively. We define a line field $\xi_{X,Y}$ on $M \backslash (z_X \cup z_Y)$ by assigning to each point $p \in M \backslash (z_X \cup z_Y)$ the line bisecting the pair $(X(p), Y(p))$ in T_pM, relative to the metric g. The line field $\xi_{X,Y}$ is called a *proto-line field*. For such a line field, we have the following helpful criterion for computing its Hopf Index at an isolated singularity $x \in z_X \cup z_Y$,

$$\mathrm{h\,ind}_{\xi_{X,Y}}(x) = \frac{1}{2}\left(\mathrm{ind}_X(x) + \mathrm{ind}_Y(x)\right).$$

For this, we refer the reader to ([BSS16],Proposition 11).

Example 2.0.10 (Baseball Line Field)**.** We construct a line field on the 2-sphere with four singularities, each of projective index 1. It is called a baseball line field, for the stitching on baseball is similar to behaviour of the line field. This is drawn from ([CG17],Example 2.9). First, we establish a proto-line field on the plane, that is, $\mathbb{R}^2$ with its standard metric. Consider the vector fields X and Y on the plane given by, at each point $(x, y) \in \mathbb{R}^2$,

$$X(x,y) = \begin{pmatrix} x+y \\ y-x \end{pmatrix}, \quad Y(x,y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let $\xi_{X,Y}$ be the induced proto-line field. Then $\xi_{X,Y}$ has a single singularity at the origin, with projective index $+1$ (this is easily computed with the above formulae). For more details, we refer the reader to the so called 'Lemon' proto-line field ([BSS16], Example 3). Visually, this is the line field of Hopf index 1/2 from Figure 2.0.4. We use this example to construct a line field on the 2-disk, parallel to the boundary, with two singularities, each of projective index $+1$. This line field intuitively looks like,

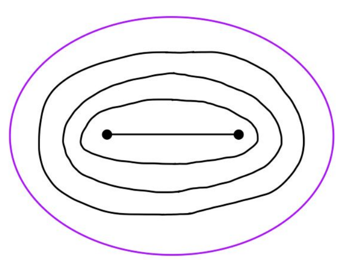


Figure 2.0.7: Here we've sketched the integral curves of the line field on the disk, and used purple to denote its boundary.

By gluing together two copies of this disk along their common boundary, we obtain a line field on the 2-sphere with four non-orientable singularities, each of projective index $+1$. And indeed, $2\chi(S^2) = 4$.

Example 2.0.11. This example is taken from [CG17]. Let n be even at least 2. Consider a line field on $D^n \subset \mathbb{R}^n$ given by rays emanating from the origin. If $n = 2$, this looks visually as follows,

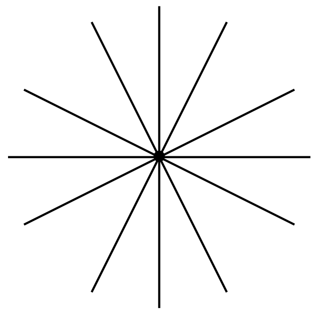


Figure 2.0.8: Emanating Rays.

By identifying antipodal points on the boundary of D^n, this descends to a line field on $\mathbb{R}P^n$ with a single orientable singularity of projective index $+2$. As n is even, we have $2\chi(\mathbb{R}P^n) = 2$, and the Theorem is confirmed. We remark that this example is of interesting consideration if one tries to refine the degeneracy of Theorem 2.0.4 in odd dimensions, for if n is odd, we have $\chi(\mathbb{R}P^n) = 0$, and one has a line field, again with a single singularity, and constructing an alternative index definition which retains such a high level of compatibility seems unlikely (as remarked in [CG17],Remark 4.2).

Chapter 3

Orbifold Theory

Orbifolds are spaces that are locally modelled on finite quotients of Euclidean space. They were first formally introduced by Satake in the 50's, under the title of V-manifolds (see [Sat56]), and futher developed upon by Thurston in the 70's. A modern perspective is to view an orbifold as a special kind of differentiable stack, which one can view as a Morita equivalence class of certain Lie groupoids, where a choice of Lie groupoid is equivalent to choosing an atlas on the underlying topological space, the 'coarse' quotient. In this Chapter, we first recall some basic definitions and properties regarding orbifolds, together with Satake's Poincaré-Hopf and Gauss Bonnet Theorem (see [SAT57]). The second half is devoted to the perspective of orbifolds as groupoids, concluding with a discussion of intersection theoretic results in the category of orbifolds.

3.1 Basic Theory

In this section, we follow the overall structure of [ALR07] closely, but our notation and technical definitions are drawn from [KL14]. We shall now define orbifolds.

Definition 3.1.1. *Let X be a paracompact Hausdorff topological space. Fix $n \in \mathbb{N}$.*

1. *An orbifold chart of dimension n for an open subset $U \subseteq X$ consists of the following data; an open non-empty connected subset $\widehat{U} \subseteq \mathbb{R}^n$, a finite group G of smooth automorphisms of $\widehat{U}$, a G-invariant, onto map $\widehat{U} \to U$ which induces a homeomorphism from $\widehat{U}/G$ onto $U \subseteq X$. We shall formally frame this data as a tuple $(\widehat{U}, G)$, leaving U to be clear from notational convention. We call the pair $(\widehat{U}, G)$ a local model (or chart) on X, which uniformises the open set U. We summarise with a triangle.*

$$\begin{tikzcd} \widehat{U} \arrow[r] \arrow[d] & U \\ \widehat{U}/G \arrow[ur, "\cong"'] & \end{tikzcd}$$

2. *By an embedding $\lambda : (\widehat{U}, G) \hookrightarrow (\widehat{V}, H)$ between two local models X we mean a smooth*

embedding $\lambda : \widehat{U} \to \widehat{V}$ for which,

$$\begin{tikzcd} \widehat{U} \arrow[r] \arrow[dr] & \widehat{V} \arrow[d] \\ & X \end{tikzcd}$$

is commutative.

3. *An n-dimensional orbifold atlas on X consists of a collection of n-dimensional local models $\mathcal{U} = \{(\widehat{U}, G)\}$ which cover X and are locally compatible in the following sense: for any two local models $(\widehat{U}, G)$, $(\widehat{V}, H)$, and a point $x \in U \cap V$, there exists an open neighbourhood $W \subseteq U \cap V$ of x and a local model $(\widehat{W}, K)$ for W such that we have two embeddings, $\lambda_1 : (\widehat{W}, K) \to (\widehat{U}, G)$ and $\lambda_2 : (\widehat{W}, K) \to (\widehat{V}, H)$.*

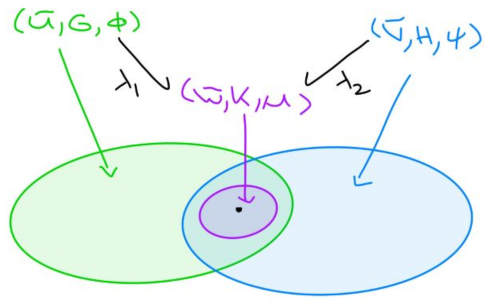


Figure 3.1.1: Local compatibility of charts.

4. *We say that an atlas $\mathcal{U}$ refines another atlas $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart of $\mathcal{V}$. We call two atlases equivalent if they admit a common refinement.*

Remark 3.1.1. Consider a quotient $\widehat{U}/G$ modelling an open subset of X. The assumptions prescribed in the above definition yield that G acts effectively on $\widehat{U}$, that is, if $g \cdot x = x$ for all $x \in \widehat{U}$, then g is the identity element of G. This particular consequence is highlighted in our forthcoming definition.

Remark 3.1.2. For completeness, let us momentarily diverge to a technical result. Some authors will state that, for an embedding $\lambda : \widehat{U} \to \widehat{V}$, there is an associated group monomorphism $G \to H$, with respect to which λ is equivariant. This actually follows from our definition, and is an important technical result for the theory.

Proposition 3.1.1. *For two embeddings $\lambda, \mu : (\widehat{U}, G) \rightrightarrows (\widehat{V}, H)$, there exists a unique $h \in H$ for which $\mu = h \circ \lambda$. In the special case for which we view an element $g \in G$ as an embedding of the chart $(\widehat{U}, G)$ into itself, the two embeddings λ and $\lambda \circ g$ yield a unique $h \in H$ for which $\lambda \circ g = h \circ \lambda$. We denote this h by $\lambda(g)$, and hence associate to our embedding $\lambda : \widehat{U} \to \widehat{V}$ an injective group homomorphism $\lambda : G \to H$.*

Proof. This result is proved in the appendix of a paper by Moerdijk and Pronk, [MP97]. □

The proposition above tells us that an embedding is equivariant with respect to its associated group monomorphism.

With this observation, we now make a general definition (following [KL14]) about mappings between local models (not necessarily on the same space).

Definition 3.1.2. *A smooth map between local models $(\widehat{U}_1, G_1)$ and $(\widehat{U}_2, G_2)$ is given by a smooth map $\widehat{f} : \widehat{U}_1 \to \widehat{U}_2$ and a homomorphism $\rho : G_1 \to G_2$ so that $\widehat{f}$ is ρ-equivariant. We make no assumption on ρ, it need not be injective or surjective.*

In this way, embeddings of charts on an orbifold can be viewed as smooth maps of such. We now come to a fundamental definition.

Definition 3.1.3. *An effective orbifold $\mathcal{X}$ of dimension n is a paracompact Hausdorff space X equipped with an equivalence class of n-dimensional orbifold atlases. We write $\mathcal{X} = (X, [\mathcal{U}])$.*

This definition is slightly more general than Satake's original definition of orbifolds, as 'V-manifolds'. In particular, we do not require that the fixed point set of each local action has codimension at least 2 (this forbidds, for example, reflections through a hyperplane). We shall see that such an assumption is closely related to orientability of an orbifold, and mainly serves a technical purpose. We call such orbifolds 'codimension 2' orbifolds. We say that $\mathcal{X}$ is compact (resp. connected) if the underlying topological space X is compact (resp. connected).

We shall almost exclusively work only with effective orbifolds, but we should point out that there are several important examples of non-effective orbifolds, such as the inertia orbifold (see Chapter 4 of [ALR07]). Furthermore, we shall use fonts such as $\mathcal{X}, \mathcal{Y}, \mathcal{O}$ to denote orbifolds, and plain fonts X, Y, O to denote their respective underlying topological spaces. We shall also sometimes write $|\mathcal{O}|$ to denote the underlying topological space. To each equivalence class of orbifold atlases on X, there is a unique associated maximal atlas. In particular, by an abuse of notation, we shall tacitly work with a fixed maximal atlas (see Chapter 1 of [ALR07] for details), and we write the datum as a pair $\mathcal{X} = (X, \mathcal{U})$. Before we proceed to examples and the notion of morphism, we need a few more basic definitions, namely the notion of boundary, and orientability.

Definition 3.1.4. *An orbifold $\mathcal{X}$ with boundary is defined similarly to the above, except that we allow each $\widehat{U}$ to be a connected open subset of $[0, \infty) \times \mathbb{R}^{n-1}$. For an orbifold $\mathcal{X}$, the boundary $\partial\mathcal{X}$ consists of points x in the underlying topological space for which there is a chart $\widehat{U}/G$ about x, so that x corresponds to an orbit in $(\widehat{U} \cap \partial\mathbb{R}^n_+)/G$. An orbifold is closed if it is compact, and its orbifold boundary is empty.*

Definition 3.1.5. *An orbifold $\mathcal{X}$ with a given atlas $\mathcal{U}$ is locally orientable if the atlas $\mathcal{U} = \{(\widehat{U}, G)\}$ is so that each group G consists of orientation preserving automorphisms. It is orientable if the embeddings of charts preserve orientation. We say that it is oriented if a orientation for each connected open subset $\widehat{U}$ has been chosen.*

Remark 3.1.3. The chart domain $\widehat{U}$ in a local model pair $(\widehat{U}, G)$ is required to be a connected open subset of Euclidean space. An equivalent definition is to allow chart domains to be connected smooth manifolds of a fixed dimension. Indeed, if $\mathcal{U}$ is an atlas consisting of pairs $(\widehat{U}, G)$ where $\widehat{U}$ is a connected smooth manifold, then there is a canonical refinement of $\mathcal{U}$ to an atlas consisting of charts whose domains are subsets of Euclidean space, the refinement being given by an argument involving the exponential map and invariant metric on each chart. We shall often pass between these two equivalent definitions when working with examples, for ease of presentation. For more details, we refer the reader to ([Sch15],Page 9). Finally, In a similar fashion to the above we can define complex orbifolds.

Fix an orbifold $\mathcal{X} = (X,\mathcal{U})$. If each local action is free, then X is, in addition, a locally Euclidean topological space, and so a manifold. In particular, the fixed points amongst the local data are thus a distinguishing difference, and are therefore called singular points. We make this notion precise. Let $x \in X$, and $\widehat{U}/G$ a chart about x. Choose a representative $\widehat{x}$, and consider the associated isotropy group $G_{\widehat{x}}$. This group is, up to conjugacy, independent of the choices made, and we denote its isomorphism class by G_x, called the local group at x (see [ALR07], Definition 1.5). If $|G_x| \neq 1$, then we call x a singular point. The set of all points in X with non-trivial local group is called the singular set of X, denoted $\Sigma(\mathcal{X})$. Points with trivial local group are called regular points. It is known that the collection of regular points is a smooth manifold which forms an open dense subset of the underlying topological space. In low dimensions, we can completely classify the structure of the singular points. We use the following result regarding the local structure to do so.

Theorem 3.1.1. *An orbifold $\mathcal{X}$ is locally modelled on $\mathbb{R}^n/G$, where G acts as a finite subgroup of $O(n)$.*

Proof. (Sketch) The proof here is drawn from ([Coo00],Theorem 2.3, Page 24). Let $x \in |\mathcal{X}|$. Let $(\widehat{U}, G)$ be a local model about x, and $\widehat{x}$ a representative. Consider the associated isotropy group, $G_{\hat{x}}$; a finite group of diffeomorphisms $\widehat{U} \to \widehat{U}$ fixing $\widehat{x}$. Choose a $G_{\widehat{x}}$-invariant Riemannian metric on $\widehat{U}$, say by averaging. The exponential map yields a $G_{\widehat{x}}$-equivariant diffeomorphism from an open neighbourhood of the origin in $T_{\widehat{x}}\widehat{U}$ to a $G_{\widehat{x}}$-invariant neighbourhood $\widehat{U}_{\widehat{x}}$ of $\widehat{x}$ in $\widehat{U}$. The action of $G_{\widehat{x}}$ on $T_{\widehat{x}}\widehat{U}$ is linear and as a subgroup of $O(n)$, and therefore the action of $G_{\widehat{x}}$ on $\widehat{U}_{\widehat{x}}$ is conjugate, via the exponential map, to a linear action. The claim follows. □

For details regarding the exponential map, we refer the reader to ([Bre72], Page 305). It follows that for a local model pair $(\widehat{U}, G)$ about x, we may assume that $G = G_x$ acts linearly and as a subgroup of $O(n)$. Several authors refer to such a pair, writing say $(\mathbb{R}^n, G)$, as a linear chart. The finite subgroups of $O(2)$ are understood, so we have the following description of local models of 2-dimensional orbifolds.

Corollary 3.1.1 (Singular Types of 2-orbifolds)**.** *For a 2-orbifold $\mathcal{O}$, let $x \in \mathcal{O}$ be a singular point. Then the local group G_x is a finite subgroup of $O(2)$, and either*

1. *G_x is a cyclic rotation group, $\mathbb{Z}_k$ for some k,which yields a cone point of angle $2\pi/k$.*

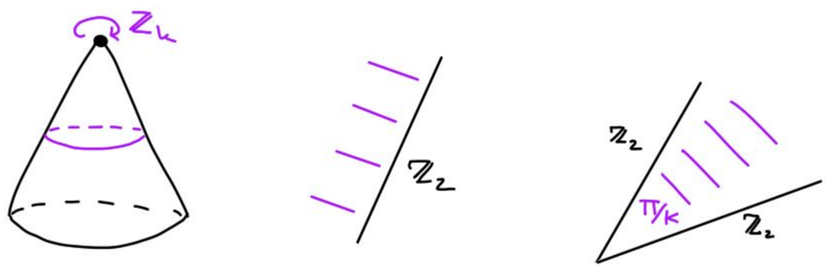


Figure 3.1.2: Singularities in dimension 2.

