# Riemann Mapping Theorem

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**Abstract** *The purpose of this report is to reproduce the proof of Riemann Mapping Theorem. We assume the readers are familiar with the basic results in real and complex analysis upto and including the Residue Theorem. Following the approach of Riesz and Fejer, we define $\mathcal{F}$ to be a family of holomorphic injective functions, uniformly bounded by 1 on a given domain. After showing that $\mathcal{F}$ is not empty by constructing a function in $\mathcal{F}$, we will use Montel's theorem to show that there is a sequence in $\mathcal{F}$ that converges locally uniformly to some holomorphic injective function in $f : U \to B_1(0)$ where at $z_0$ such that $f_n(z_0) = 0$ for all $n$, $|f_n(z_0)| \leq |f'(z_0)|$. We then use Hurwitz's theorem to show injectivity of $f$ and proof by contradiction to show that such $f$ is surjective to the unit disk.*

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# 1 Notations

Here we list some of notation that will be used throughout this report.

If $A$ is a set, $\overline{A}$ will denote the closure of A.

The unit disk is denoted by $B_1(0) = \{z \in \mathbb{C} : |z| < 1\}$.

If $f : U \to \mathbb{C}$ is a function and $A \subset U$ a set, then $f(A)$ will denote image set of $f$.

If $\gamma : [a,b] \to \mathbb{C}$ is a curve, $\gamma* := \gamma([a,b])$ is the image set of the curve.

The sequence $(a_n)_{n\in\mathbb{N}}$ will be written $(a_n)$.

The series $\sum_{n=0}^{\infty} a_n$ will be denoted $\sum a_n$.

If $(f_n)$ is a sequence of functions then it's subsequence $(f_{n_k})_{k\in\mathbb{N}}$ will be denoted $(f_{n_k})$ or sometimes $(f_n^i)$ to mean the $i$th subsequence of $(f_n)$.

If $[a,b]$ is a real interval and $P = \{a_0...a_n\}$ a partition of $[a,b]$, we denote the $i$th subinterval by $\Delta_i := [a_{i-1}, a_i]$.

The annulus centred at $z_0$ with inner radius $r_1$, outer radius $r_2$ is the set $A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$.

# 2 Basic Results

Here we state the relevant definitions and some of the basic results from real and complex analysis for the sake of completeness. These results will be used throughout the proofs of some of the main theorems involved in proving the Riemann Mapping Theorem.

## 2.1 General Cauchy Theorem

**Definition 1.** *Let $U$ be an open subset of $\mathbb{C}$ and $\gamma_1 : [a_0, b_0] \to U$ and $\gamma_2 : [a_1, b_1] \to U$ be closed curves in $U$. We say $\gamma_1$ is* homotopic *to $\gamma_2$ if there exists a continuous function $\phi : [0,1] \times [0,1] \to U$ such that*

$\phi(t, 0) = \gamma_1(a_0 + t(b_0 - a_0))$ *for all* $t \in [0,1]$
$\phi(t, 1) = \gamma_1(a_1 + t(b_1 - a_1))$ *for all* $t \in [0,1]$
$\phi(0, s) = \phi(1, s)$ *for all* $s \in [0,1]$.

*The function $\phi$ is called a* homotopy *from $\gamma_1$ to $\gamma_2$ in U. If $\gamma_1$ is homotopic to a constant curve, then $\gamma_1$ is said to be* null homotopic in U.

**Definition 2.** *a* simply connected set *is a set in which every closed, pieces wise smooth curve is null homotopic.*

**Theorem 1** (Homotopy Theorem)**.** *Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a holomorphic function. If $\gamma_1$ and $\gamma_2$ are curves in $U$, homotopic to each other then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

.

The proof of the Homotopy Theorem is not included in this article but can be found in page 93 of, *Complex Analysis* by *E. M. Stein and R. Sharkarchi.*

As a Corollary, we obtain the Cauchy's theorem for simply connected sets.

**Theorem 2** (Cauchy)**.** *Let $S \subset \mathbb{C}$ be an open and simply connected set, $f : S \to \mathbb{C}$ a holomorphic function and $\gamma$ a closed piecewise smooth curve in U, then*

$$\int_{\gamma} f(z)dz = 0$$

.

**Theorem 3** (Morera)**.** *Suppose $f$ is a continuous complex function in an open set $U$ such that*

$$\int_{\partial\triangle} f(z)dz = 0$$

*for every closed triangle $\triangle \in U$. Then $f$ is holomorphic*

The proof of Morera's Theorem can be found in page 224 of *Real and Complex Analysis 2nd edition* by *W. Rudin.*

**Definition 3.** *Let* $0 \leq r_1 < r_2 \leq \infty$ *and* $z_0 \in \mathbb{C}$*. Let* $f : A(z_0, r_1, r_2) \to \mathbb{C}$ *be a holomorphic function. Suppose* $\sum_1^\infty \alpha_{-k}(z - z_0)^{-k} + \sum_0^\infty \alpha_k(z - z_0)^k$ *is the Laurent Series of* $f$ *about* $z_0$*. Then* $\alpha_{-1}$ *is the* residue of $f$ at $z_0$ *denoted* $\alpha_{-1}$*.*

**Theorem 4** (Residue Theorem)**.** *Let* $U \subset \mathbb{C}$ *be an open set and* $A \subset U$*. Suppose* $U/A$ *is open. Let* $f : U/A \to \mathbb{C}$ *be a holomorphic function. Suppose* $f$ *has a isolated singularity at each* $a \in A$*. Let* $\gamma$ *be a closed piecewise smooth curve in* $U/A$ *which is null homotopic in* $U$*. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{k=1}^{N}(Res_{a_k} f) Ind_\gamma(a_k)$$

.