2. G_x is a reflection of order 2, and x is a mirror point.

3. G_x is a dihedral group D_{2k}, of order $2k$, giving a corner point x.

In particular, the singular set of an orientable 2-orbifold is discrete, consisting only of cone points. Furthermore, we see that the underlying space of a 2-orbifold is a topological 2-manifold (potentially with boundary).

Example 3.1.1 (A Note on Orientation)**.** Earlier we mentioned that an orientable orbifold satisfies Satake's fixed point condition. Indeed, let $\mathcal{X}$ be an orientable (locally orientable is all that is required) orbifold, and suppose $(\mathbb{R}^n, G)$ is a linear chart about a point. If G fixes a hyperplane $V \subseteq \mathbb{R}^n$, which is to say that the fixed point set has codimension 1, consider $V^{\perp}$. The G-action restricts to $V^{\perp}$ in an effective manner, and thus if G is not acting trivially, it must act as $\mathbb{Z}_2$ via reflection about V, which is non-orientable.

Note that, in particular, if one is working with a codimension 2 orbifold of dimension 2, the singular set just consists of points. We now come to the notion of a smooth map. We follow ([KL14], Page 7) and define a smooth map of orbifolds as follows.

Definition 3.1.6. *A smooth map $f : \mathcal{X} \to \mathcal{Y}$ between orbifolds is given by a continuous map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ with the property that for each $x \in |\mathcal{X}|$, there are local models $(\widehat{U}, G)$ about x, $(\widehat{V}, H)$ about $|f|(x) \in |\mathcal{Y}|$, together with a smooth map $\widehat{f} : (\widehat{U}, G) \to (\widehat{V}, H)$ of local models, so that the diagram,*

$$\begin{array}{ccc} \widehat{U} & \xrightarrow{\widehat{f}} & \widehat{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow[|f|]{} & V \end{array}$$

is commutative. Smooth maps can be composed, and a diffeomorphism $f : \mathcal{X} \to \mathcal{Y}$ is a smooth map with a smooth inverse. In this case, G_x is isomorphic to $G_{f(x)}$.

Remark 3.1.4. Historically, there have been issues with the notion of a smooth map between orbifolds first provided by Satake. For example, the desired property that the pullback of an orbifold vector bundle by a smooth map is an orbifold vector bundle, may not always hold (see section 4.4 of [CR01] and section 2.4 of [ALR07]). Fortunately, the issues are resolved by introducing another notion of a map between orbifolds, that of the Chen-Ruan good map (see section 4.4 of [CR01]) or, equivalently, the Moerdijk-Pronk strong map (see, for example, section 5 of [MP97]). In this way, upon considering the correct notion of an orbifold morphism, the theory of orbifolds begins to distinguish itself from its manifold counterpart. A diffeomorphism in our language is a good map, and so compatible with both sheaf and bundle type constructions.

Example 3.1.2. Define $X = \mathbb{R}\times\mathbb{C}$, and consider a uniformising structure $(\mathbb{R}\times\mathbb{C}, \mathbb{Z}_4, \pi : \mathbb{R}\times\mathbb{C} \to (\mathbb{R}\times\mathbb{C})/\mathbb{Z}_4)$ where $\mathbb{Z}_4$ acts on $\mathbb{C}$ by multiplication of $\sqrt{-1}$. Define a map $\widehat{f} : \mathbb{R} \to \mathbb{R}\times\mathbb{C}$ by $\widehat{f}(t) = (t, t^2)$. Then $f := \pi \circ \widehat{f}$ is a smooth map between orbifolds.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\widehat{f}"] \arrow[d] & \mathbb{R}\times\mathbb{C} \arrow[d, "\pi"] \\
\mathbb{R} \arrow[r, "f"'] & (\mathbb{R}\times\mathbb{C})/\mathbb{Z}_4
\end{tikzcd}$$

This way of constructing a smooth map generalises to constructing morphisms between global quotients M/G, which we'll discuss in the example below.

Later on, we shall be concerned with generalisations of classical theorems to the setting of orbifold. In particular, we shall need the notion of a metric.

Definition 3.1.7. *A Riemannian metric on an orbifold $\mathcal{O} = (O, \mathcal{U})$ is given by a collection of Riemannian metrics on the chart domains $\widehat{U}$'s so that each G acts isometrically on $\widehat{U}$, and the embeddings of charts on $\mathcal{O}$ are isometries with respect to these metrics.*

Existence is taken care by a generalisation of the usual partition of unity argument, which involves averaging (see Proposition 2.20 in [MM03]). We shall now furnish a collection of examples, varying in degree of complexity.

Example 3.1.3 (Quotient Orbifolds)**.** Recall that if a compact Lie group G acts smoothly and freely on a manifold M, then M/G can be equipped with a smooth structure so that the projection $M \to M/G$ is a principal G-bundle. Suppose now G acts smoothly (say a left action), effectively, but only almost freely (i.e., finite stabilisers). The quotient M/G is then an orbifold. The underlying topological space is M/G equipped with the quotient topology. Fix $x \in M$. By the differentiable slice theorem ([Bre72], page 308), there exists a a G_x-invariant neighbourhood U of x along with a G-equivariant map $G \times_{G_x} U \to M$ which is a diffeomorphism onto an open neighbourhood N of the orbit of x (so N is a G-space). Identify U with an open subset of $\mathbb{R}^n$, denoted $\widehat{U}$, via a diffeomorphism $f : \widehat{U} \to U$. We let G_x act on $\widehat{U}$ so that f is G_x-equivariant. An orbifold chart about x is given by $(\widehat{U}, G_x, \varphi)$ where $\varphi : \widehat{U} \to M/G$ is defined as follows; observe that $(G \times_{G_x} U)/G$ is homeomorphic to N/G and the former is identified with U/G_x. Now by definition, φ must map onto an open subset of M/G, and so it is defined via $\widehat{U} \to U \to U/G_x \to N/G$. Collecting such charts as x runs over M, we obtain an orbifold atlas on the orbit space M/G, where local compatibility of our charts is taken care of by sufficiently shrinking our open sets U. The resulting orbifold, again denoted by M/G, is called an effective quotient orbifold. A special case is for which G is a finite group, in this case such an orbifold is called an effective global quotient

This example, and its varies subfamilies, motivates a definition.

Definition 3.1.8. *An orbifold $\mathcal{O}$ is called good (or developable) if $\mathcal{O} = M/G$ for some manifold M and discrete group G. We say $\mathcal{O}$ is very good if G is a finite group. Orbifolds that are not good are called bad.*

We'll now provide examples of very good, and bad orbifolds, amongst others.

Example 3.1.4 (Coordinate Reflection on the Torus)**.** We now specialise to examples from a class called Toroidal orbifolds, these are orbifolds where we consider a quotient of the n-torus by a finite subgroup $G \subset GL_n(\mathbb{Z})$ acting smoothly. Let $\mathbb{T}^n = (S^1)^n = (\mathbb{R}/\mathbb{Z})^n$ be the n-torus, and consider the action of $\mathbb{Z}_2$ on $\mathbb{T}^n$ generated by the involution τ which acts by complex conjugation on each coordinate. The resulting orbit space $\mathbb{T}^n/\mathbb{Z}_2$ is an orbifold with 2^n singular points; the singular points having coordinates chosen from $\{0, 1/2\}$. A special case is the so called pillowcase orbifold. View the torus $\mathbb{T}^2 = S^1 \times S^1$ as a submanifold of $\mathbb{R}^3$. Let $\mathbb{Z}_2$ act on $\mathbb{T}^2$ via $(z, w) \mapsto (\bar{z}, \bar{w})$. The orbifold $\mathbb{T}^2/\mathbb{Z}_2$ has underlying topological space (homeomorphic to) S^2, and four singular points, each with local group $\mathbb{Z}_2$. We may visually interpret this action as a rotation by π around an axis,

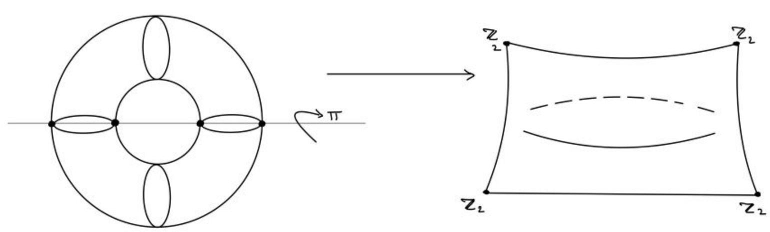


Figure 3.1.3: Our pillowcase.

This example allows us to realise S^2 as the underlying coarse space of a "flat" orbifold (we'll make sense of this later, once we've discussed curvature, for now, we refer the reader to here).

Example 3.1.5 (Mirror)**.** Let $\mathbb{Z}_2$ act on $\mathbb{R}^n$ by reflection through a hyperplane. The quotient $\mathbb{R}^n/\mathbb{Z}_2$ is an orbifold, which we call a mirror. Its singular set is exactly the hyperplane, and has codimension 1. This example is not an orbifold within the stricter definition of Satake. Moreover, this particular consideration allows us to construct an interesting class of examples; manifolds with boundary realised as orbifolds without. Let M be an n-manifold with boundary. We may provide M with the structure of an orbifold as follows. We consider each point $x \in \partial M$ to be modelled on the quotient $\mathbb{R}^n/\mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ is generated by reflection about the hyperplane in the half space model about x. The resulting orbifold has singular set ∂M,

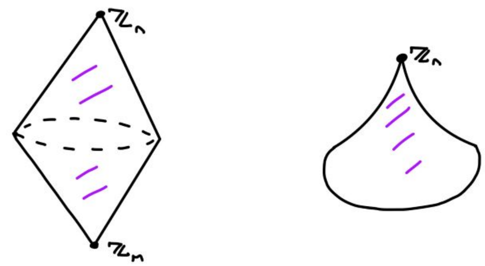


Figure 3.1.4: An awkward rugby ball, and Thurston's teadrop, a 'bad' orbifold.

and the singular points are thought of as 'mirror' points, with local group $\mathbb{Z}_2$. (See Example 1.3.3 at [Car19]).

Example 3.1.6 (Orbifold Structures on the 2-sphere)**.** In this example, we pass to and from identification with complex structure. This example closely follows ([BH13], Example 1.4, page 587). Identify the 2-sphere S^2 with $\mathbb{C} \cup \{\infty\}$. Let $V_0 = \mathbb{C} \subset S^2$ and $V_\infty = S^2 \backslash \{0\}$. Let n and m be two positive integers. We define an orbifold structure on S^2 with the datum of two charts. Define mappings, $q_0 : \mathbb{C} \to V_0$ by $q_0(z) = z^m$ and $q_\infty : \mathbb{C} \to V_\infty$ by $q_\infty(w) = 1/w^n$. Let G_0 be the group of all rotations of order m fixing 0, acting on $\mathbb{C}$ (identified with the cyclic group of order m). Let G_∞ be the group of all rotations of $\mathbb{C}$ fixing 0 whose order is n (identified with the cyclic group of order n). There are two uniformising systems, $(\mathbb{C}, G_0, q_0)$ and $(\mathbb{C}, G_\infty, q_\infty)$, which cover S^2. We model the intersection $V_0 \cap V_\infty$ with $\mathbb{C}\backslash\{0\}$ and the restriction of q_0, call it $q_\#$. There are then two embeddings; into the q_0 chart this is simply the inclusion map $\mathbb{C}\backslash\{0\} \to \mathbb{C}$, and for the q_∞ chart, we define $\lambda : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by $\lambda(z) = (1/z)^{m/n}$, then $q_\infty \circ \lambda = q_\#$. We thus have a (complex) orbifold structure on S^2. By varying choices of n and m, we obtain rather famous examples, for example, set $n \neq 1$, $m = 1$, then the corresponding orbifold is called a 'teardrop'. In fact, this orbifold is good (i.e., a global quotient) if and only if $m = n$, in which case it is the canonical quotient $S^2/\mathbb{Z}_m$ by rotations.

Example 3.1.7 (Symmetric Product)**.** Let M be a smooth manifold. Let n be a positive integer, at least 2. Consider the product $M^n = M \times \cdots \times M$ (n times). Consider the symmetric group of degree n, denoted S_n, acting on M^n by permutation of coordinates. The quotient M/S_n is an orbifold, a global quotient at that. Furthermore, the diagonal in the product is the fixed point set. This singular set therefore looks like a copy of M inside of M^n.

Example 3.1.8 (Gorenstein Orbifolds)**.** We say that an n-dimensional complex orbifold $\mathcal{X}$ is Gorenstein (or an SL-orbifold) if all local groups G_x are subgroups of $SL(n, \mathbb{C})$. Gorenstein orbifolds are of particular interest, for example, they are involved in the so called Crepant resolution conjecture and, in particular, have $\mathbb{Z}$-graded Chen-Ruan cohomology, with their canonical bundle (top exterior power of the cotangent bundle) being an honest line bundle! (We refer the reader to [CR00] for details on the Chen-Ruan cohomology).

3.2 Bundles on Orbifolds

We now come to several standard geometric constructions, namely those of bundles, forms and (a little bit of) cohomology. An orbifold bundle is locally a G-bundle for varying finite G. We require that sections of orbifold bundles consist of locally equivariant sections, compatible with the underlying embedding data. A common way to define a bundle over an orbifold is first locally over each chart, and then glue together the quotients. Let us provide a general definition, drawn from [KL14].

Definition 3.2.1. *An orbifold fiber bundle with total space $\mathcal{O}_1$, base space $\mathcal{O}_2$ and fiber F (a smooth manifold), consists of a smooth map of orbifolds $\pi : \mathcal{O}_1 \to \mathcal{O}_2$ for which,*

1. *The underlying map $|\pi|$ is surjective,*
2. *For each $p \in |\mathcal{O}_2|$, there is a local model $(\widehat{U}, G_p)$ about p, where G_p denotes the local group at p, together with an action of G_p on F and a diffeomorphism $(\widehat{U} \times F)/G_p \to \mathcal{O}_1|_{|\pi|^{-1}(U)}$ so that the diagram,*

$$\begin{tikzcd}
(\widehat{U} \times F)/G_p \arrow[r] \arrow[d] & \mathcal{O}_1 \arrow[d] \\
\widehat{U}/G_p \arrow[r] & \mathcal{O}_2
\end{tikzcd}$$

is commutative.

Definition 3.2.2. *A smooth section s of an orbifiber bundle $\pi : \mathcal{O}_1 \to \mathcal{O}_2$ consists of a smooth map $s : \mathcal{O}_2 \to \mathcal{O}_1$ such that $\pi \circ s$ is the identity on $\mathcal{O}_2$.*

In particular, the local lifts of a smooth section consist of equivariant sections of the form $\widehat{U} \to \widehat{U} \times F$.

Remark 3.2.1. Let us comment on a general perspective. For a fixed atlas $\mathcal{U}$ on an orbifold $\mathcal{O}$, we may consider the disjoint union $\coprod \widehat{U}$ of chart domains (this is the object space of a groupoid induced by $\mathcal{O}$). Bundles and sheaves (and their associated constructions), may be viewed as a sequence of data defined over this union, satisfying various compatibility conditions. This approach is taken by several authors, and is demonstrated below. However, each example satisfies our underlying definition given above.

Construction 3.2.1 (Recovering $|\mathcal{O}|$ via our atlas)**.** Fix an orbifold $\mathcal{O} = (O, \mathcal{U})$. Let $(\widehat{U}, G)$ and $(\widehat{V}, H)$ be overlapping charts, $x \in U \cap V$. By hypothesis, there is a third chart $(\widehat{W}, K)$, $x \in W \subseteq U \cap V$, and two embeddings $\lambda_1 : (\widehat{W}, K) \to (\widehat{U}, G)$ and $\lambda_2 : (\widehat{W}, K) \to (\widehat{V}, H)$. We may use λ_1 and λ_2 to produce a diffeomorphism,

$$\lambda_{12} := \lambda_2 \circ \lambda_1^{-1} : \lambda_1(\widehat{W}) \to \lambda_2(\widehat{W}).$$

Related to an embedding is an associated group monomorphism with respect to which the embedding is equivariant. In this case there are two injections $K \to G$ and $K \to H$, and we may view λ_{12} as a K-equivariant diffeomorphism. We glue $\widehat{U}/G$ and $\widehat{V}/H$ together according to λ_{12}, that is, we say $[\widehat{u}] \sim [\widehat{v}]$ if $\lambda_{12}(\widehat{u}) = \widehat{v}$, and this is well defined. In this fashion, we may consider the following quotient space,

$$Y := \left(\coprod_{\widehat{U} \in \mathcal{U}} \widehat{U}/G \right) \Big/ \sim$$

This is homeomorphic to $|\mathcal{O}|$, with the homeomorphism given by piecing together the collection of mappings $\{\phi : \widehat{U} \to |\mathcal{O}|\}$ induced via $\mathcal{U}$.

We shall use the core principle underlying the above construction to formulate examples of orbibundles. The reader can certainly guess the approach that will be taken (thinking of cocyles, gluing and so on). Our main example will be the tangent bundle of an orbifold, together with its projectivisation, whom we will consider in the following chapter.

Example 3.2.1 (Tangent Orbibundle)**.** We shall construct the tangent bundle of an orbifold (an orbivector bundle). Let $\mathcal{O} = (O, \mathcal{U})$ be an orbifold of dimension n. Over each local model $(\widehat{U}, G)$, consider the tangent bundle $T\widehat{U} \to \widehat{U}$. The G-action on $\widehat{U}$ lifts canonically via the differential to $T\widehat{U} \cong \widehat{U} \times \mathbb{R}^n$, namely, for $g \in G$ and $(\widehat{x}, v) \in T\widehat{U}$, we have

$$g(\widehat{x}, v) = (g(\widehat{x}), (dg)_{\widehat{x}} v).$$

The projection $T\widehat{U} \to \widehat{U}$ is an equivariant mapping, and we have a G-bundle. There is a canonical map $T\widehat{U}/G \to \widehat{U}/G \cong U$. Define,

$$TO := \left(\coprod_{\widehat{U} \in \mathcal{U}} T\widehat{U}/G \right) \Big/ \sim$$

where $[(\widehat{x}, v)] \in T\widehat{U}/G$ is equivalent to $[(\widehat{y}, q)] \in T\widehat{V}/H$ if there is a model $(\widehat{W}, K)$ in $\mathcal{U}$ with embeddings $\lambda_1 : (\widehat{W}, K) \to (\widehat{U}, G)$ and $\lambda_2 : (\widehat{W}, K) \to (\widehat{V}, H)$ for which we have a pair $(\widehat{w}, u) \in T\widehat{W}$ so that $\lambda_1(\widehat{w}) = \widehat{x}$, $\lambda_2(\widehat{w}) = \widehat{y}$ and $(d\lambda_1)_{\widehat{w}} u = v$ and $(d\lambda_2)_{\widehat{w}} u = q$. Topologise TO with the quotient topology. The collection of charts $(T\widehat{U}, G, \pi_{\widehat{U}} : T\widehat{U} \to T\widehat{U}/G)$ yields an orbifold atlas on TO, and the resulting orbifold is denoted $T\mathcal{O}$, an orbifold of dimension $2n$. The collection of maps $T\widehat{U}/G \to U$ yield a projection $p : T\mathcal{O} \to \mathcal{O}$, which is a smooth map of orbifolds. For $x \in O$, the fiber $p^{-1}(x)$ looks like $\mathbb{R}^n/G_x$, and thus fibers above non-singular points are ordinary vector spaces. Suppose we are given a G-equivariant map $v_{\widehat{U}} : \widehat{U} \to T\widehat{U}$ over each chart $(\widehat{U}, G)$, for which the collection $\{v_{\widehat{U}}\}$ is compatible with embeddings, in the sense that if $\lambda : (\widehat{U}, G) \to (\widehat{V}, H)$ is an embedding of charts, then $\lambda_* v_{\widehat{V}} = v_{\widehat{U}}$, where $\lambda_* v_{\widehat{V}}$ denotes the pullback vector field. We may then produce a section $v : \mathcal{O} \to T\mathcal{O}$. Indeed, all that is needed is to define the underlying continuous map. For $x \in O$, we define $v : O \to TO$ by $v(x) = [(\widehat{x}, v_{\widehat{U}}(\widehat{x})] \in TO$ where $\widehat{x}$ is a representative of x in a chart $\widehat{U}/G$, and this is both a well-defined and continuous function. The tangent bundle of an orbifold is an example of an orbifold vector bundle, an 'orbivector' bundle. By definition the tangent space at $p \in |\mathcal{O}|$ is (the isomorphism class of) the orbivector space $(T_{\widehat{p}}\widehat{U}, G_p)$, where $\widehat{p}$ is a representative of p in a model pair $(\widehat{U}, G_p)$. The tangent cone $C_p|\mathcal{O}|$ at p is by definition isomorphic to the quotient $T_{\widehat{p}}\widehat{U}/G_p$.

Remark 3.2.2. For a global quotient orbifold M/G, if one takes the perspective which allows chart domains to be smooth manifolds (see 3.1.3), then we can think of a vector field on the orbifold M/G as a G-equivariant section $M \to TM$. Often, we shall employ this perspective when giving examples, for ease of presentation.

Example 3.2.2 (Projectivised Orbibundle)**.** We may construct the projectivisation of $T\mathcal{O}$. Over each local model $(\widehat{U}, G)$, consider the projectivised tangent bundle $PT\widehat{U} \to \widehat{U}$. The G action extends canonically to $PT\widehat{U}$, by

$$g(\widehat{x}, \langle v \rangle) = (g(\widehat{x}), \langle dg_{\widehat{x}} v \rangle).$$

We remark that the action on the fiber space, in this case $\mathbb{R}P^{n-1}$, need not be effective (consider scalar matrices, for example). By gluing together quotients $PT\widehat{U}/G$ as outlined above, we obtain an orbifold $PT\mathcal{O}$, together with a projection $p : PT\mathcal{O} \to \mathcal{O}$, an example of an orbifold fiber bundle. A section $\xi : \mathcal{O} \to PT\mathcal{O}$ is a *line field* on $\mathcal{O}$. We shall consider such maps in more detail in Chapter 4.

Equipped with the notion of tangent bundle, we can make sense of the differential of a smooth map of orbifolds. In particular, one can talk of immersions and submersions. We briefly mention this. Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a smooth map of orbifolds. Given a point $p \in |\mathcal{O}_1|$, we have local models $(\widehat{U}_1, G_1)$, $(\widehat{U}_2, G_2)$ and an equivariant lift $\widehat{f} : \widehat{U}_1 \to \widehat{U}_2$. Let $\widehat{p}$ be a representative of p. We may then consider the differential $d\widehat{f}_{\widehat{p}} : T_{\widehat{p}}\widehat{U}_1 \to T_{\widehat{f}(\widehat{p})}\widehat{U}_2$. This data is, up to isomorphism, only dependent on our basepoint, and we have a mapping $df_p : T_p\mathcal{O}_1 \to T_{|f|(p)}\mathcal{O}_2$. More so, in locality we have an equivariant bundle map $d\widehat{f} : T\widehat{U}_1 \to T\widehat{U}_2$, the collection of which piece together to yield a smooth map $df : T\mathcal{O}_1 \to T\mathcal{O}_2$. We refer the reader to [KL14] for more detail.

Definition 3.2.3. *We shall say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a submersion at p (respectively, an immersion at p), if the differential $df_p : T_p\mathcal{O}_1 \to T_{|f|(p)}\mathcal{O}_2$ is surjective (respectively, injective). We say that f is a submersion (respectively, immersion) if it is a submersion at all points of p (respectively, an immersion at all points of p).*

With the notion of an immersion, we can make sense of a suborbifold.

Definition 3.2.4. *A suborbifold $\mathcal{O}$ is given by an orbifold $\mathcal{O}'$ and an immersion $f : \mathcal{O}' \to \mathcal{O}$ for which $|f|$ maps $|\mathcal{O}'|$ homeomorphically onto its image in $|\mathcal{O}|$. Let us remark that given an open subset of $|\mathcal{O}|$, there is a canonically induced orbifold structure.*

We shall now construct differential forms. It is clear how to define them, namely as sections of exterior powers of the cotangent bundle of an orbifold. Let us remark (as in [ALR07]), if we are given any continuous functor F from vector spaces to vector spaces, we can construct an orbivector bundle $F(T\mathcal{O}) \to \mathcal{O}$ with fibers $F(T_{\widehat{x}}\widehat{U})/G_x$. Thus we have access to the cotangent bundle of an orbifold, $T^*\mathcal{O}$, and its exterior powers. These can all be built in detail as above. We omit a detailed description. We define 0-forms on an orbifold to simply be smooth real-valued functions. For $k > 0$,

Definition 3.2.5. *A differential k-form on an orbifold $\mathcal{O}$ is a section $\mathcal{O} \to \bigwedge^k(T^*\mathcal{O})$. As usual, we write the space of k-forms on $\mathcal{X}$ as $\Omega^k(\mathcal{O})$. The wedge product of forms on an orbifold is defined, furthermore, by naturality, we have a well-defined exterior derivative $d : \Omega^k(\mathcal{X}) \to \Omega^{k+1}(\mathcal{X})$, and in particular, taking the cohomology of the complex*

$$\cdots \xrightarrow{d} \Omega^{k-1}(\mathcal{O}) \xrightarrow{d} \Omega^k(\mathcal{O}) \xrightarrow{d} \Omega^{k+1}(\mathcal{O}) \xrightarrow{d} \cdots$$

we obtain the de Rham cohomology $\mathcal{O}$ of an orbifold, $H^*_{dR}(\mathcal{O})$.

Let us now briefly discuss the integration of differential forms over an oriented n-orbifold $\mathcal{O}$. It is akin to integration on a manifold. Let $U \subset |\mathcal{O}|$ be an open subset uniformised by a model pair $(\widehat{U}, G)$. A compactly supported n-form ω on U is (naturally identified with) a compactly supported G-invariant n-form $\widehat{\omega}$ on $\widehat{U} \subseteq \mathbb{R}^n$. The integration of ω on U is defined by,

$$\int_U^{\mathrm{orb}} \omega := \frac{1}{|G|} \int_{\widehat{U}} \widehat{\omega}.$$

Example 3.2.3 (Global Quotient)**.** Let $\mathcal{O} = M/G$ be a orientable global quotient orbifold, whom we think of as being uniformised by a single chart, allowing, for the moment, our domains to be smooth manifolds (refer to Remark 3.1.3). A differential form on M/G is then a G-invariant differential form on M. Let $\omega \in \Omega^n(M)$ be G-invariant, and suppose ω has compact support. Then,

$$\int_{\mathcal{O}}^{\mathrm{orb}} \omega = \frac{1}{|G|} \int_M \omega.$$

In fact, we can think of $T\mathcal{O}$ as the quotient orbifold TM/G.

Let us now consider the global case. We have $\mathcal{O}$ with a cover $\{U_\alpha\}$ of uniformised open sets; we may choose, via paracompactness, (see Lemma 3.4.1 in [Car19]) a smooth partition of unity $\{\rho_\alpha\}$ subordinate to this cover, then integrate a compactly supported n-form ω on $\mathcal{O}$ as,

$$\int_{\mathcal{O}}^{\mathrm{orb}} \omega := \sum_\alpha \int_{U_\alpha}^{\mathrm{orb}} \rho_\alpha \omega.$$

In exactly the same way as for manifolds, this definition is independent of the choice of partition of unity. We have (see Theorem 3.4.2 in [Car19]),

Theorem 3.2.1 (Stokes' Theorem)**.** *Let $\mathcal{O}$ be an oriented n-dimensional orbifold with boundary, and $\omega \in \Omega^{n-1}(\mathcal{O})$ a compactly supported $(n-1)$-form. Then,*

$$\int_{\mathcal{O}}^{orb} d\omega = \int_{\partial\mathcal{O}}^{orb} \omega.$$

Proof. This is a trivial consequence of Stokes' theorem in the setting of manifolds applied to local model pairs. ☐

We shall now state a collection of classical results generalised to the setting of orbifolds.

Theorem 3.2.2. *The following results were proved by Satake in [Sat56];*

1. *For $\mathcal{O}$ a closed, orientable, n-orbifold the pairing,*

$$\int : H^k_{dR}(\mathcal{O}) \otimes H^{n-k}_{dR}(\mathcal{O}) \to \mathbb{R}$$
$$(\omega, \tau) \mapsto \int_{\mathcal{O}}^{orb} \omega \wedge \tau$$

is non-degenerate. In particular, compact orientable orbifolds satisfy Poincaré duality, $H^k_{dR}(\mathcal{O}) \cong (H^{n-k}_{dR}(\mathcal{O}))^*$.

2. *For a closed orbifold $\mathcal{O}$, there is an isomorphism,*

$$H^*_{dR}(\mathcal{O}) \cong H^*(|\mathcal{O}|; \mathbb{R})$$

where the right hand side denotes the singular cohomology of the underlying topological space, with real coefficients.

Included above is a de Rham Theorem in the setting of orbifolds. It implies, in particular, that the orbifold de Rham cohomology does not detect singular points (for example, consider a point with the trivial action of a finite group; all the group data is lost upon passing to the orbifold de Rham cohomology). Taking the perspective that our orbifold $\mathcal{O}$ consists of two pieces of data;

1. Geometric data; the underlying topological space $|\mathcal{O}|$, the 'coarse quotient'.

2. Singular data; the set of all points in $|\mathcal{O}|$ with non-trivial local group, $\Sigma(\mathcal{O})$.

The orbifold de Rham cohomology is insufficient; it loses too much information. This suggests we search for an alternative cohomology theory, one which at least detects the presence of singular points. This brings us to the following philosophy (inspired by Section 4.3 of [Gin13]) "The correct characteristic zero (co)homology invariants of an orbifold $\mathcal{X}$ are those of its inertia orbifold (possibly up to some regrading)". We shall not pursue this further, but simply refer the reader to [CR00], which discusses the so called 'Chen-Ruan' cohomology of an orbifold admitting an almost complex structure. This cohomology theory is distinct even at the most basic level, for example, the 0-th Chen-Ruan cohomology of a point orbifold $G \curvearrowright \{\mathrm{pt}\}$ has dimension equal to the number of conjugacy classes of G, that is, the number of irreducible complex representations of the group G.

Let us now turn to a different type of question. Suppose we are given an effective orbifold $\mathcal{O}$. Is there a general theme regarding the global structure of $\mathcal{O}$? The answer is affirmative. In fact, all effective orbifolds look like the quotient of a smooth manifold by a compact lie group (with a suitable action, of course). To see this, we must first establish the candidate smooth manifold, for this, consider a model $(\widehat{U}, G)$ on $\mathcal{O}$. Choose a Riemannian metric on $\mathcal{O}$. We may then, on $\widehat{U}$, consider the corresponding orthonormal frame bundle of $\widehat{U}$,

$$\mathrm{Fr}(\widehat{U}) = \{(\widehat{x}, B) \mid B \in O(T_{\widehat{x}}\widehat{U})\}.$$

The differential of a smooth mapping yields an action of G on $\mathrm{Fr}(\widehat{U})$. This G-action is free, and so the quotient $\mathrm{Fr}(\widehat{U})/G$ is a smooth manifold. There is a right $O(n)$ action on the quotient $\mathrm{Fr}(\widehat{U})/G$, induced by the canonical $O(n)$ action on the frame bundle. It can be shown that one can glue together the quotients $\mathrm{Fr}(\widehat{U})/G$ to obtain a smooth manifold $\mathrm{Fr}(\mathcal{O})$, called the frame bundle of $\mathcal{O}$, with a well-defined $O(n)$-action. We then have the following result.

Theorem 3.2.3. *For an effective orbifold $\mathcal{O}$, its frame bundle $Fr(\mathcal{O})$ is a smooth manifold with a smooth, effective and almost free $O(n)$-action. Moreover, there is a diffeomorphism of orbifolds $\mathcal{O} \cong Fr(\mathcal{O})/O(n)$.*

Proof. We refer the reader to Theorem 1.23 in [ALR07]. □

Suppose we apply the construction above to a global quotient orbifold. We then have two 'presentations' (this shall be made a little bit more precise later), for now we shall be concerned with simply an outline of the details.