## 2.2 Properties of Mobius Transformations

Mobius transformations are used to construct an example of a holomorphic injective function bounded by 1 in a given open simply connected set. It is necessary to show that the family of functions on $U$ that are holomorphic, injective and bounded by 1 is not empty, for if such family was empty, the Riemann mapping theorem is trivially false.

**Definition 4.** *Let* $U \to \mathbb{C}$ *be an open, simply connected set. The function* $f : U \to \mathbb{C}$ *is a mobius transformation if for some* $a, b, c, d \in \mathbb{C}$, $f(z) = \frac{az+b}{cz+d}$*.*

**Proposition 1.** *Let* $U \to \mathbb{C}$ *be an open set and* $a, b, c, d \in \mathbb{C}$ *such that* $\frac{-d}{c} \notin U$ *and* $ad - bc \neq 0$*. If* $f : U \to \mathbb{C}$ *is defined by* $f(z) = \frac{az+b}{cz+d}$ *for all* $z \in \mathbb{C}$*, then* $f$ *is injective and holomorphic in U.*

*Proof.* Holomorphicity is trivial since the mapping $z \mapsto az + b$ and $z \mapsto cz + d$ are holomorphic and for all $z \in \mathbb{C}$, $cz + d \neq 0$ as $\frac{-d}{c} \notin U$.

Suppose that for some $w, z \in \mathbb{C}$, $f(z) = f(w)$, then we get $(ad - bc)z = (ad - bc)w$. By hypothesis $(ad - bc) \neq 0$ so $z = w$. So $f$ is injective. □

## 2.3 Sequences of Functions

In defining a non-empty family of holomorphic, injective functions, bounded by 1, we wish to show that such family always contains a sequence of functions converging to a function with the same properties (such is the content of Montel's Theorem and Weierstrass Theorem). Here we give the sufficient conditions for the limit of a sequence to inherit the properties of the functions in that sequence.

**Definition 5.** *Let* $U \subset \mathbb{C}$ *be open,* $(f_n)_\mathbb{N}$ *a sequence of functions on* $U$*. The sequence* $(f_n)_\mathbb{N}$ *is said to be* pointwise convergent to f *if for each* $z \in U$*, the sequence* $(f_n(z))_\mathbb{N}$ *converges to* $f(z)$*.*

**Definition 6.** *Let $U \subset \mathbb{C}$ be open, and $f_n : U \to \mathbb{C}$ for each $n \in \mathbb{N}$ functions on $U$. The sequence $(f_n)_{\mathbb{N}}$ is said to be* uniformly convergent to f in U *if for each $\varepsilon$, there exists $N \in \mathbb{N}$ such that for each $z \in U$, for all $n \in \mathbb{N}$ with $N < n$ we have $|f_n(z) - f(z)| < \varepsilon$.*

**Definition 7.** *Let $U \subset \mathbb{C}$ be open, and $f_n : U \to \mathbb{C}$ for each $n \in \mathbb{N}$ functions on $U$. The sequence $(f_n)_{\mathbb{N}}$ is said to be* locally uniformly convergent to f *if for each $z \in U$ there exist $0 < r$ such that $B_r(z) \subset U$ and the sequence $(f_n)_{\mathbb{N}}$ converges uniformly to $f$ in $B_r(z)$.*

**Theorem 5.** *Let $[a, b]$ be a closed interval in $\mathbb{R}$ and $(f_n)_{\mathbb{N}}$ a sequence of integrable functions from $[a, b]$ to $\mathbb{C}$. If $(f_n)_{\mathbb{N}}$ converges uniformly to $f : [a, b] \to \mathbb{C}$, then $f$ is integrable and $\int_a^b f(x)dx = lim_{n\to\infty} \int_a^b f_n(x)dx$.*

*Proof.* Since the integral of complex valued function $f$ of real variables are defined in-terms of the individual integral of the real and imaginary components of $f$, it is sufficient to prove the case for the sequence of real valued functions of real variable.

Suppose $(f_n)$ is a sequence of integrable functions from the interval [a,b] to $\mathbb{R}$, uniformly convergent to $f : [a, b] \to \mathbb{R}$. Let $\varepsilon$ be given and $N \in \mathbb{N}$ such that for all $x \in [a, b]$, $N < n$, we have $|f_n(x) - f(x)| < \varepsilon$. Pick $N < n$ and let $P = \{a_0, ..., a_n\}$ be a partition of $[a, b]$ such that $U(P, f_n) - L(P, f_n) < \varepsilon$. We have $f_n(x) - \varepsilon < f(x) < f_n(x) + \varepsilon$ for all $x \in [a, b]$ and thus for each subinterval $\Delta_i$ we have $inf_{x\in\Delta_i} f_n(x) - \varepsilon < inf_{x\in\Delta_i} f(x) \leq sup_{x\in\Delta_i} f(x) < sup_{x\in\Delta_i} f_n(x) + \varepsilon$.

We Thus have $0 \leq \int^- f(x)dx - \int_- f(x)dx \leq U(P, f) - L(P, f) < U(P, f_n + \varepsilon) - L(P, f_n - \varepsilon) = U(P, f_n) - L(P, f_n) + \varepsilon(b - a) < \varepsilon(1 + b - a)$.
Since $0 < \varepsilon$ was arbitrary, $\int^- f(x)dx = \int_- f(x)dx$ and so $f$ is Rieman Integrable.