Proposition 3.2.1. *Let M be a compact manifold with a smooth, almost free and effective action of G, a compact Lie group. Then the frame bundle $Fr(M)$ of M has a smooth almost free $G \times O(n)$ action such that the following diagram is commutative,*

$$\begin{tikzcd} Fr(M) \arrow[r, "/O(n)"] \arrow[d, "/G"'] & M \arrow[d, "/G"] \\ Fr(M/G) \arrow[r, "/O(n)"'] & M/G \end{tikzcd}$$

and one has $Fr(M)/G \cong Fr(M/G)$.

Proof. This is Proposition 1.25 in [ALR07]. □

In light of the established results, we see that for a global quotient M/G, there are two presentations, namely M/G and $\mathrm{Fr}(M/G)/O(n)$, which give rise to the same orbifold structure. Why should we be concerned with alternative presentations? We illustrate an outline in the following two examples.

Example 3.2.4. This example is inspired by ([Cav12],Page 23). Consider the morphisms between two global quotient orbifolds $[M/G] \to [N/H]$. Certainly, the data of a smooth map $f : M \to N$ and a Lie group homomorphism $\Phi : G \to H$ with respect to which it is equivariant, induces a smooth map of orbifolds (for a detailed proof, we refer the reader to [PR20], Proposition 4.1). Constraining all morphisms to arise in this fashion can fail to capture necessary data. The issue arises from the fact that the global quotients are specific presentations of the orbifolds, and only by considering 'Morita equivalent' presentations, will we enlargen our collection of morphisms. To illustrate this, consider the following elementary example. Let e denote the additive identity in the abelian group $(\mathbb{Z}, +)$. Let $M = S^1$, $G = \{e\}$, $N = \mathbb{R}$ and $H = \mathbb{Z}$. We let G act trivially on M, and H act on N by translations. Then M/G and N/H are both copies of the circle. View the data from an orbifold perspective, and constrain all morphisms to arise as induced maps from M to N. The identity map of the 1-sphere, presented on either side in different ways, ought to appear in our collection of morphisms. However, any smooth map $f : S^1 \to \mathbb{R}$ is homotopically trivial, so has degree zero. Consider the triangle,

$$\begin{tikzcd} S^1 \arrow[r] \arrow[dr] & \mathbb{R} \arrow[d] \\ & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

The degree is multiplicative under composition, and the identity is of non-zero degree, therefore no map can descend to the identity. This issue is resolved via the notion of Morita equivalence, which we shall discuss soon.

Example 3.2.5 (An Alternative Viewpoint of $T\mathcal{X}$)**.** For an orbifold $\mathcal{X}$, let us provide an alternative perspective on the tangent orbibundle $T\mathcal{X} \to \mathcal{X}$. As in [ALR07], we identify the tangent bundle of $\mathcal{X}$ with the quotient $T\mathrm{Fr}(\mathcal{X})/O(n) \to \mathrm{Fr}(\mathcal{X})/O(n)$. Of course, we can now also identify sections. In particular, a vector field $\mathcal{X} \to T\mathcal{X}$ can be thought of as an $O(n)$-equivariant section $\mathrm{Fr}(\mathcal{X}) \to T\mathrm{Fr}(\mathcal{X})$.

3.3 The Orbifold Euler Characteristic and Coverings

We'll now turn our attention to more combinatorial invariants of an orbifold, and several applications. Recall that for a manifold M, we have an integer isomorphism invariant called the Euler characteristic of M, denoted $\chi(M)$. We'll now define the orbifold analogue of this result, called the orbifold Euler characteristic, initially introduced by Satake, and further developed upon by Thurston. The notational style here follows [Sea08].

Definition 3.3.1. *Let $\mathcal{O} = (O, \mathcal{U})$ be an orbifold. We call a triangulation $\mathcal{T}$ of O compatible if the order of the isotropy group is a constant function on the interior of each simplex $\sigma \in \mathcal{T}$. Orbifolds admit good triangulations (see [MP99]). Let N_σ denote the order of an isotropy group on the interior of the simplex $\sigma \in \mathcal{T}$. The orbifold Euler characteristic, denoted $\chi^{\mathrm{orb}}(\mathcal{O})$, is*

$$\chi^{\mathrm{orb}}(\mathcal{O}) = \sum_{\sigma \in \mathcal{T}} \frac{(-1)^{\dim(\sigma)}}{N_\sigma} \in \mathbb{Q}.$$

If we are to call this an invariant, we should make precise what it is invariant under. Clearly diffeomorphic orbifolds have the same orbifold Euler characteristic. Furthermore, the orbifold Euler characteristic is 'compatible' with orbifold covering space theory, as we shall shortly see. Before we do so, let us state some properties of the orbifold Euler characteristic. The first and most basic property is additivity.

Proposition 3.3.1. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are subsets of $\mathcal{O}$ such that $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ and $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_1 \cap \mathcal{O}_2$ correspond to subcomplexes of the triangulation, then*

$$\chi^{\mathrm{orb}}(\mathcal{O}) = \chi^{\mathrm{orb}}(\mathcal{O}_1) + \chi^{\mathrm{orb}}(\mathcal{O}_2) - \chi^{\mathrm{orb}}(\mathcal{O}_1 \cap \mathcal{O}_2)$$

Proof. This is immediate from the definition, for more details we refer the reader to ([FS09], Page 5). □

We should also mention how to compute the characteristic in the simplest case.

Example 3.3.1. If M/G is a global quotient orbifold, for G finite, we have

$$\chi^{\text{orb}}(M/G) = \chi(M)/|G| \in \mathbb{Q}.$$

In particular, the orbifold Euler characteristic need not agree with the Euler characteristic of the underlying coarse space (i.e., consider $S^2/\mathbb{Z}_k$ where $\mathbb{Z}_k$ acts by rotations). Once we've defined orbifold coverings and established a basic result, this property will become immediate.

We now define our notion of covering, which is compatible with our invariant defined above.

Definition 3.3.2. *A covering of an orbifold $\mathcal{O}_2$ is an orbifold $\mathcal{O}_1$ together with a projection $\pi : |\mathcal{O}_1| \to |\mathcal{O}_2|$ between the underlying spaces such that at each point $p \in |\mathcal{O}_2|$, one has a neighbourhood $p \in U \cong \widehat{U}/G$ for which each connected component V of $p^{-1}(U)$ is isomorphic to $\widehat{U}/H$, where $H \leq G$, and π restricted to V is locally the canonical map $\widehat{U}/H \to \widehat{U}/G$, i.e.*

$$\begin{tikzcd} V \arrow[r, "\sim"] \arrow[d, "\pi"'] & \widehat{U}/H \arrow[d] \\ U \arrow[r, "\sim"'] & \widehat{U}/G \end{tikzcd}$$

commutes. We define the number of sheets of the covering π to be the number of points in the pre-image of a regular point.

An orbifiber bundle $\pi : \mathcal{O}_1 \to \mathcal{O}_2$ with zero-dimensional fiber is a covering map of orbifolds (see page 8 of [KL14]). Futhermore, we have the following fibration result in the setting of orbifolds.

Theorem 3.3.1. *A proper surjective submersion $f : \mathcal{O}_1 \to \mathcal{O}_2$, with $\mathcal{O}_2$ connected, defines an orbifiber bundle with discrete fibers. In particular, a proper surjective local diffeomorphism to a connected orbifold is a covering map with finite fibers.*

Proof. We refer the reader to Lemma 2.9 of [KL14]. □

Example 3.3.2. If we have a global quotient M/G, and $H \leq G$, then the canonical map $M/H \to M/G$ is the prototypical example of an orbifold covering. To give a concrete example, let $G = \mathbb{Z}_2$ act on S^1 by reflection. The quotient $S^1/\mathbb{Z}_2$ is an interval with two singular points, each of local group $\mathbb{Z}_2$, which we think of as an interal with mirrored endpoints. The quotient map $S^1 \to S^1/\mathbb{Z}_2$ is a two sheeted covering of orbifolds.

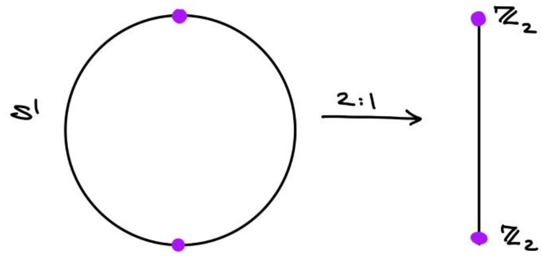


Figure 3.3.1: A two-sheeted orbifold covering.

The map $S^1 \to S^1/\mathbb{Z}_2$ may alternatively be viewed as a branched two sheeted covering, with branch points of ramification index 2 the singular points on the interval. This example carries a hint of generality. In general, for an orbifold $\mathcal{O}$, consider the 'mirror points', $\Sigma_{\text{mir}}(O)$, i.e., where the local model is a quotient of Euclidean space by reflection through a hyperplane. By doubling the underlying coarse space $|\mathcal{O}|$ along the set of mirror points, one obtains the so called local orientation cover. To illustrate this, if we start with the 'stacky' interval above, we see that the local orientation cover is simply the circle (it is not in general a manifold, but it is an orbifold with no mirror points, so locally orientable).

Example 3.3.3. Consider a line orbivector bundle $\pi : L \to \mathcal{O}$. The bundle is locally of the form $(\widehat{U} \times \mathbb{R})/G_x$ where $(\widehat{U}, G_x)$ is a local model on $\mathcal{O}$. The fiber above $x \in |\mathcal{O}|$ is of the form $\mathbb{R}/G_x$. By means of a Riemannian metric on $\mathcal{O}$, we may consider the associated orbi-sphere bundle $S(L) \to \mathcal{O}$. This is an example of an orbifold fiber bundle with finite fibers, where each local lift is a local diffeomorphism.

Let us now demonstrate that the established notions of orbifold Euler characteristic and orbifold coverings are in fact compatible.

Proposition 3.3.2. *If $\pi : \mathcal{O}_1 \to \mathcal{O}_2$ is a k-sheeted orbifold covering map, then*

$$\chi^{\text{orb}}(\mathcal{O}_1) = k\chi^{\text{orb}}(\mathcal{O}_2).$$

The reader is invited to first check it with the example provided above. Furthermore, neighbourhoods satisfying the core propertiy in the definition of an orbifold covering are sometimes called elementary neighbourhoods. It is known that if one takes a simply connected model pair, we always have an elementary neighbourhood (see Proposition 4.2 in [Liu22]).

Proof. We refer the reader to ([Car19], Proposition 2.4.2). □

For a global quotient M/G, consider the canonical projection $M \to M/G$, an orbifold covering. Applying the result above yields $\chi(M) = |G|\chi^{\text{orb}}(M/G)$. We now derive a formulae for the orbifold Euler characteristic in the setting of 2-orbifolds.

Proposition 3.3.3. *Let $\mathcal{O}$ be a closed, orientable 2-orbifold $\mathcal{O}$, with k cone points $\{x_1, \ldots, x_k\} \subset |\mathcal{O}|$. For a singular point x_ℓ, write q_ℓ for its order. Then,*

$$\chi^{\mathrm{orb}}(\mathcal{O}) = \chi(|\mathcal{O}|) - \sum_{i=1}^{k} \left(1 - \frac{1}{q_i}\right).$$

Proof. About each cone point x_i, we choose, by means of a linear model pair, an orbifold 2-disk $\widehat{D}_i/G_{x_i}$, and we ensure the collection of such is pairwise disjoint. We may define,

$$N := \mathcal{O} - \bigsqcup_{i=1}^{k} \widehat{D}_i/G_{x_i}.$$

Then N is a manifold, whose boundary consists of circles bounding cones, and of course $\chi(S^1) = 0$. In particular, by additivity, we have,

$$\chi^{\mathrm{orb}}(\mathcal{O}) = \chi(N) + \sum_{i=1}^{k} \frac{1}{q_i}.$$

On the other hand, as a topological space, the quotient $\widehat{D}_i/G_{x_i}$ is homeomorphic to a 2-disk, and so $\chi(|\mathcal{O}|) = \chi(N) + k$. Thus,

$$\chi^{\mathrm{orb}}(\mathcal{O}) = \chi(|\mathcal{O}|) - \sum_{i=1}^{k} \left(1 - \frac{1}{q_i}\right).$$

We are done. □

Remark 3.3.1. For $\mathcal{O}$ above, one can view $\mathcal{O} - \Sigma(\mathcal{O})$ as N with open collars attached to ∂N.

Example 3.3.4 (Euler Characteristic of a Bad Orbifold)**.** Consider Thurston's teadrop $\mathcal{O}$; we have a single cone point of order $p > 1$ and $|\mathcal{O}| \cong S^2$. In particular,

$$\chi^{\mathrm{orb}}(\mathcal{O}) = 1 + \frac{1}{p}.$$

Example 3.3.5. This example is inspired from Ian Agol's answer here. Any rational number $m/n \in \mathbb{Q}$ can be obtained as the orbifold Euler characteristic of some orbifold. Indeed, consider the sphere S^{2n} under a rotation action by $\mathbb{Z}_n$, this is an orbifold with characteristic $1/n$. Then take the product of any manifold with Euler characteristic $m \in \mathbb{Z}$ (consider a connected sum of disks, for example) to obtain an orbifold with orbifold Euler characteristic m/n.

3.4 Satake's Poincaré-Hopf and Gauss-Bonnet for Orbifolds

The Euler characteristic of a manifold is a topological invariant which appears in several highlighted theorems, such as the generalised Gauss-Bonnet Theorem, and the classical Poincaré-Hopf Index Theorem, amongst other things. In the previous section, we defined the orbifold

Euler characteristic. The purpose of this section is to illustrate its corresponding role in the orbifold versions of the previous two theorems, both of which were proved by Satake in the 50's. Before we proceed, let us motivate our forthcoming definitions. Consider a closed orientable 2-orbifold $\mathcal{O}$. Let $v : \mathcal{O} \to T\mathcal{O}$ be a vector field on $\mathcal{O}$. Consider a local model pair $(\widehat{D}, G_x)$ uniformising a neighbourhood of a cone point $x \in \Sigma(\mathcal{O})$, where $\widehat{D}$ is 2-disk, and G_x acts as $\mathbb{Z}_k$ by rotations, for $k = |G_x|$. The restriction (and then pullback) of v to this model pair yields a $\mathbb{Z}_k$-invariant vector field $\widehat{v} : \widehat{D} \to T\widehat{D}$. In particular, we have, for $\mathbb{Z}_k = \langle g \rangle$, $\widehat{v}(0) = \widehat{v}(g(0)) = g\widehat{v}(0)$. Therefore the vector field v necessarily vanishes at the cone point x. We thus conclude (as in [Ham18], Proposition 2.3)

Proposition 3.4.1. *Let $\mathcal{O}$ be a closed connected orientable 2-orbifold. Let v be a vector field on $\mathcal{O}$. Then v necessarily vanishes at all the cone points of $\mathcal{O}$. In particular, if v is nowhere vanishing, then $\mathcal{O}$ is a torus.*

This is intuitively clear, we cannot comb our field around a cone point without losing some form of regularity. On the other hand, a compatible triangulation of $\mathcal{O}$ looks like $V - E + F$, except some vertices are weighted with their isotropy. In particular, we see that, we should, in developing the notion of an 'index' for a vector field on an orbifold, consider weighting a local index calculation with isotropy. This is what we shall do.

Definition 3.4.1. *Fix an arbitrary orbifold $\mathcal{O}$. Let $v : \mathcal{O} \to T\mathcal{O}$ be a vector field on $\mathcal{O}$ with an isolated zero $x \in \mathcal{O}$. Let $(\widehat{U}, G)$ be a model pair about x. Write $v_{\widehat{U}} : \widehat{U} \to T\widehat{U}$ for the corresponding vector field, whom has a zero at $\widehat{x}$. We define the orbifold index of v at x, denoted* $\operatorname{orb}\operatorname{ind}_v(x)$*, to the the rational number,*

$$\operatorname{orb}\operatorname{ind}_v(x) := \frac{1}{|G_x|}\operatorname{ind}_{v_{\widehat{U}}}(\widehat{x}) \in \mathbb{Q}.$$

This definition is well-defined, for the index is a diffeomorphism invariant (and vector fields between charts are related via pullback).

Let us provide an elementary example.

Example 3.4.1. Consider a G-invariant vector field $v : \mathbb{R}^2 \to \mathbb{R}^2$, where G is a finite subgroup of $O(2)$. Suppose v has an isolated non-degenerate zero at the origin. The origin has isotropy $|G|$. Then, $\operatorname{orb}\operatorname{ind}_v(0) = \pm\frac{1}{|G|}$. We present this example without compactness, but it is clear that suitable adjustments can be made.

We now present Satake's Poincaré-Hopf Index Theorem, as given in ([SAT57], Theorem 3).

Theorem 3.4.1. *For a closed codimension 2 orbifold $\mathcal{O}$ and a vector field $v : \mathcal{O} \to T\mathcal{O}$ with isolated singularities $x_1, \ldots, x_q \in \mathcal{O}$, we have,*

$$\chi^{\mathrm{orb}}(\mathcal{O}) = \sum_{i=1}^{q} \operatorname{orb}\operatorname{ind}_v(x_i).$$

Proof. We work here only with a global quotient, and refer the reader to [SAT57] for the general case. Let M be a connected, closed smooth manifold. Let G be a finite group acting on M by diffeomorphisms. Let v be a vector field on the global quotient orbifold M/G, with isolated zeros $x_1, \ldots, x_q \in M/G$. To give a vector field v on M/G is to give a G-invariant vector field $\overline{v}$ on M (here, $\overline{v}$ is the pullback of v by the orbifold covering $\pi : M \to M/G$ of degree $|G|$). Each zero x in M/G of v corresponds to $|G|/|G_x|$ zeroes of $\overline{v}$ in M, each of which have the same index with respect to $\overline{v}$, for the index is a local diffeomorphism invariant. By the classical Poincaré-Hopf Index Theorem, we deduce that,

$$\chi(M) = \sum_{i=1}^{q} \sum_{y \in \pi^{-1}(x_i)} \operatorname{ind}_{\overline{v}}(y) = \sum_{i=1}^{q} \frac{|G|}{|G_{x_i}|} \operatorname{ind}_{\overline{v}}(x_i) = |G| \sum_{i=1}^{q} \operatorname{orb}\operatorname{ind}_v(x_i).$$

Therefore,

$$\chi^{\text{orb}}(M/G) = \sum_{i=1}^{q} \operatorname{orb}\operatorname{ind}_v(x_i).$$

□

Corollary 3.4.1. *Let $\mathcal{O}$ be a closed orientable 2-orbifold. Let v be a vector field on $\mathcal{O}$ with isolated zeroes $x_1, \ldots, x_q \in \mathcal{O}$. By renumbering if necessary, write $x_1, \ldots, x_k$ for the cone points of $\mathcal{O}$, orders $q_1, \ldots, q_k$ respectively. Then,*

$$\chi(|\mathcal{O}|) = \sum_{i=1}^{q} \operatorname{orb}\operatorname{ind}_v(x_i) + \sum_{i=1}^{k} \left(1 - \frac{1}{q_i}\right).$$

Remark 3.4.1. Although here we have specialised to tangent bundles and sections thereof, by making appropriate definitions, it seems one can form a corresponding generalisation of those core intersection theoretic results outlined in Chapter 1.

Let us point out that there are certain facts that hold true in the smooth category, which are patently false for orbifolds. For example, if M is a closed manifold with zero Euler characteristic, then M admits a nowhere vanishing vector field (see Theorem 1.2.4). This is false for orbifolds. For example, consider the pillowcase orbifold $\mathbb{T}^2/\mathbb{Z}_2$. This is a closed 2-orbifold with four cone points. It is a global quotient of a surface of genus 1, so $\chi^{\text{orb}}(\mathbb{T}^2/\mathbb{Z}_2) = 0$. Yet, as discussed earlier, any vector field necessarily vanishes at the cone points. Later on we'll see how the notion of a *line field* can 'resolve' an issue like this, for the special case where the action linearised yields that of a scalar matrix.

We now furnish some basic examples of Satake's Poincaré-Hopf Theorem.

Example 3.4.2. This example extends Example 1.2.1 into the setting of orbifolds. Consider the 2-sphere under an action of the cyclic group of order k by rotations. To be precise, let, as usual, (x, y, z) be the standard global coordinates on $\mathbb{R}^3$ and S^2 the 2-sphere, realised as an embedded submanifold of $\mathbb{R}^3$. Let N and S denote the North and South poles respectively. Define a vector vield $v : S^2 \to TS^2$ on the 2-sphere by $v = -y\partial_x + x\partial_y$. This vector field has zeroes at the poles, and both are non-degenerate with index +1. Let $G := \mathbb{Z}_k$ be identified with the k-th roots of unity. Define the action of G on S^2 by

$$e^{it} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for suitable t. The quotient $S^2/\mathbb{Z}_k$ is an orbifold. The poles are the fixed points of the action, in particular, the poles are cone points of order k. We can easily check directly that v is a G-invariant vector field on S^2 (the differential of a linear map is canonically identified with the linear map itself, and it reduces to a simple matrix calculation). We thus have a vector field on the orbifold $S^2/\mathbb{Z}_k$. Now, $\chi^{\mathrm{orb}}(\mathbb{S}^2/\mathbb{Z}_k) = \frac{2}{k}$. On the other hand,

$$\operatorname{orb}\operatorname{ind}_v(N) = \frac{1}{k}, \qquad \operatorname{orb}\operatorname{ind}_v(S) = \frac{1}{k}.$$

This checks out with Satake's Poincaré-Hopf Index Theorem.

Example 3.4.3 (Symmetric Product)**.** Let M be a closed smooth manifold, and v a vector field on M with a single isolated zero at $x \in M$. Let $\mathbb{Z}_2$ act on $M \times M$ by permutation of coordinates. There is $\mathbb{Z}_2$-invariant vector field on $M \times M$ induced by v, with a singularity at $(x, x) \in M \times M$, a point which has local group of order 2. Call this vector field v^2. Easily,

$$\chi^{\mathrm{orb}}(M \times M/\mathbb{Z}_2) = \frac{\chi(M)^2}{2} = \frac{\operatorname{ind}_v^2(x)}{2} = \operatorname{orb}\operatorname{ind}_{v^2}(x, x)$$

where we recall that for suitable mappings f, g, one has $\deg(f \times g) = \deg(f)\deg(g)$.

We'll conclude this section with a brief discussion regarding the Gauss-Bonnet Theorem. Let us fix a closed orientable Riemannian 2-orbifold. Over each model pair $(\widehat{U}, G)$, there is a notion of curvature and area element. As embeddings of charts (and the local actions themselves) are by isometries with respect to the local metrics, we obtain a well-defined curvature function K on $\mathcal{O}$, together with our canonical area form, denoted dA. We have the following Gauss-Bonnet Theorem,

Theorem 3.4.2. *Let $\mathcal{O}$ be a closed orientable Riemannian 2-orbifold, with $q_1, \ldots, q_k$ denoting orders of the cone points of $\mathcal{O}$. Write K for our curvature functon, and dA for our area element. Then,*

$$\int_{\mathcal{O}} K\,dA = 2\pi\chi^{\mathrm{orb}}(\mathcal{O}).$$

In particular,

$$\int_{\mathcal{O}} K\,dA + 2\pi\sum_{i=1}^{k}\left(1 - \frac{1}{q_i}\right) = 2\pi\chi(|\mathcal{O}|).$$

This can be proved by excising 'geodesic' cone neighbourhoods about the singularities, leaving a manifold with boundary, and then applying the usual Gauss-Bonnet Theorem for manifolds with boundary, together with a limiting process, and employing additivity of the orbifold Euler-Satake characteristic. What is particularly interesting about the second equality above is that the second summand on the left can be viewed as the an error correction, namely the "curvature concentrated at cone points". For more details, we refer the reader to ([Coo00], Page 31, Proposition 2.17).

Example 3.4.4 (Flat Orbifolds)**.** A flat orbifold is one whom admits a metric of zero curvature away from singular points. A family of examples which has generated some interest are those flat orbifolds whose underlying topological space is an n-sphere. We already are well-versed with an example. The pillowcase $\mathbb{T}^2/\mathbb{Z}_2$ (or more generally, any Toroidal orbifold) is a flat orbifold, whose underlying topological space is homeomorphic to S^2.

Finally, let us make general commentary following ([BG07], Page 121).

Remark 3.4.2. Let $\mathcal{O}$ be an orientable closed $2n$-dimensional orbifold. One can define the Euler class $e^{\mathrm{orb}} \in H^{2n}(\mathcal{O}, \mathbb{Q})$, and this cohomology class can be represented by the top invariant curvature form Ω of the Riemannian curvature. In this language, Satake's result may be phrased as,

$$\chi^{\mathrm{orb}}(\mathcal{O}) = \langle e^{\mathrm{orb}}, [\mathcal{O}] \rangle = \int_{\mathcal{O}} \Omega$$

where $[\mathcal{O}] \in H_{2n}(\mathcal{O}, \mathbb{Q})$ denotes the fundamental class of $\mathcal{O}$. In order to make this precise, one needs to discuss characteristic classes, connections, and Chern-Weil theory (amongst other details) on orbifolds, all of which can be done. We refer the reader to Chapter 4 of [BG07] for more details.

3.5 Orbifolds as Groupoids

As briefly mentioned in the Introduction and start of this Chapter, there are more or less two ways to think about an orbifold. Previously, we defined things in the first/classical way, using charts and atlases. It is clear that this quickly becomes clumsy, and it is easy to write down wrong definitions (consider even the simplest notion, that of a smooth map). It turns out that there is an alternative, more high powered perspective of an orbifold, and that is as a type of differentiable stack. Whilst this approach (at least to the author) forgoes some geometric intuition, it makes up for it in elegance once it is up an running. Rather than proceed directly with the language of stacks, we shall opt to introduce the incarnation of orbifolds as groupoids, and demonstrate, briefly, how one passes to the standing definition of an orbifold. Ultimately, then, we shall think of as a stack as an 'equivalence class' of groupoids. To explain the need for equivalence at once, recall from Example 3.2.4 that in certain cases, one must consider different presentations of an orbifold in order to get enough morphisms. This phenomena highlights the need for an ability to consider alternate presentations. Throughout this section, we follow closely section 4.3 of [BG07], and section 1.4 of [ALR07].

Definition 3.5.1. *A groupoid $\mathcal{G}$ is a (small) category in which every arrow is invertible.*

To be a little bit more precise, a groupoid $\mathcal{G}$ consists of a set of objects G_0 and set of arrows G_1, with five natural structure maps; the source and target maps $s, t : G_1 \rightrightarrows G_0$, a composition map $m : G_1 \times_{G_0} G_1 \to G_1$, a unit map $u : G_0 \to G_1$, and finally an inverse map $i : G_1 \to G_1$. For an arrow $g \in G_1$ with $s(g) = x$ and $t(g) = y$ we shall write $g : x \to y$. We write g^{-1} for $i(g)$ and $g \circ h$ for $m(g, h)$. The usual identities must be satisfied (see for example section 3 of [MP97]). A topological groupoid is a groupoid in which both the set of objects and arrows are topological spaces, and the structure maps are continuous. Going one step further,

Definition 3.5.2. *A Lie groupoid $\mathcal{G}$ is a groupoid whose objects G_0 and arrows G_1 both admit the structure of smooth manifolds, with the additional property that the structure maps of $\mathcal{G}$ are all smooth and further, our source and target maps $s, t : G_1 \to G_0$ are submersions.*

That the source and target maps are required to be submersions is so that the domain of the multiplication map $G_1 \times_{G_0} G_1$ is a manifold, and consequently it makes sense to say that the composition map m is smooth. We point out that sometimes it is useful to think of G_0 as a base space, and the groupoid $\mathcal{G}$ is written as $G_1 \implies G_0$. Let us provide some examples of Lie groupoids.

Example 3.5.1 (Action Groupoid)**.** Let a smooth manifold M be equipped with a smooth left action of a Lie group K. We define a Lie groupoid $K \ltimes M$ with objects $(K \ltimes M)_0 = M$ and arrows $(K \ltimes M)_1 = K \times M$. The source map $s : K \times M \to M$ is projection onto the second factor, the target map $t : K \times M \to M$ is the group action. Thus arrow $(k, x) \in (K \ltimes M)_1$ is of the form,

$$x \xrightarrow{\quad (k,x) \quad} k \cdot x$$

The composition map m is defined in the natural way, with respect to our action. We call such a Lie groupoid an action groupoid. Note that by specialising our Lie group or manifold in the obvious way, we may view a manifold as a Lie groupoid (the so called 'unit groupoid', whose arrows are all units), or alternatively, a Lie group as a Lie groupoid (the set of objects being a single point).

Example 3.5.2 (A Groupoid of Germs; see Example 5.32 in [MM03])**.** Let M be a smooth manifold. By a 'local transition' on M, we mean a diffeomorphism between two open subsets of M. For the set of all local transitions on M, we write C^∞_M. A pseudogroup of local transitions on M is a subset P of local transitions on M for which,

1. $\mathrm{Id}_V \in P$ for any open set $V \subseteq M$.
2. If $f, f' \in P$, then the composition $f' \circ f|_{f^{-1}(\mathrm{dom}(f'))} \in P$ and inverse $f^{-1} \in P$.
3. If f is a transition on M and (V_α) is an open cover of $\mathrm{dom}(f)$ for which each restriction $f|_{V_\alpha} \in P$, then $f \in P$.

For a pseudogroup of local transitions P, we can associate a groupoid $\Gamma(P)$ whose objects are points of M, and arrows between $x, y \in M$ are given by

$$\Gamma(P)_1(x, y) = \{\mathrm{germ}_x f \mid f \in P, x \in \mathrm{dom}(f),\ f(x) = y\}$$

Multiplication is defined naturally, by composing transitions. The set of arrows $\Gamma_1(P)$ may be equipped with the sheaf topology, upon doing so, the groupoid $\Gamma(P)$ becomes effective.

Example 3.5.3 (Fundamental Groupoid)**.** Suppose M is a connected manifold. The fundamental groupoid of M, denoted $\Pi(M)$, has as objects points of M, $\Pi(M)_0 = M$. An arrow $g \in \Pi(M)_1$ with $s(g) = x$ and $t(g) = y$ is given by a homotopy class of paths from x to y. Note then that composition is defined naturally, and inversion of an arrow is simply given by walking along in the opposite direction. If we consider all arrows with source and target $x \in M$ (i.e. self loops of x), then we capture the fundamental group of M, based at x, $\pi_1(M, x)$.

In the previous example we saw that self loops of an object had significance; we can make some general definitions and remarks regarding such loops.

Definition 3.5.3. *Let $\mathcal{G}$ be a Lie groupoid with objects G_0 and arrows G_1. For an object $x \in G_0$, the set of all arrows with source and target x is called (because of a canonical group structure) the isotropy group (or local group) at x, and is denoted by G_x. The set $ts^{-1}(x)$ of targets of arrows with source x is called the orbit of x. The orbit space $|\mathcal{G}|$ of $\mathcal{G}$ is by definition of the quotient space $G_0/\sim$ where $x \sim y$ if and only if x and y are in the same orbit (i.e. there is an arrow from x to y). We call $\mathcal{G}$ a (groupoid) represenation of $|\mathcal{G}|$.*

In order to make the connection to orbifolds, we must restrict our attention to classes of Lie groupoids. The classes of interest are as follows;

Definition 3.5.4. *Let $\mathcal{G}$ be a Lie groupoid, with set of objects G_0 and arrows G_1.*

1. *We call $\mathcal{G}$ proper if the map $(s, t) : G_1 \to G_0 \times G_0$ is a proper map (i.e. the preimage of any compact set is compact).*

2. *We call $\mathcal{G}$ a foliation groupoid if for each $x \in G_0$, the isotropy group G_x is discrete.*

3. *We call $\mathcal{G}$ étale if the source and target maps $s, t : G_1 \rightrightarrows G_0$ are local diffeomorphisms.*

Note that if $\mathcal{G}$ is étale, then $\dim \mathcal{G} = \dim G_0 = \dim G_1$ is well-defined. Next, for an arbitrary Lie groupoid $\mathcal{G}$ each isotropy group G_x is a Lie group. To see this, we note that $G_x = (s, t)^{-1}(x, x) = s^{-1}(x) \cap t^{-1}(x) \subset G_1$ and that, by hypothesis, s and t are submersions (i.e. their differential is everywhere onto), which implies that G_x is a smooth submanifold of G_1 (refer to Theorem 9.9 of [Tu]), the assumption that our structure maps are smooth implies that the natural group operations are smooth on G_x, so it is a Lie group. If we assume that $\mathcal{G}$ is proper, then each G_x is clearly a compact Lie group. A compact discrete Lie group is a finite group, so that if we assume $\mathcal{G}$ is a proper foliation Lie groupoid, then each G_x is a finite group. Clearly an étale Lie groupoid is a foliation groupoid, and so we have the following proposition,

Proposition 3.5.1. *If $\mathcal{G}$ is a proper étale Lie groupoid, then for each $x \in G_0$, the isotropy group G_x is finite.*

The reason for restricting to a special case in the above proposition will soon become apparent. A useful property of proper étale Lie groupoids is as follows.