For each $n \in \mathbb{N}$ let $\varepsilon_n = sup_{x\in[a,b]} |f(x) - f_n(x)|$ then $lim_{n\to\infty} \varepsilon_n = 0$.
So $0 \leq lim_{n\to\infty} | \int_a^b f(x)dx - \int_a^b f_n(x)dx| < lim_{n\to\infty} \int_a^b |f(x) - f_n(x)|dx \leq lim_{n\to\infty} \int_a^b \varepsilon_n dx = lim_{n\to\infty} \varepsilon_n(b - a) = 0$ □

**Theorem 6.** *Let $U \subset \mathbb{C}$ be an open set, $\gamma$ a closed piecewise smooth curve in $U$ and $(f_n)_{\mathbb{N}}$ a sequence of integrable functions from $U$ to $\mathbb{C}$. If $(f_n)_{\mathbb{N}}$ converges uniformly to $f : U \to \mathbb{C}$, then $f$ is integrable and $\int_a^b f(x)dx = lim_{n\to\infty} \int_a^b f_n(x)dx$.*

**Theorem 7** (Weierstrass)**.** *Let $U \subset \mathbb{C}$ be open, and $(f_n)_{\mathbb{N}}$ a sequence of holomorphic functions on $U$ that converges locally uniformly to $f : U \to \mathbb{C}$ then $f$ is holomorphic.*

*Proof.* Pick $z_0 \in U$ and $0 < r$ such that $B_r(z_0) \subset U$ and $(f_n)$ converges uniformly to $f$ in $B_r(z_0)$. Let $\gamma$ be an arbitrary closed, piecewise smooth curve in $B_r(z_0)$. We have by the previous theorem $\int_\gamma f(z)dz = lim_{n\to\infty} \int_\gamma f_n(z)dz$. Since $\gamma$ was an arbitrary curve, we have by Morera's Theorem, $f$ is holomorphic on $B_r(z_0)$. Since $z_0 \in U$ was arbitrary, we have that $f$ is holomorphic on $U$. □

# 3 Montel's Theorem

Here we present the necessary theorems required to prove Montel's theorem.

**Definition 8.** *Let $(f_n)$ be a sequence of complex valued functions defined on the metric space $E$. We say that $(f_n)$ is* pointwise bounded *if for each $x \in E$ the sequence $(f_n(x))$ is bounded.*

*The sequence $(f_n)$ is* uniformly bounded *in $E$ if there exist $0 < M$ such that $|f_n(x)| < M$ for all $n \in \mathbb{N}$ and $x \in E$.*

**Definition 9.** *A family of complex functions $\mathcal{F}$ on the set $S$, subset of the metric space $X$ is said to be* pointwise equicontinuous *if for a given $0 < \varepsilon$ and $x \in S$, there exist $0 < \delta$ such that for all $f \in \mathcal{F}$, $|f(x) - f(y)| < \varepsilon$ for all $y \in S$ with $|x - y| < \delta$.*

*We say $\mathcal{F}$ is* uniformly equicontinuous *if for a given $0 < \varepsilon$ there exist a single $0 < \delta$ such that for all $f \in \mathcal{F}$ and $x, y \in S$, $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$.*

**Lemma 1.** *Let $X$ be a metric space, $(f_n)$ a pointwise bounded sequence of function from $X$ to $\mathbb{C}$. If $E$ is a countable subset of $\mathbb{C}$ then $(f_n)$ has a subsequence $(f_{n_k})$ that is pointwise convergent in $E$.*

*Proof.* Enumerate the countable set $E$ by $E = \{x_0, x_1 ...\}$. Since $(f_n(x_0))$ is a bounded sequence in $\mathbb{C}$, there is a convergent subsequence which we denote by $(f_n^{(0)}(x_0))$ then we define $(f_n^{(1)})$ to be such that $(f_n^{(1)}(x_1))$ is a convergent the subsequence of $(f_n^{(0)}(x_1))$. Suppose for some $k \in \mathbb{N}$, for all $i \in \mathbb{N}$ with $i < k$, $(f_n^{(i)})$ is the subsequence of $(f_n^{(i-1)})$ such that $(f_n^{(i)}(x_i))$ is a convergent subsequence of $(f_n^{i-1}(x_i))$. Then we have the there exists $(f_n^{(k+1)})$ such that $(f_n^{(k+1)}(x_{k+1}))$ is a convergent subsequence of $(f_n^{(k)}(x_{k+1}))$. By Induction, such sequence of functions exists for all $k \in \mathbb{N}$. Define $(g_n)$ by $g_n(z) = f_n^n(z)$ for all $n \in \mathbb{N}$ and $z \in \mathbb{E}$. Then $(g_n)$ is pointwise convergent on $E$. $\square$

**Lemma 2.** *Let $X$ be a metric space. If $K \in X$ is a compact set, then there exists a countably dense subset of $K$.*

*Proof.* For each $q \in \mathbb{Q}^+$ the collection of sets $\{B_q(z) : z \in K\}$ is an open cover of $K$ and thus there exists a finte subcover $\{B_q(z_1^q), ..., B_q(z_{n_q}^q)\}$. Since the set of rational numbers are countable and the countable union of finite sets are again countable, we have that the set $\mathbb{U} = \bigcup_{q \in \mathbb{Q}^+} \{z_1^q, ..., z_{n_q}^q\}$ is a countable subset of $K$.