Construction 3.5.1. Let $\mathcal{G}$ be a proper étale Lie groupoid. We shall describe a way in which the (finite) isotropy group G_x of $x \in G_0$ acts as a group of diffeomorphisms on a neighbourhood of x. Let $g \in G_x$ be fixed, then, because s and t are local diffeomorphisms, there exists an open neighbourhood V_g of $g \in G_1$ for which both s and t map V_g diffeomorphically onto an open neighbourhood U_x of x. Let $j : U_x \to V_g$ be the local inverse to the source map $s|_{V_g} : V_g \to U_x$. Define a diffeomorphism $\widehat{g} = t|_{V_g} \circ j : U_x \to U_x$. We obtain a group homomorphism $G_x \to \mathrm{Diff}(U_x)$ defined by $g \to \widehat{g}$. In this way, an arrow $g : x \to x$ yields a well defined germ of a diffeomorphism about x.

Definition 3.5.5. *An orbifold groupoid $\mathcal{G}$ is a proper étale Lie groupoid. We call an orbifold groupoid $\mathcal{G}$ effective if, for each $x \in G_0$, there exists an open neighbourhood U_x about x such that the associated group homomorphism $G_x \to \mathit{Diff}(U_x)$ is injective.*

In what is to come, we shall justify the title 'orbifold groupoid'. In order to do so, we need the notion of Morita equivalence. First, a few definitions. Given that we view a Lie groupoid as a sort of 'smooth category', a homomorphism of Lie groupoids should be a smooth functor. Precisely,

Definition 3.5.6. *A homomorphism of Lie groupoids $\phi : \mathcal{K} \to \mathcal{G}$ consists of a pair of smooth maps $\phi_0 : K_0 \to G_0$, $\phi_1 : K_1 \to G_1$ which together commute with all the structure maps.*

If homomorphisms are functors, then we must have natural transformations. Let us quickly mention this, for completeness. If $\phi, \psi : \mathcal{K} \rightrightarrows \mathcal{G}$ are homomorphisms of Lie groupoids, a natural transformation α from ϕ to ψ, denoted $\alpha : \phi \implies \psi$, is given by a smooth map $\alpha : K_0 \to G_1$ for which $s \circ \alpha = \phi_0$ and $t \circ \alpha = \psi_0$. By natural, it is meant that if $k : x \to x'$ is an arrow in K_1, the following diagram commutes,

$$\begin{tikzcd} \phi_0(x) \arrow[r, "\phi_1(k)"] \arrow[d, "\alpha(x)"'] & \psi_0(x) \arrow[d, "\alpha(x')"] \\ \psi_0(x) \arrow[r, "\psi_1(k)"'] & \psi_0(x') \end{tikzcd}$$

Definition 3.5.7. *A homomorphism $\phi : \mathcal{K} \to \mathcal{G}$ of Lie groupoids is called an equivalence if,*

1. *(Essentially Surjective) The map*

$$t\pi_1 : G_{1s} \times_\phi K_0 \to G_0$$

defined on $G_{1s} \times_\phi K_0 = \{(g,k) \,|\, s(g) = \phi_0(k)\}$ is a surjective submersion.

2. *(Fully faithful) The diagram*

$$\begin{tikzcd} K_1 \arrow[r, "\phi_1"] \arrow[d, "(s{,}t)"'] & G_1 \arrow[d, "(s{,}t)"] \\ K_0 \times K_0 \arrow[r, "\phi_0 \times \phi_0"'] & G_0 \times G_0 \end{tikzcd}$$

is a fibered product of manifolds.

If we unwrap these conditions; the first means that any object in G_0 can be connected by an arrow in G_1 to the image of ϕ_0. The second condition means that ϕ produces a diffeomorphism

$$K_1(y,z) \to G_1(\phi_0(y), \phi_0(z))$$

from the space of arrows between y and z in K_0 and the space of arrows between $\phi_0(y)$ and $\phi_0(z)$ in G_0. Thus an equivalence is a smooth equivalence of categories. We call ϕ strong if $\phi_0 : K_0 \to G_0$ is a surjective submersion. We point out that an equivalence ϕ yields a homeomorphism of the underlying orbit spaces, $|\phi| : |\mathcal{K}| \to |\mathcal{G}|$. We now come to the notion of Morita equivalence.

Definition 3.5.8. *We say that two Lie groupoids $\mathcal{H}$ and $\mathcal{G}$ are Morita equivalent if there exists a third Lie groupoid $\mathcal{K}$ and two equivalences,*

$$\mathcal{H} \stackrel{\psi}{\leftarrow} \mathcal{K} \stackrel{\phi}{\rightarrow} \mathcal{G}$$

Let us make two remarks. If $\phi : \mathcal{K} \to \mathcal{G}$ is an equivalence, then $\mathcal{K}$ is Morita equivalent to $\mathcal{G}$ via strong equivalences (see Definition 1.43 in [ALR07]). If ϕ is an equivalence of orbifold groupoids, then $\phi_0 : K_0 \to G_0$ is a local diffeomorphism (see Lemma 2.1 in [ALR07]). We will now explain the connection between our standing definition of an orbifold (a space with charts) and the content of Definition 3.5.5, in which we called a proper étale Lie groupoid an 'orbifold groupoid'. We shall pass from an orbifold to an orbifold groupoid, and vice versa. Upon considering Morita equivalent Lie groupoids and isomorphic orbifolds, this passage is well defined.

First, we will show how one goes from an effective orbifold to an effective orbifold groupoid. Let $\mathcal{X} = (X,\mathcal{U})$ be an effective orbifold with a fixed atlas $\mathcal{U} = \{(\widehat{U}_i, G_i, \phi_i)\}$ on X. Define $\widehat{U} = \coprod_i \widehat{U}_i$. Let $\mathcal{P}_{\mathcal{X}}$ denote the pseudogroup of local diffeomorphisms of $\widehat{U}$ generated by the embeddings and their inverses. Let $\mathcal{G}(\widehat{U})$ denote the groupoid of germs of diffeomorphisms of this pseudogroup $\mathcal{P}_{\mathcal{X}}$, as in Example 3.5.2, i.e. objects $\widehat{U}$ and arrows germs of the embeddings. Consider the projection map $\phi : \widehat{U} \to X$ defined by taking the union of the ϕ_i. If $x_i \in \widehat{U}_i$ and $x_j \in \widehat{U}_j$ are such that $x_i \sim x_j$, then there is an embedding $\lambda_{ij} : \widehat{U}_i \to \widehat{U}_j$ for which $\lambda_{ij}(x_i) = x_j$, then, because $\phi_j \circ \lambda_{ij} = \phi_i$, we see that $\phi(x_i) = \phi(x_j)$. This implies that ϕ yields a well-defined map from the space of orbits $|\mathcal{G}(\widehat{U})| \to X$. In this sense, we say that the groupoid $\mathcal{G}(\tilde{U})$ represents the orbifold X. (A nice point to skip to now would be Definition 3.5.9). In fact, even more is true,

Proposition 3.5.2. *Let $\mathcal{X} = (X,\mathcal{U})$ be an effective orbifold with a fixed atlas $\mathcal{U}$, then $\mathcal{G}(\widehat{U})$ is an effective orbifold groupoid. Moreover, if $\mathcal{X}' = (X',\mathcal{U}')$ is another effective orbifold with a fixed atlas $\mathcal{U}'$, then $\mathcal{G}(\widehat{U})$ is Morita equivalent to $\mathcal{G}(\widehat{U}')$ if and only if the orbifolds $\mathcal{X}$ and $\mathcal{X}'$ are isomorphic.*

Proof. See Proposition 5.29 in [MM03]. □

Now we will show how to go from an effective orbifold groupoid to an effective orbifold. Let $\mathcal{G}$ be an effective orbifold groupoid. By Proposition 3.5.1, for each $x \in G_0$, the isotropy group G_x is finite. Futhermore, for any $x \in G_0$, there exists an open neighbourhood U_x of x in G_0 with an action of G_x such that there is an isomorphism of étale Lie groupoids, $\mathcal{G}|_{U_x} \cong G_x \ltimes U_x$ (see for example, Corollary 5.31 in [MM03]). This allows us to construct an orbifold atlas on the orbit space $|\mathcal{G}|$, which is both Hausdorff and paracompact. Let $\pi : G_0 \to |\mathcal{G}|$ denote the quotient projection. For $x \in G_0$, we choose the neighbourhood U_x so that we have a diffeomorphism $\phi_x : U_x \to \widehat{U}_x \subseteq \mathbb{R}^n$, for $n = \dim \mathcal{G}$. Let G_x act on $\widehat{U}_x$ so that ϕ_x is G_x-equivariant. An orbifold atlas $\mathcal{U}$ on $|\mathcal{G}|$ consists of charts of the form, $\{(\widehat{U}_x, G_x, \pi \circ \phi_x^{-1})\}$. Embeddings of charts look as follows. If V_y and U_x are two such neighbourhoods and $V_y \overset{\iota_y}{\hookrightarrow} U_x$, then the embedding

$$\lambda_{xy} : (\widehat{V}_y, G_y, \pi \circ \psi_y^{-1}) \to (\widehat{U}_x, G_x, \pi \circ \phi_x^{-1})$$

is defined by $\lambda_{xy} = \phi_x \circ \iota_y \circ \psi_y^{-1}$. Note that the resulting orbifold represents the groupoid $\mathcal{G}$, for its underlying topoogical space is exactly $|\mathcal{G}|$. Our discussion may be summarised, along with a Theorem 1.45 from [ALR07] (originally appearing in [MP97]),

Theorem 3.5.1. *If $\mathcal{G}$ is an effective orbifold groupoid, then its space of orbits $|\mathcal{G}|$ admits the structure of an effective orbifold. Two effective orbifold groupoids $\mathcal{G}$ and $\mathcal{H}$ represent the same effective orbifold up to isomorphism if and only if they are Morita equivalent.*

This roughly describes the bridge between the two vantage points. Equipped with our current theory, one may provide a new definition of an orbifold (one which makes it easy to drop the condition of an effective action). First, we specify the data akin to an atlas.

Definition 3.5.9. *An orbifold structure on a paracompact Hausdorff topological space X is given by an orbifold groupoid $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to X$. If $\phi : \mathcal{K} \to \mathcal{G}$ is an equivalence, then $|\phi| : |\mathcal{K}| \to |\mathcal{G}|$ is a homeomorphism, and $f \circ |\phi| : |\mathcal{K}| \to X$ is said to define an equivalent orbifold structure on X.*

A modern definition is now as follows.

Definition 3.5.10. *An orbifold $\mathcal{X}$ is a paracompact Hausdorff space X equipped with an equivalence class of orbifold structures. A specific choice of structure is given by the datum of an orbifold groupoid $\mathcal{G}$, and a homeomorphism $f : |\mathcal{G}| \to X$, called a presentation of $\mathcal{X}$.*

3.6 Intersection Theory on Deligne-Mumford Stacks

In this section, we wish to very briefly present interesection theory on an orbifold, contrasting those results developed in Chapter 1. We shall follow the notes [Beh02] by Kai Behrend extremely closely, and omit several technical details. The purpose of this section is to simply demonstrate that key results in Chapter 1 admit vast generalisations. Furthermore, we are motivated by the following slogan,

Remark 3.6.1. "Topological stacks are the right formalism for dealing with orbifolds and topological groupoids" - Angelo Vistoli.

We shall take as an imprecise definition the following. A differentiable stack $\mathfrak{X}$ is a Morita equivalence class of Lie groupoids (or more precisely, is the quotient stack of a Lie groupoid $X_1 \rightrightarrows X_0$, denoted $[X_0/X_1]$). A choice of presenting Lie groupoid can be thought of as choosing an 'atlas', for the coarse space. If the presenting Lie groupoids are proper and étale, then we have an orbifold, a so called (differentiable) stack of Deligne-Mumford type. We remark that the usual way to proceed is by first defining a category fibered in groupoids (or a prestack, over Diff, in this case), then a stack (which is a prestack satisfying some descent data), and then equip ourselves with the data of an 'atlas', so becoming 'differentiable'. Then one establishes a passage to the description above. We shall not recall these definitions here, and be content with choosing presentation of our differentiable stack, and ensuring invariance under Morita equivalence. For details, we refer the reader to [Gin13] and [Beh02]. We may set up the de Rham cohomology of a differentiable stack $\mathfrak{X}$ as follows. First, we introduce the simplicial nerve of a Lie groupoid. Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. Recall that a simplicial manifold is a simplicial object in Diff. We associate a simplicial manifold to $X_1 \rightrightarrows X_0$ as follows. For $p \geq 0$, let X_p be the manifold consisting of composable sequences of arrows in X_1, of length p, that is,

$$X_p = X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \ (p \ \text{times}).$$

There are now $(p+1)$ canonical maps, *face maps*, $\partial_i : X_p \to X_{p-1},\ i = 0, \dots, p$ where ∂_0 leaves out the first arrow, ∂_p the last, and ∂_i, $1 \leq i \leq p-1$ is given by composing two succesive arrows, the pair located at $(i, i+1)$. As pointed out in [Beh02], one has the following relations; $\partial_i \partial_j = \partial_{j-1}\partial_i : X_p \to X_{p-2}$, $0 \leq i, j \leq p$. The data given above is summarised as,

$$\cdots \qquad \cdots \qquad X_2 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; X_1 \;\substack{\longrightarrow \\ \longrightarrow}\; X_0$$

On each manifold X_p, we can make sense of differential forms $\Omega^q(X_p)$ for $q \geq 0$. By pulling back the maps above, we obtain a 'cosimplicial set',

$$\Omega^q(X_0) \;\substack{\longrightarrow \\ \longrightarrow}\; \Omega^q(X_1) \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \Omega^q(X_2) \qquad \cdots\cdots\cdots$$

and therefore, we have an induced complex,

$$\Omega^q(X_0) \xrightarrow{\ \partial\ } \Omega^q(X_1) \xrightarrow{\ \partial\ } \Omega^q(X_2) \xrightarrow{\ \partial\ } \cdots$$

where the operator $\partial : \Omega^q(X_{p-1}) \to \Omega^q(X_p)$ is given by,

$$\sum_{i=0}^{p} (-1)^i \partial_i^* = \partial_1^* - \partial_2^* + \cdots + (-1)^p \partial_p^*.$$

We call this complex the Čech complex associated to the sheaf of q-forms and groupoid $X_1 \rightrightarrows X_0$. The corresponding cohomology groups are called Čech cohomology groups of $X_1 \rightrightarrows X_0$, denoted $H^k(X = X_1 \rightrightarrows X_0, \Omega^q)$. The first point to move to is check invariance (up to isomorphism) under Morita equivalence.

Proposition 3.6.1. *Any Morita equivalence of Lie groupoids, write $f_\bullet : X_\bullet \to Y_\bullet$, induces isomorphisms on Čech cohomology groups, $f^* : H^k(Y, \Omega^q) \to H^k(X, \Omega^q)$. In particular, Morita equivalent Lie groupoids have canonically isomorphic Čech cohomology groups with values in Ω^q.*

Proof. We refer the reader to Corollary 3 in [Beh02]. □

We can in particular, make the following definition.