To show that $\mathbb{U}$ is dense in $K$, pick an arbitrary $x \in K$ and suppose that $x \notin \mathbb{U}$. Let $0 < \varepsilon$ be given and $r \in \mathbb{Q}$ with $0 < r < \varepsilon$. Since $\{B_r(z_1^r), ..., B_r(z_{n_r}^r)\}$ is an open cover of $K$, we can select $z_i^r \in \{z_1^r ... z_{n_r}^r$ such that $x \in B_r(z_i^r)$. So we have $|x - z_i^r| < r < \varepsilon$. Since $\varepsilon$ was arbitrary, for each $\varepsilon$ there exists $z \in \mathbb{U}$ with $z \neq x$ such that $|x - z| < \varepsilon$ meaning $x$ is a limit point of $\mathbb{U}$. $\square$

Here we prove the Arzela-Ascoli's Theorem. While a few different versions of the theorem exists the following is the easiest version to prove.

**Theorem 8** (Arzela-Ascoli)**.** *If $K$ is a compact subset of some metric space $X$ and $(f_n)$ is a sequence of continuous function locally uniformly bounded and unifotmly equicontinuous on $K$, then $(f_n)$ contains a uniformly convergent subsequence $(f_{n_k})$.*

*Proof.* Let $\mathbb{U}$ be a countably dense subset of $K$ and $(g_k) = (f_{n_k})$ a subsequence of $(f_n)$ pointwise convergent in $\mathbb{U}$ which exists by Lemma 1. Let $0 < \varepsilon$ be given and $0 < \delta$ such that for all $z, w \in K$ and for all $n \in \mathbb{N}$ we have $|f_n(z) - f_n(w)| < \frac{\varepsilon}{3}$ for $|z - w| < \delta$. Let $q \in \mathbb{Q}$ be such that $0 < q < \delta$, then $\{B_q(w) : w \in \mathbb{U}\}$ is an open cover of the compact set $K$ and so we select $\{w_1...w_m\} \subset \mathbb{U}$ such that $\{B_q(w_1)...B_q(w_m)\}$ is a finite subcover of $K$. For each $i \in \{1...m\}$, denote by $N_i$ the natural number for which $|g_{k_1}(w_i) - g_{k_2}(w_i)| < \frac{\varepsilon}{3}$ for all $N_i < k_1, k_2$ and take $N = max\{N_1...N_m\}$.

Choose arbitrary $z_0 \in K$ and pick $w_i \in \{w_1...w_m\}$ such that $z \in B_q(w_i)$ and $k_1, k_2 \in \mathbb{N}$ where $N < k_1, k_2$. We then have by pointwise convergence and uniformly equicontinuity, $|g_{k_1}(z_0) - g_{k_2}(z_0)| < |g_{k_1}(z_0) - g_{k_1}(w_i)| + |g_{k_1}(w_i) - g_{k_2}(w_i)| + |g_{k_2}(w_i) - g_{k_2}(z_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Since $\varepsilon$ was arbitrary, for each $\varepsilon$ there exist $N \in \mathbb{N}$ such that for each $z \in K$, for all $k \in \mathbb{N}$ with $N < k$, $|f_{n_k}(z) - f_{n_k}(z)| < \varepsilon$. $\square$

**Definition 10.** *A family of functions $\mathcal{F}$ is said to be a* normal *if each sequence $(f_n)$ of functions $f_n \in \mathcal{F}$ contains a subsequence $(f_{n_k})$ which converges uniformly on every compact subset $K$ of $U$.*

**Lemma 3.** *If $\mathcal{F}$ is locally bounded in $U$, then it's pointwise equicontinuous on $U$.*

*Proof.* Pick $z_0 \in U$ and let $0 < r$ be such that $\overline{B_r(z_0)} \in U$ and $\mathcal{F}$ is uniformly bounded in $B_r(z)$. Let $M$ be such that for all $f$ in $\mathcal{F}$ and $z \in U$, $|f(z)| < M$. Pick $z \in B_r(z_0)$ such that $|z - z_0| < \frac{r}{2}$. Denote by $\gamma$ a closed, piecewise smooth curve whos image is $\partial B_r(z_0)$. Then for each $f \in \mathcal{F}$, we have by Cauchy's integral formula, $f(z) - f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z_0}dw = \frac{1}{2\pi i}\int_\gamma \frac{(z-z_0)f(w)}{(w-z)(w-z_0)}dw$ and by Cauchy's approximation $|f(z) - f(z_0)| < r|z-z_0|sup_{w\in\gamma*}|\frac{f(w)}{r(\frac{r}{2})}| < \frac{2|z-z_0|M}{r}$ for all $z \in B_{\frac{r}{2}}(z_0)$ as $z$ with $|z - z_0| < \frac{r}{2}$ was arbitrary. Let $\varepsilon$ be given and let $\delta = min\{\frac{\varepsilon r}{2M}, \frac{r}{2}\}$ then if $z \in U$ such that $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \frac{2\delta M}{r} < \varepsilon$. Since $f \in \mathcal{F}$, $z \in B_{\frac{r}{2}}(z_0)$ were arbitrary, for each $f \in \mathcal{F}$ and $z \in B_{\frac{r}{2}}(z_0)$, $|f(z) - f(z_0)| < \varepsilon$ if $|z - z_0| < \delta$. Since $z_0 \in U$ and $\varepsilon$ were arbitrary, such $\delta$ exists for every $\varepsilon$ and $z_0$, thus $\mathcal{F}$ is pointwise equicontinuous. $\square$