Definition 3.6.1. *For a differentiable stack $\mathfrak{X}$, we define the Čech cohomology groups associated the sheaf of q-forms by,*

$$H^k(\mathfrak{X}, \Omega^q) = H^k(X_1 \rightrightarrows X_0, \Omega^q)$$

where $X_1 \rightrightarrows X_0$ is a presentation of $\mathfrak{X}$.

We now discuss the de Rham complex. The exterior differential $d : \Omega^q(X_p) \to \Omega^{q+1}(X_p)$ for all $p \geq 0$ yields a double complex.

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \\
\Omega^2(X_0) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^2(X_1) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^2(X_2) \arrow[r, "\partial"] \arrow[u, "d"] & \cdots \\
\Omega^1(X_0) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^1(X_1) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^1(X_2) \arrow[r, "\partial"] \arrow[u, "d"] & \cdots \\
\Omega^0(X_0) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^0(X_1) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^0(X_2) \arrow[r, "\partial"] \arrow[u, "d"] & \cdots
\end{tikzcd}$$

In order to obtain a singly graded complex we set (we refer the reader to [Bot82], Page 90, for the general construction),

$$C^n_{dR}(X) = \bigoplus_{p+q=n} \Omega^q(X_p)$$

and define a differential $\delta : C^n_{dR}(X) \to C^{n+1}_{dR}(X)$ by,

$$\delta(w) = \partial(w) + (-1)^p d(w)$$

for each $w \in \Omega^p_q(X)$ with $p + q = n$. The complex $C^\bullet_{dR}(X)$ is titled the de Rham complex of $X_1 \rightrightarrows X_0$, where we write $H^n_{dR}(X)$ for the de Rham cohomology groups. We now, as previously, move to the following.

Proposition 3.6.2. *Morita equivalent Lie groupoids have canonically isomorphic de Rham cohomology groups.*

Proof. We refer the reader to Definition 9 in [Beh02]. □

This allows us to make the following definition.

Definition 3.6.2. *For a differentiable stack $\mathcal{X}$, we define its de Rham cohomology*

$$H^k_{dR}(\mathfrak{X}) = H^k_{dR}(X_1 \rightrightarrows X_0)$$

where $X_1 \rightrightarrows X_0$ is a presentation of $\mathfrak{X}$.

Remark 3.6.2. A very natural question to ask is if orbifold 'geometry' is simply a special case of equivariant geometry, at least for effective orbifolds, for we know they can be expressed as quotients of a smooth manifold by a compact Lie group. As an answer to this question, it is pointed out in [Beh02] that the de Rham cohomology of a quotient stack $[M/G]$ is equal to its equivariant cohomology, namely write $H^*_G(X)$ for the equivariant cohomology of M/G, given via the Cartan complex. Then for G compact, there is an isomorphism $H^*_G(M) \to H^*_{dR}(G \times M \rightrightarrows M) = H^*_{dR}([M/G])$ where $G \times M \rightrightarrows M$ is the action/transformation groupoid.

We now recall the multiplicative structure given on the data above. Let $\Omega^q(X_p)$ and $\eta \in \Omega^{q'}(X_{p'})$. Then we set,

$$\omega \cup \eta = (-1)^{qp'} \pi_1^* \omega \wedge \pi_2^* \eta \in \Omega^{q+q'}(X_{p+p'})$$

where the maps $\pi_1 : X_{p+p'} \to X_p$ and $\pi_2 : X_{p+p'} \to X_{p'}$ are, to be informal, defined as follows. For π_1, we map an arrow sequence $(\phi_1, \dots, \phi_{p+p'})$ in $X_{p+p'}$ to the first p arrows $(\phi_1, \dots, \phi_p)$ and for π_2, to a sequence of length p', given by $(\phi_{p+1}, \dots, \phi_{p+p'})$. As discussed in [Beh02], there is a cup product, $H^n_{dR}(X) \otimes H^m_{dR}(X) \to H^{n+m}_{dR}(X)$ induced via $\cup$. Indeed, the following relation holds, $\delta(\omega \cup \eta) = \delta(\omega) \cup \eta + (-1)^{p+q} \omega \cup \delta(\eta)$. Recall that we call a differentiable stack $\mathfrak{X}$ proper if there is a presenting groupoid $X_1 \rightrightarrows X_0$ for which $(s,t) : X_1 \to X_0 \times X_0$ is proper, and the coarse underlying space is proper (that is, the diagonal map is proper).

Theorem 3.6.1 (Integration and Poincaré-Duality)**.** *Let $\mathfrak{X}$ be a proper oriented Deligne-mumford stack. We have a well-defined integral,*

$$\int_{\mathfrak{X}} : H^n_{dR}(\mathfrak{X}) \to \mathbb{R}$$

given by integration over a presenting groupoid, and furthermore, the induced pairing,

$$\begin{gathered} H^k(\mathfrak{X}) \otimes H^{n-k}(\mathfrak{X}) \to \mathbb{R} \\ \omega \otimes \tau \mapsto \int_{\mathfrak{X}} \omega \cup \tau. \end{gathered}$$

is non-degenerate.

Proof. We refer the reader to Corollary 25 in [Beh02], where compactly supported cohomology coincides with the usual de Rham cohomology as the stack $\mathfrak{X}$ is assumed proper. □

Remark 3.6.3. In order to define integration, one needs a partition of unity. The corresponding notion for a groupoid is defined in [Beh02], but there is a subtle aspect; their existence is not guaranteed unless one can pass to a Morita equivalent groupoid.

Let us consider a proper representable morphism $f : \mathfrak{D} \to \mathfrak{X}$ of oriented differentiable Deligne-mumford stacks. As in [Beh02], for presenting groupoids $X_1 \rightrightarrows X_0$ of $\mathfrak{X}$ and $Y_1 \rightrightarrows Y_0$ of $\mathfrak{D}$, this is given by the data of a morphism $f_\bullet : X_\bullet \to Y_\bullet$ presenting f, for which our base change $f_0 : X_0 \to Y_0$ is a proper map, and the square

$$\begin{tikzcd} Y_1 \arrow[r, "s"] \arrow[d, "f_1"'] & Y_0 \arrow[d, "f_0"] \\ X_1 \arrow[r, "s"'] & X_0 \end{tikzcd}$$

is cartesian. Assume that both $\mathfrak{X}$ and $\mathfrak{D}$ are proper. We have a map $\Omega^q(X_p) \to \Omega^q(Y_p)$ given by pulling back differential forms, and an induced map $f^* : H^*(\mathfrak{X}) \to H^*(\mathfrak{D})$. For $\dim \mathfrak{D} = k$, we get a map,

$$H^k(\mathfrak{X}) \to \mathbb{R}$$
$$\gamma \mapsto \int_{\mathfrak{D}} f^*\gamma$$

and thus by Poincaré-duality, a class $\mathrm{cl}(\mathfrak{D}) \in H^{n-k}(\mathfrak{X})$, the 'class of $\mathfrak{D}$', generalising the Poincaré-dual of a submanifold, a tool we used extensively within the smooth category, to the category of orbifolds. For $\mathfrak{X}$ above, and our presentation $X_1 \rightrightarrows X_0$, we can consider the associated tangent bundles, giving the data of a Lie groupoid $TX_1 \rightrightarrows TX_0$, and the induced stack is called the tangent stack of $\mathfrak{X}$, and there is a canonical morphism $T\mathfrak{X} \to \mathfrak{X}$. In particular, as stated in [Beh02], by taking the class of the zero section, and pulling back (by the zero section) to $\mathfrak{X}$, we obtain a class $e(T_{\mathfrak{X}}) \in H^n(\mathfrak{X})$. The Euler number of $\mathfrak{X}$ is given by,

$$e(\mathfrak{X}) = \int_{\mathfrak{X}} e(T_{\mathfrak{X}}).$$

In particular, for two *representable* morphisms $\mathfrak{D} \to \mathfrak{X}$ and $\mathfrak{Z} \to \mathfrak{X}$, with complimentary dimension, we define their intersection number to be given by,

$$\int_{\mathfrak{X}} \mathrm{cl}(\mathfrak{D}) \cup \mathrm{cl}(\mathfrak{Z}) \in \mathbb{R}.$$

We now conclude this section, and close our chapter, with a statement generalising the result 1.1.1 in Chapter 1, to the category of orbifolds.

Theorem 3.6.2 (Intersection Theory on Orbifolds)**.** *Consider a cartesian diagram of proper differentiable stacks, of Deligne-Mumford type,*

$$\begin{tikzcd} \mathfrak{M} \arrow[r] \arrow[d] & \mathfrak{Z} \arrow[d] \\ \mathfrak{D} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

where all maps are proper and representable, and that for all $w \in \mathfrak{M}$*, we have* $T_{\mathfrak{M},w} = T_{\mathfrak{D},w} \cap T_{\mathfrak{Z},w} \subset T_{\mathfrak{X},w}$ *(a property which can be defined by pullback to a presentation* $X_0 \to \mathfrak{X}$*). Further, suppose that* $\dim \mathfrak{D} + \dim \mathfrak{Z} = \dim \mathfrak{X}$*, and* $\mathfrak{D}$ *and* $\mathfrak{Z}$ *intersect transversally (see [Beh02] for more details). Then,*

$$\int_{\mathfrak{X}} \mathrm{cl}(\mathfrak{D}) \cup \mathrm{cl}(\mathfrak{Z}) = \#\mathfrak{M} = \sum_{x \in \mathfrak{M}/\cong} \frac{1}{\#\mathrm{Aut}(x)}.$$

Proof. We refer the reader to Proposition 28 in [Beh02], and Example 26 in [Beh02]. □

Remark 3.6.4. Given the results above, we observe that one can generalise Satake's Poincaré-Hopf Theorem to an intersection theoretic result on orbifolds. Moreover, a Lefschetz fixed point theorem for orbifolds is given in [Beh02].

Chapter 4

Poincaré-Hopf for Orbiline Fields

We work with an effective codimension 2 orbifold $\mathcal{O}$ of dimension $n \geq 2$. Our definitions in this Chapter all align with those given in Chapter 3. We define a line field ξ on an orbifold $\mathcal{O}$ to be a section of the projectivised tangent orbibundle, denoted $PT\mathcal{O}$. A line field with a finite number of singularities is a line field defined on a suborbifold whose complement consists of a finite set of points. To each singularity, we assign a rational number called the orbifold projective index. From an intersection theory perspective, one may view this (locally) as an honest intersection number (oriented in even dimensions, mod 2 in odd), where we weight points with additional data, namely their isotropy. For a closed orbifold of even dimension, the sum of these rational numbers is shown to be equal to twice the orbifold Euler characteristic, an invariant compatible with orbifold covering space theory. In odd dimensions, the sum vanishes mod 2, and equality is obtained in this fashion. The results thus obtained represent a straightforward generalisation of the work of Crowley and Grant in [CG17].

4.1 Line Fields on an Orbifold

Fix a(n effective codimension 2) closed, and connected orbifold $\mathcal{O} = (O,\mathcal{U})$. Let $PT\mathcal{O} \to \mathcal{O}$ denote the projectivisation of the tangent orbibundle of $\mathcal{O}$ (recall Example 3.2.2).

Definition 4.1.1. *A line field ξ on $\mathcal{O}$ is a smooth section $\xi : \mathcal{O} \to PT\mathcal{O}$. Let $k \in \mathbb{N}$. We say that a line field ξ has singularities $\Sigma(\xi) := \{x_1, \ldots, x_k\} \subset |\mathcal{O}|$ if it is defined only on the suborbifold $\mathcal{O} - \{x_1, \ldots, x_k\}$ (see Definition 3.2.4).*

Let ξ be a line field on $\mathcal{O}$. The line field ξ induces a canonical a subbundle of the tangent orbibundle $L_\xi \subseteq T\mathcal{O}$. Fix a Riemannian metric on $\mathcal{O}$. By means of the metric, we may consider the associated sphere bundle $\pi : S(L_\xi) \to \mathcal{O}$ with zero-dimensional fiber, which is, in particular, a two sheeted orbifold covering (we refer the reader to Theorem 3.3.1). We construct an associated vector field v_ξ on $S(L_\xi)$ by means of a local definition. For each chart $\widehat{U}/G$ on $\mathcal{O}$, consider the associated sphere bundle $S(L_\xi^{\widehat{U}}) \to \widehat{U}$, the local lift of π. Let $(x,w) \in S(L_\xi^{\widehat{U}})$. Define $v_\xi^{\widehat{U}}$ on $S(L_\xi^{\widehat{U}})$ by,

$$v_\xi^{\widehat{U}}(x,w) = w \in T_x\widehat{U}$$

where we are using the identification $T_x\widehat{U} \cong T_{(x,w)}(S(L_\xi^{\widehat{U}}))$, for the fiber is zero dimensional. In particular, each local lift of π is a local diffeomorphism. Let $g \in G$. Then, $v_\xi^{\widehat{U}}(g\cdot(x,w)) = g\cdot w = g \cdot v_\xi^{\widehat{U}}((x,w))$ so that $v_\xi^{\widehat{U}}$ is G-equivariant. The collection of these vector fields are compatible with the induced embedding data on $S(L_\xi)$. We denote the induced vector field on $S(L_\xi)$ by v_ξ. The following is now immediate.

Proposition 4.1.1. *If $\mathcal{O}$ admits a nowhere vanishing vector field, then it admits a line field. On the other hand, if $\mathcal{O}$ admits a globally defined line field ξ, then $\chi^{\mathrm{orb}}(\mathcal{O}) = 0$, but $\mathcal{O}$ need not admit a nowhere vanishing vector field.*

Proof. The first statement is a triviality. If a closed orbifold admits a nowhere vanishing vector field, then its Euler-Satake characteristic vanishes. This is a consequence of Satake's Poincaré-Hopf Index Theorem (see, [SAT57], and [FS10]). Let ξ be a globally defined line field on $\mathcal{O}$. As above, there is an associated a double cover $S(L_\xi) \to \mathcal{O}$ together with a nowhere vanishing vector field v_ξ on $S(L_\xi)$. By the previous remark and the multiplicativity of the Euler-Satake characteristic under coverings, we have,

$$0 = \chi^{\mathrm{orb}}(S(L_\xi)) = 2\chi^{\mathrm{orb}}(\mathcal{O}).$$

To conclude, it suffices to construct an example where we have a globally defined line field, but no nowhere vanishing vector field. Let $\mathbb{T}^2 = S^1 \times S^1$ and $\mathbb{Z}_2 = \langle g \rangle$ act on $\mathbb{T}^2$ by conjugation on each coordinate. The quotient $\mathbb{T}^2/\mathbb{Z}_2$ is an orbifold with four singular points, each of local group $\mathbb{Z}_2$. We have $T(\mathbb{T}^2) \cong \mathbb{T}^2 \times \mathbb{R}^2$. We consider the line field ξ induced via the line subbundle $\mathbb{T}^2 \times \mathbb{R} \times \{0\}$. We observe that this induces a line field on the orbifold $\mathbb{T}^2/\mathbb{Z}_2$. Let v be a nowhere vanishing vector field on $\mathbb{T}^2/\mathbb{Z}_2$, tantamount to a $\mathbb{Z}_2$-invariant vector field on $\mathbb{T}^2$. Let x be a singular point. Then, $v(x) = v(g \cdot x) = g \cdot v(x) = -v(x)$, and so $v(x) = 0_x$. $\square$

Proposition 4.1.1 demonstrates that in some cases, the existence of globally defined line fields allow conclusions which the ordinary result for vector fields does not. This contrasts Corollary 2.0.1, where a manifold admits a line field if and only if it admits a nowhere vanishing vector field. We now define our local tools. Recall the definition of index about a zero from Chapter 3, Definition 3.4.1.

Definition 4.1.2. *Let $v : \mathcal{O} \to T\mathcal{O}$ be a vector field with an isolated zero at $x \in |\mathcal{O}|$. Let $\widehat{U}/G$ be a chart about x, $v_{\widehat{U}}$ the corresponding G-invariant vector field on $\widehat{U}$, and $\widehat{x} \in \widehat{U}$ a representative of x. We define the orbifold normal index of v at x, denoted* $\operatorname{orb\,ind}_v^{\perp}(x)$, *to be*

$$\operatorname{orb\,ind}_v^{\perp}(x) := \frac{1}{|G_x|}\operatorname{ind}_{v_{\widehat{U}}}^{\perp}(\widehat{x}) \in \mathbb{Q}.$$

The definition given is independent of the choices made, for the classical construction is invariant under diffeomorphisms, and the integer $|G_x|$ depends only on x.

Our notation for the index of a vector field on an orbifold is non-standard. We make a similar definition for the projective analogue.

Definition 4.1.3. *Let x be a singular point of a line field ξ on $\mathcal{O}$. Let $\widehat{U}/G$ be a chart about x, and let $\widehat{x} \in \widehat{U}$ be a representative of x. We define the orbifold projective index of ξ at x, denoted* $\operatorname{orb}\operatorname{p}\operatorname{ind}_\xi(x)$*, to be,*

$$\operatorname{orb}\operatorname{p}\operatorname{ind}_\xi(x) := \frac{1}{|G_x|}\operatorname{p}\operatorname{ind}_{\xi_{\widehat{U}}}(\widehat{x}) \in \mathbb{Q}$$

where $\xi_{\widehat{U}}$ is the corresponding G-invariant line field. We define the orbifold projective normal index of ξ at x, denoted $\operatorname{orb}\operatorname{p}\operatorname{ind}^\perp_\xi(x)$*, to be*

$$\operatorname{orb}\operatorname{p}\operatorname{ind}^\perp_\xi(x) := \frac{1}{|G_x|}\operatorname{p}\operatorname{ind}^\perp_{\xi_{\widehat{U}}}(\widehat{x}) \in \mathbb{Q}.$$

Similarly, the definitions given are well-defined, for the projective index is a diffeomorphism invariant. Note that the usual projective indices appearing above depend on the dimension of the orbifold.

Corollary 4.1.1. *Let v and ξ be as in Definitions 4.1.2, 4.1.3. Set $n := \dim \mathcal{O} \geq 2$. Then, we have,*

$$\begin{aligned}
\operatorname{orb}\operatorname{ind}^\perp_v(x) &= \operatorname{orb}\operatorname{ind}_v(x) + \frac{(-1)^{n-1}}{|G_x|} \\
\operatorname{orb}\operatorname{p}\operatorname{ind}^\perp_\xi(x) &= \begin{cases} \operatorname{orb}\operatorname{p}\operatorname{ind}_\xi(x) - 2/|G_x|, & \text{for } n \text{ even;} \\ \operatorname{orb}\operatorname{p}\operatorname{ind}_\xi(x) = 0 \in \mathbb{Z}/2, & \text{for } n \geq 3 \text{ odd.} \end{cases}
\end{aligned}$$

Proof. The relationships above are trivial consequences of the results developed in ([CG17], Lemmata 3.4, 3.8 and 3.9) by Crowley and Grant, and summarised in Chapter 2 as Lemmata 2.0.2, 2.0.3, all applied locally within an orbifold chart. For example, if $(\widehat{U}, G)$ is a model pair about a singularity x of a vector field v, and $v_{\widehat{U}}$ the corresponding vector field on $\widehat{U}$ with a singularity at $\widehat{x}$, one has

$$\begin{aligned}
\operatorname{orb}\operatorname{ind}^\perp_v(x) &= \frac{1}{|G_x|}\operatorname{ind}^\perp_{v_{\widehat{U}}}(\widehat{x}) \\
&= \frac{1}{|G_x|}\left(\operatorname{ind}_{v_{\widehat{U}}}(\widehat{x}) + (-1)^{n-1}\right) \\
&= \operatorname{orb}\operatorname{ind}_v(x) + \frac{(-1)^{n-1}}{|G_x|}.
\end{aligned}$$

□

We shall now state and prove the corresponding generalisation. Recall that for an odd-dimensional closed orbifold, its Euler-Satake characteristic necessarily vanishes (see [SAT57]).

Theorem 4.1.1. *Let $\mathcal{O}$ be a closed orbifold of dimension at least 2. Let ξ be a line field on $\mathcal{O} - \{x_1, \ldots, x_q\}$. Then,*

$$2\chi^{\text{orb}}(\mathcal{O}) = \sum_{j=1}^{q} \operatorname{orb}\operatorname{p}\operatorname{ind}_\xi(x_j) \in \mathbb{Q}$$

where equality is congruence modulo 2 in odd dimensions.

Proof. If $n := \dim \mathcal{O}$ is odd, then the result follows at once from Corollary 4.1.1. Suppose now that $n \geq 2$ is even. Fix a Riemannian metric on $\mathcal{O}$. About each singularity x_j, we choose an orbifold disk $(\widehat{D}_j, G_{x_j})$ where G_{x_j} is the local group at x_j, acts linearly on $\widehat{D}_j$ as a subgroup of $O(n)$, and x_j corresponds to the origin in $\widehat{D}_j$. We arrange our collection of disks $\widehat{D}_j/G_{x_j}$ so that they are pairwise disjoint. Define,

$$N := \mathcal{O} - \bigsqcup_{j=1}^{q} \mathrm{int}(\widehat{D}_j)/G_{x_j}.$$

Then N canonically admits the structure of an orbifold with boundary, induced from $\mathcal{O}$. In particular, ∂N consists of those points in $\partial \widehat{D}_j/G_{x_j}$ for $j \in \{1, \ldots, q\}$. The restriction $\xi|_N$ yields a globally defined line field on N, and therefore, by means of our metric, an associated double cover $p : \widehat{N} \to N$, together with a vector field $v_{\xi|_N}$ on $\widehat{N}$. Each boundary component of $\widehat{N}$ is in correspondence with a base point singularity of ξ. Indeed, $\partial \widehat{N} = p^{-1}(\partial N)$. To each boundary component (or to be more precise, to each connected component thereof), we glue copies of the corresponding quotient $\widehat{D}_j/G_{x_j}$ along an equivariant homeomorphism of $(n-1)$-spheres (which induces a diffeomorphism of orbifold boundaries). If the singularity x_j is non-orientable, only a single copy is required. If the singularity x_j is orientable, two copies are required (see Definition 2.0.2). This procedure yields a closed orbifold $\widehat{\mathcal{O}}$, together with a map $\pi : \widehat{\mathcal{O}} \to \mathcal{O}$ extending $p : \widehat{N} \to N$. On each glued in disk, by means of working on the associated chart, we radially (see Example 1.2.2) extend the associated vector field $v_{\xi|_N}$, and as G_{x_j} acts as a subgroup of $O(n)$, the radial extension is G_{x_j}-invariant. In particular, we obtain a vector field v on $\widehat{\mathcal{O}}$ with isolated zeroes $\{\pi^{-1}(x_j) \ : \ j \in \{1, \ldots, q\}\}$. Let k denote the number of non-orientable singularities. Then v has $2q - k$ isolated zeroes, each of which admits a neighbourhood uniformised by the quotient of a disk. By construction, the order of the isotropy group at each point of $\pi^{-1}(x_j)$ is the same as that of the base point x_j. In particular, as a consequence of ([CG17],Lemma 4.1) (which is summarised as Lemma 2.0.5), we have, for each $j \in \{1, \ldots, q\}$,

$$\mathrm{orb\,p\,ind}_{\xi}^{\perp}(x_j) = \sum_{y \in \pi^{-1}(x_j)} \mathrm{orb\,ind}_{v}^{\perp}(y).$$

Before we conclude, we establish a relationship between $\chi^{\mathrm{orb}}(\widehat{\mathcal{O}})$ and $\chi^{\mathrm{orb}}(\mathcal{O})$. First, we have $\chi^{\mathrm{orb}}(\widehat{N}) = 2\chi^{\mathrm{orb}}(N)$. Now, $\mathcal{O} = N \cup \left(\bigsqcup_{j=1}^{q} \widehat{D}_j/G_{x_j} \right)$. The boundary of each orbifold disk is realised as the global quotient of an $(n-1)$-sphere (odd dimensional). In particular,

$$\begin{aligned} \chi^{\mathrm{orb}}(\mathcal{O}) &= \chi^{\mathrm{orb}}(N) + \sum_{j=1}^{q} \chi^{\mathrm{orb}}(\widehat{D}_j/G_{x_j}) \\ &= \chi^{\mathrm{orb}}(N) + \sum_{j=1}^{q} \frac{1}{|G_{x_j}|}. \end{aligned}$$

as each disk $\widehat{D_j}$ is contractible. Now, the closed orbifold $\widehat{\mathcal{O}}$ is obtained from $\widehat{N}$ by gluing $2q - k$ orbifold disks, of varying singular order. By renumbering if necessary, write $x_1, \ldots, x_k$ for the non-orientable singularities. Then,

$$\begin{aligned}
\chi^{\text{orb}}(\widehat{\mathcal{O}}) &= \chi^{\text{orb}}(\widehat{N}) + \sum_{j=1}^{k} \frac{1}{|G_{x_j}|} + \sum_{j=k+1}^{q} \frac{2}{|G_{x_j}|} \\
&= \chi^{\text{orb}}(\widehat{N}) + \sum_{j=1}^{q} \frac{2}{|G_{x_j}|} - \sum_{j=1}^{k} \frac{1}{|G_{x_j}|} \\
&= 2\chi^{\text{orb}}(N) + \sum_{j=1}^{q} \frac{2}{|G_{x_j}|} - \sum_{j=1}^{k} \frac{1}{|G_{x_j}|} \\
&= 2\left(\chi^{\text{orb}}(\mathcal{O}) - \sum_{j=1}^{q} \frac{1}{|G_{x_j}|}\right) + \sum_{j=1}^{q} \frac{2}{|G_{x_j}|} - \sum_{j=1}^{k} \frac{1}{|G_{x_j}|} \\
&= 2\chi^{\text{orb}}(\mathcal{O}) - \sum_{j=1}^{k} \frac{1}{|G_{x_j}|}
\end{aligned}$$

We therefore have, by Satake's Poincaré-Hopf Theorem 3.4.1 and Corollary 4.1.1,

$$\begin{aligned}
2\chi^{\text{orb}}(\mathcal{O}) - \sum_{j=1}^{k} \frac{1}{|G_{x_j}|} = \chi^{\text{orb}}(\widehat{\mathcal{O}}) &= \sum_{j=1}^{q} \sum_{y \in \pi^{-1}(x_j)} \operatorname{orb} \operatorname{ind}_v(y) \\
&= \sum_{j=1}^{q} \sum_{y \in \pi^{-1}(x_j)} \left(\operatorname{orb} \operatorname{ind}_v^{\perp}(y) + \frac{1}{|G_y|}\right) \\
&= \sum_{j=1}^{k} \frac{1}{|G_{x_j}|} + \sum_{j=k+1}^{q} \frac{2}{|G_{x_j}|} + \sum_{j=1}^{q} \sum_{y \in \pi^{-1}(x_j)} \operatorname{orb} \operatorname{ind}_v^{\perp}(y) \\
&= \sum_{j=1}^{k} \frac{1}{|G_{x_j}|} + \sum_{j=k+1}^{q} \frac{2}{|G_{x_j}|} + \sum_{j=1}^{q} \operatorname{orb} \operatorname{p} \operatorname{ind}_\xi(x_j) \\
&= \sum_{j=1}^{k} \frac{1}{|G_{x_j}|} + \sum_{j=k+1}^{q} \frac{2}{|G_{x_j}|} + \sum_{j=1}^{q} \left(\operatorname{orb} \operatorname{p} \operatorname{ind}_\xi(x_j) - \frac{2}{|G_{x_j}|}\right) \\
&= \sum_{j=1}^{q} \operatorname{orb} \operatorname{p} \operatorname{ind}_\xi(x_j) - \sum_{j=1}^{k} \frac{1}{|G_{x_j}|}
\end{aligned}$$

We thus have,

$$2\chi^{\text{orb}}(\mathcal{O}) = \sum_{j=1}^{q} \operatorname{orb} \operatorname{p} \operatorname{ind}_\xi(x_j).$$

□

Remark 4.1.1. Let us make a few remarks about geometric operations on orbifolds. For compact orbifolds with boundary, one has access to collar neighbourhoods about the boundary.

For a proof, we refer the reader to ([PS08], Page 6). In particular, given compact orbifolds $\mathcal{X}$ and $\mathcal{Y}$, together with a diffeomorphism $f : \partial\mathcal{Y} \to \partial\mathcal{X}$ (as orbifolds), the adjunction space $X \cup_f Y$ admits the structure of a closed orbifold. One such proof is similar to the case of manifolds, utilising collar neighbourhoods (see [Lee], Theorem 9.29, Page 224). For a brief discussion regarding surgery and connected sums on orbifolds, we refer the reader to ([KL14], Page 10). Finally, note that in the extension of the vector field over the glued in orbifold disks, one can ensure smoothness, using the following result locally.

Theorem 4.1.2 ([Bre72], Page 317)**.** *Let G be a compact Lie group acting smoothly on the manifolds M and N. Let $\varphi : M \to N$ be a continuous equivariant map. Then φ can be approximated by a smooth equivariant map $\psi : M \to N$ which is equivariantly homotopic to φ by a homotopy approximating the constant homotopy. Moreover, if φ is already smooth on the closed invariant set $A \subset M$, then ψ can be chosen to coincide with φ on A, and the homotopy between φ and ψ to be constant there.*

We conclude this chapter with some examples.

Example 4.1.1. We provide an infinite family of orbifolds, all of whom admit globally defined line fields, but none of which admit a globally defined vector field. Let n be a positive integer greater than 1. Consider $\mathbb{T}^n = (S^1)^n$ and $\mathbb{Z}_2$ acting on $\mathbb{T}^n$ by complex conjugation on each coordinate. The quotient $\mathbb{T}^n/\mathbb{Z}_2$ is an orbifold, and there are 2^n singular points. The tangent bundle of $\mathbb{T}^n$ as a manifold is trivial, and choosing a trivial line subbundle determines a mapping $\mathbb{T}^n \to \mathbb{R}P^{n-1}$, which is identified with a line field on $\mathbb{T}^n$. The $\mathbb{Z}_2$ action linearised is a scalar multiple of the identity matrix, and therefore acts trivially on real projective $(n-1)$-space. In particular, any such line field descends to the orbifold $\mathbb{T}^n/\mathbb{Z}_2$, the induced map $\mathbb{T}^n \to \mathbb{R}P^{n-1}$ being $\mathbb{Z}_2$-equivariant. In particular, the orbifold $\mathbb{T}^n/\mathbb{Z}_2$ admits a globally defined line field, and $\chi^{\mathrm{orb}}(\mathbb{T}^n/\mathbb{Z}_n) = 0$, as expected. On the other hand, $\mathbb{T}^n/\mathbb{Z}_2$ does not admit a nowhere vanishing vector field. Indeed, any such $\mathbb{Z}_2$-equivariant section $\mathbb{T}^n \to T(\mathbb{T}^n)$ necessarily vanishes at the singular points.

Example 4.1.2. Let $\mathbb{Z}_k$ act on S^2 by rotations. Extending Example 1.2.1, one can define a line field ξ on the orbifold $S^2/\mathbb{Z}_k$ with two singularities at the cone points, each of projective index $2/k$, and indeed $2\chi^{\mathrm{orb}}(S^2/\mathbb{Z}_k) = 4/k$. In a natural way, this example can also be adjusted to both the teardrop and akward rugby ball of Figure 3.1.4.

Example 4.1.3. Following Example 3.3.5, given $q = 2m/n \in \mathbb{Q}$, we construct a line field on a orbifold whose projective index at a singularity is $q \in \mathbb{Q}$. Let $\mathbb{Z}_{2n}$ act on S^{2n} by rotations, and let v be a vector field on S^{2n}, invariant under rotations, with zeroes at the poles, each of index $+1$. There is an induced line field ξ on the orbifold $S^{2n}/\mathbb{Z}_{2n}$ with two singularites, each of projective index $1/n$. Let M be a compact manifold with Euler characteristic m, and write ν for a line field on M of projective index $2m$, obtained via generation with a vector field whom has a single isolated zero. The pair (ξ, ν) induce a line field on the orbifold $S^{2n}/\mathbb{Z}_{2n} \times M$, which has a single singularity, of projective index $2m/n$. Moreover, $2\chi^{\mathrm{orb}}(S^{2n}/\mathbb{Z}_{2n} \times M) = 2m/n$.

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