**Lemma 4.** *If $\mathcal{F}$ is pointwise equicontinuous in $U$, then it is uniformly equicontinuous on each compact subsets of $U$.*

*Proof.* Let $K$ be an arbitrary compact subset of $U$. Let $\varepsilon$ be given and for each $z \in K$ let $\delta(z)$ be such that for every $f \in \mathcal{F}$ and $w \in K$, $|f(z) - f(w)| < \varepsilon$ if $|z - w| < \delta(z)$. The collection of sets $\{B_{\frac{\delta(z)}{2}}(z) : z \in K\}$ is an open cover of the compact set $K$. Let $\{B_{\frac{delta(z_1)}{2}}(z_1)...B_{\frac{\delta(z_n)}{2}}(z_n)\}$ be a finte subcover of $K$. We then choose $\delta =$

$min\{\frac{\delta(z_1)}{2}...\frac{\delta(z_n)}{2}\}$. Pick aritrary $z_0, z \in K$ such that $|z - z_0| < \delta$ and let $z_i \in \{z_1...z_n\}$ be such that $z_0 \in B_{\frac{\delta(z_i)}{2}}(z_i)$. We then have $|z - z_i| < |z - z_0| + |z_0 - z_i| < \delta + \frac{\delta(z_i)}{2} < \frac{\delta(z_i)}{2} + \frac{\delta(z_i)}{2} = \delta(z_i)$. Let $f \in \mathcal{F}$, we have $|f(z) - f(z_0)| < |f(z) - f(z_i)| + |f(z_i) - f(z_0)| < 2\epsilon$. Since $f \in \mathcal{F}$ and $z, z_0 \in K$ with $|z - z_0| <$was arbitrary, for every $f \in \mathcal{F}$ and $z, z_0 \in K$ we have $|f(z) - f(z_0)| < \varepsilon$ if $|z - z_0| < \delta$. Since $0 < \varepsilon$ was arbitrary, $\mathcal{F}$ is equicontinuous on $K$. As $K$ was an arbitrary compact subset of $U$, $\mathcal{F}$ is uniformly equicontiunous on every compact subset of $U$. □

**Theorem 9** (Montel's Theorem)**.** *Let $U \in \mathbb{C}$ be an open connected set. If $\mathcal{F}$ is a family holomorphic functions on $U$ locally bounded, then $\mathcal{F}$ is a normal family in $U$.*

*Proof.* Let $(f_n)$ be a sequence of functions from $\mathcal{F}$. Construct a sequence of compact subsets $(K_n)$ of $U$ such that $K_n \in K_{n+1}$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} K_n = U$.

Suppose there exists some $m \in \mathbb{N}$ such that for all $i \leq m$, $(f_n^i)$ is a subsequence of $(f_n)$ that is uniformly convergent on $K_i$ and is a subsequence of $(f_n^{i-1})$. Then $(f_n^m)$ is a subsequence of $(f_n)$ and thus locally bounded and pointwise equicontinuous. So the restriction of $(f_n^m)$ to the compact set $K_{m+1}$ is uniformly equicontinuous and so by Arzela-Ascoli's theorem, $(f_n^m)$ has a subsequence $(f_n^{m+1})$ that is uniformly convergent in $K_{m+1}$. Since $(f_n)$ is locally bounded and uniformly equicontinuous on $K_1$, by Arzela-Ascoli's theorem, $(f_n)$ has a subsequence $(f_n^1)$, uniformly convergent in $K_1$. By Induction such subsequence of $(f_n)$ exists for all $m \in \mathbb{N}$.

Construct the subsequence $(g_m)$ of $(f_n)$ by Cantor's diagonalization method such that $g_m = f_m^m$ for all $m \in \mathbb{N}$ where $(f_n^m)$ are the subsequence of $(f_n)$ as defined before. Then we have that $(g_m)$ is uniformly convergent on $K_n$ for each $n \in \mathbb{N}$.

Let $K$ be an arbitrary compact, proper subset of $U$. Then there exists some $N \in \mathbb{N}$ such that $K \in K_N$, other wise $K \not\subset K_n$ for all $n \in \mathbb{N}$ and thus $K \not\subset \bigcup_{n\in\mathbb{N}} K_n = U$. Since $K$ is is a subset of $K_N$ and $(g_m)$ is uniformly convergent in $K_N$, $(g_m)$ is uniformly convergent in $K$. As $K$ was an arbitrary compact subset of $U$, $(g_m)$ is uniformly convergent in every compact subset of $U$.

Since $(f_n)$ was an arbitrary sequence of functions from $\mathcal{F}$, $\mathcal{F}$ is a normal family. □

## 4 Hurwitz's Theorem

Hurwit'z theorem is used to prove the injectivity of the limit function of the locally uniformly convergent sequence of functions, who's existence was proved using Montels Theorem. Hurwitz's Theorem is fundamentally a statement about the behaviour of the zeros of a sequence of holomorphic functions. Simply, Hurwitz says that if $(f_n)$ is a sequence of holomorphic functions, locally uniformly convergent to $f$, then the zeros fo $(f_n)$ converges to the zeros of $f$.

**Lemma 5.** *Let $U \subset \mathbb{C}$ be and open connected set, $A \subset U$ and $\gamma$ a closed piecewise smooth curve in $U/A$, null homotopic in $U$. Denote by $U_1$ the set $\{a \in \mathbb{C}/\gamma : Ind_\gamma(a) \neq 0\}$ and suppose that $U_1 \subset U$. If each $a \in A$ is isolated, then $A \bigcap U_1$ is finite.*

*Proof.* Suppose the contrary. Let $\phi : [0,1] \times [0,1] \to$ be a homotopy from $\gamma$ to a point. Pick arbitrary $z_0$ from $A/\phi([0,1] \times [0,1])$, then the function $\frac{1}{w-z_0}$ is holomorphic on the open set $U_0/\{z_0\}$ and $\gamma$ is null homotopic on $U_0/\{z_0\}$. So $Ind_\gamma(z_0) = \int_\gamma \frac{1}{w-z_0} dw = 0$ meaning $z_0 \not\subset U_1$. Thus we have contrapositively that $z_0 \in U_1$ implies $z_0 \in \phi([0,1] \times [0,1])$ and since $z_0 \in A/\phi([0,1] \times [0,1])$ was arbitrary, $U_1 \bigcap A \subset \phi([0,1] \times [0,1]) \bigcap A$. By hypothesis $U_1 \bigcap A$ is infinite, thus $\phi([0,1] \times [0,1]) \bigcap A$ is an infinite subset of the compact set $\phi([0,1] \times [0,1])$. We then get that there exist a limit point in $\phi([0,1] \times [0,1]) \bigcap A$ and thus in A, which contradicts the fact that every point in A is isolated. □

**Corollary 1.** *Let $U \subset \mathbb{C}$ be an open connected set, $A \subset U$ and $\gamma$ a closed piecewise smooth curve in $U/A$, null homotopic in $U$. Denote by $U_1$ the set $\{a \in \mathbb{C}/\gamma : Ind_\gamma(a) \neq 0\}$ and suppose that $U_1 \subset U$. If $f : U/A \to \mathbb{C}$ is holomorphic and $f$ has a pole or a zero of finite order at each $a \in A$ then $A \bigcap U_1$ is finite.*

**Theorem 10** (Argument Principle)**.** *Let $U \in \mathbb{C}$ be an open connected set and let $\gamma$ be a simple curve in $U$, null-homotopic in $U$. Suppose that $U_1 := \{a \in \mathbb{C}/\gamma* : Ind_\gamma(a) = 1\} \subset U$. Let $A \subset U$ be finite and $f : U/A \to \mathbb{C}$ a holomorphic function. Suppose $f$ has a pole at each $a \in A$ and no poles or zeroes on $\gamma*$. Denote by $N_f$ the number of zeros of $f$ in $U_1$ and $P_1$ the number of poles in $U_1$ counted to their multiplicities. Then*

$$N_f - P_f = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$$

*Proof.* $A \bigcap U1$ is finite by lemma 5. Let $A \bigcap U1 = \{a_1...a_n\}$ and $\{z \in U_1 : f(z) = 0\} = \{z_1...z_m\}$ and denote by $p_i$ the order of pole at $a_i$ and denote by $N_i$ the order of zero at $z_i$.

Suppose for some $k \in \{1...n\}$ there exist a holomorphic function $h_k : U/A \to \mathbb{C}$ with removeable singularities at $\{a_1...a_k\}$, poles at $\{a_{k+1}...a_n\}$ with order $\{p_{k+1}...p_n\}$ and such that

$$\frac{f'(z)}{f(z)} = \sum_1^k \frac{p_i}{z-a_i} + \frac{h_k'(z)}{h_k(z)}$$

then the holomorphic extenstion to $\{a_1...a_k\} \bigcap U/A$ is unique. Define the function $h_{k+1}$ by $h_{k+1}(z) = (z-a_{k+1})^{p_{k+1}} h_k(z)$, then $h_{k+1}$ has a removeable singularity at $a_{k+1}$. We have $\frac{h_k'(z)}{h_k(z)} = -\frac{p_{k+1}}{z-a_{k+1}} + \frac{h_{k+1}'(z)}{h_k(z)}$ and so $\frac{f'(z)}{f(z)} = \sum_1^{k+1} \frac{-p_i}{z-a_{k+1}} + \frac{h_{k+1}'(z)}{h_{k+1}(z)}$.
Since $f$ has pole order $p_1$ at $a_1$, the function $h_1$ defined by $f(z) = (z-a_1)^{p_1} f(z)$ has a removeable singularity at $a_1$ and $\frac{f'(z)}{f(z)} = \frac{-p_1}{z-a_1} + \frac{h_1'(z)}{h_1(z)}$. So by induction we have $\frac{f'(z)}{f(z)} = \sum_1^n \frac{-p_i}{z-a_i} + \frac{h_n'(z)}{h_n(z)}$ where $h_n$ is holomorphic in $U$ and $h_n(z) = (z-a_1)...(z-a_n)f(z)$ for all $z \in U/A$. This implies that $h_n(z) = 0$ for all $z \in \{z_1...z_m\}$. By analogus argument to the set of zeros of $h_n$ we have that there exists a function $h_{n+m} : U \to \mathbb{C}$ such

that $\frac{h_n'(z)}{h_n(z)} = \sum_0^m \frac{N_i}{z-z_i} + \frac{h_{n+m}'(z)}{h_{n+m}(z)}$. This gives us $\frac{f'(z)}{f(z)} = \sum_1^n \frac{-p_i}{z-a_i} + \sum_1^m \frac{N_i}{z-z_i} + \frac{h_{n+m}'(z)}{h_{n+m}(z)}$ where $h_{n+m}(z)$ is holomorphic in $U$. Thus $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_\gamma (\sum_1^n \frac{-p_i}{z-a_i} + \sum_1^m \frac{N_i}{z-z_i} + \frac{h_{n+m}'(z)}{h_{n+m}(z)}dz) = -\sum_1^m p_i + \sum_1^n N_i = -P_f + N_f$ by the residue theorem. □

**Theorem 11** (Rouche)**.** *Let $U \in \mathbb{C}$ be an open connected set, $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ holomorphic functions, and $\gamma$ a closed piecewise smooth curve null homotopic in $U$ and such that $Ind_\gamma(z) \in \{0,1\}$ for all $z \in \mathbb{C}/\gamma*$. Denote by $U_1$ the set $\{z \in \mathbb{C}/\gamma* : Ind_\gamma(a) = 1\}$, $N_f andN_{f+g}$ the number of zero in $U_1$ for $f$ and $f+g$ respectively.*

*If $|g(z)| < |f(z)|$ for all $z \in \gamma*$ then $N_f = N_{f+g}$.*

*Proof.* Let the set $U$ and the curve $\gamma : [a,b] \to \mathbb{C}$ be as described. Suppose $f$ and $g$ are holomorphic functions on $U$ such that $|g(z)| < |f(z)|$ for all $z \in \gamma*$. Let $U_0 = \{z \in U : f(z) = 0\}$ and define the function $F : U/U_0 \to \mathbb{C}$ by $F(z) = \frac{g(z)}{f(z)} + 1$. Then for all $z \in \gamma*$ we have $|F(z) - 1| = |\frac{g(z)}{f(z)}| < 1$ and so $F(z) \in B_1(1)$ for all $z \in \gamma*$

Now $\gamma(a) = \gamma(b)$ so $F \circ \gamma(a) = F \circ \gamma(b)$, furthermore $F$ is holomorphic on $\gamma \subset U/U_0$ and so $F \circ \gamma$ is a closed piecewise smooth curve in $B_1(1)$. Since $B_1(1)$ is simply connected, $F \circ \gamma$ is null homotopic in $B_1(1)$. We have that $0 \notin B_1(1)$ so the function $\frac{1}{z}$ is holomorphic in $B_1(1)$ and so by the homotopy theorem $\int_\gamma \frac{F'(z)}{F(z)}dz = \int_{F\circ\gamma} \frac{1}{z}dz = 0$.

From the Argument principle $N_{f+g} - N_f = \int_\gamma \frac{f'(z)+g'(z)}{f(z)+g(z)}dz - \int_\gamma \frac{f'(z)}{f(z)}dz = \int_\gamma \frac{\frac{g(z)}{f(z)}'}{\frac{g(z)}{f(z)}+1}dz = \int_\gamma \frac{F'(z)}{F(z)}dz = \int_{F\circ\gamma} \frac{1}{z}dz = 0$. □

**Theorem 12.** *Let $U \in \mathbb{C}$ be open connected and $fn : U \to \mathbb{C}$ holomorphic for eachn $\in \mathbb{N}$. Let $f : U \to \mathbb{C}$ be holomorphic and $(f_n)$ a sequence of functions that converges to $f$ locally uniformly on $U$. Suppose $f$ is not identically zero in $U$.*

*If $f$ has a zero order $m$ at $a \in U$ then there exists $0 < r$ such that $B_r(a) \subset U$ and for all $s \in (0,r)$ there exists $N \in \mathbb{N}$ such that for all $N < n$, the function $f_n$ has $m$ zeros in $B_s(a)$, counted according to their multiplicities.*

*Proof.* Suppose $f$ has a zero order $m$ at $a \in U$. Since $f$ is not identically zero, $a$ is an isolated zero. Let $0 < R$ be such that $B_R(a) \subset U$ and $(f_n)$ is uniformly convergent in $B_R(a)$. Let $0 < r < R$ be such that $f(z) \neq 0$ for all $z \in B_r(a)/\{a\}$. Choose $s$ with $0 < s < r$ then $\overline{B_s(a)} \subset B_r(a)$. Let $\gamma$ be the closed piecewise smooth curve whos image is $\partial B_s(a)$. We have that $\gamma*$ is compact and $|f|$ is continuous and so $|f|$ attains it's minimum value in $\gamma*$ and so let $z_0 \in \gamma*$ be such that $|f(z_0)| = min_{z\in\gamma*}|f(z)|$. Since $f(z) \neq 0$ for all $z \in B_r(a)/\{a\}$ we have that $0 < |f(z_0)|$. Let $0 < |f(z_0)| = \varepsilon$ and $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $N < n$, $|fn(z) - f(z)| < \varepsilon$ for all $z \in B_r(a)$. Then $|fn(z) - f(z)| < \varepsilon < |f(z)|$ for all $z \in \gamma*$ and so by Rouche's theorem we have that $N_{f_n} = N_f = m$. Since $s$ with $0 < s < r$ was arbitrary, such holds for all $0 < s < r$. □

Hurwitz's theorem is then a trivial corollary.

**Theorem 13** (Hurwitz)**.** *Let $U \subset \mathbb{C}$ be open connected, $(f_n)$ a sequence of holomorphic functions on $U$, locally uniformly convergent to $f : U \to \mathbb{C}$. If $f_n$ does not have any zeros in $U$ for each $n \in \mathbb{N}$, then $f(z) = 0$ for all $z \in U$ or $f(z) \neq 0$ for all $z \in U$.*

*Proof.* Suppose $f$ is not identically zero in $U$ and for some $a \in U$, $f(a) = 0$. Let m be the order of zero at $a$, then by Theorem 12 there exists some $0 < s$ such that there exist $N \in \mathbb{N}$ where $f_n$ has $m$ zeroes in $B_s(a)$. This proves the contrapostive of the theorem. □

**Theorem 14** (Open Mapping Theorem)**.** *Let $U \subset \mathbb{C}$ be open connect and $f : U \to \mathbb{C}$ a holomorphic function which not constant. Then $f(U)$ is open.*

*Proof.* Let $b \in f(U)$ be arbitrary, $g : U \to \mathbb{C}$ defined by $g(z) = f(z) - b$ for all $z \in U$. Let $a \in U$ such that $f(a) = b$. then $a$ is an isolated zero of $g$ and so let $0 < r$ such that $B_r(b) \subset f(U)$ and $g(z) \neq 0$ for all $z \in B_r(b)/\{b\}$. Let $p$ be such that $0 < p < r$ and $\gamma$ the closed piecewise smooth curve whos image is $\partial B_p(b)$. Then $\overline{B_p(b)}$ is compact, $|g|$ continuous so it achieves it's minimum in $\overline{B_p(b)}$. Let $z_0 \in \overline{B_p(b)}$ such that $|g(z_0)| = min_{z \in \overline{B_p(b)}} |g(z)|$. Let $0 < |f(z_0)| = \varepsilon$ and pick arbitrary $w \in B_\varepsilon(a)$. We have $|b - w| < \varepsilon < |f(z)|$ for all z in $\gamma * \subset \overline{B_p(b)}$. Thus by Rouche's Theorem we have that $f(z) - w = g(z) + (b - w)$ has the same number of zeros as $f(z) - b = g(z)$ in $B_p(b)$. So there exists some $z \in B_p(b)$ such that $f(z) = w$ meaning $w \in f(U)$. Since $w \in B_p(b)$ was arbitrary, $B_p(b) \subset f(U)$ and $b \in f(U)$ was arbitrary so for each $b \in F(U)$, there exist an open disk centred at $b$, contained in $f(U)$. Thus $f(U)$ is open. □

## 5 Proof of the Riemann Mapping Theorem

**Definition 11.** *A function $f : U \to \mathbb{C}$ is biholomorphic if it is bijective, holomorphic and the inverse is also holomorphic.*

The condition that the inverse of a bijective holomorphic function be holomorphic in the definition is actually redundent as every bijective holomorphic function has a holomorphic inverse. Such is the content of the inverse function theorem.

**Theorem 15** (Inverse Function Theorem)**.** *Let $U \in \mathbb{C}$ be an open connected set, $f : U \to \mathbb{C}$ injective and holomorphic function. Let $g : f(U) \to \mathbb{C}$ be the inverse of $f$. Then $g$ is holomorphic and $f'(z) \neq 0$ for all $z \in U$.*

*Proof.* Neither functions $f$ and $g$ can be constant for if $f$ is constant, $f$ is not injective and if $g$ is constant, then $g$ is not the inverse function of $f$.

Let $U_0$ be an arbitrary open subset of $U \subset \mathbb{C}$. Then $f|_{U_0}$ holomorphic so by the open mapping theorem $f(U_0)$ is open in $\mathbb{C}$. Thus for every open set $U_0 = g(f(U_0))$, $f(U_0)$ is open, hence $g : f(U_0) \to \mathbb{C}$ is continuous.

Now fix $a \in U$ and write $b = f(a)$. Suppose (for the sake of contradiction) $f'(a) = 0$, then there exists $0 < r$ such that for all $w \in B_r(b)$, $f - w$ has 2 zeros counting multiplicities, by Rouche's theorem. We show that there necessarily exists some $w_0 \in B_r(b)$ such that $f - w_0$ has 2 distinct zero in $B_r(z_0)$ which will complete the contradiction. Suppose for all $w \in B_r(b)$, $f - w$ has a single zero of order 2. We have that $g(B_r(b))$ is open. Pick aribitrary $z_0 \in g(B_r(b))$, then $f - f(z_0)$ has a zero order 2 at $z_0$. If $\sum a_n(z - z_0)^n$ is the Taylor series of $f - f(z_0)$ about $z_0$ (convergent in some $B_{r_0}(z_0) \subset g(B_r(b))$, then $f(z_0) - w = a_0 = 0$ and $f'(z_0) = a_1 = 0$. Since $z_0 \in g(B_r(z_0))$ was arbitrary, $f'(z) = 0$ for all $z$ in the open connected set $g(B_r(z_0))$ (where the connectedness is given since $g$ is continuous and $B_r(z_0)$ is connected). Since $f$ is holomorphic, $f$ is constant in $g(B_r(z_0))$, which contradicts the fact that $f$ is injective in $U$. So there necessarily exists some $w_0 \in B_r(b)$ such that $f - w_0$ has 2 distinct zero in $B_r(z_0)$. But again that means $f$ is not injective in U, so our assumption that $f'(a) = 0$ must be wrong. Thus $f'(a) = 0$.

Define $H : U \to \mathbb{C}$ by $H(z) := \begin{cases} \frac{z-a}{f(z)-b} & \text{if } z \neq a \\ \frac{1}{f'(a)} & \text{if } z = a \end{cases}$ Then since $f$ is injective $f(z) \neq b$ for all $z \neq a$ and since $f'(a) \neq 0$ and $f$ is continuous, we have that $H$ is continuous on U. Since $g : V \to \mathbb{C}$ is continuous we have that $H \circ g(w) = \begin{cases} \frac{f^{-1}(w)-f^{-1}(w)}{w-b} & \text{if } w \neq b \\ \frac{1}{f'(f^{-1}(b))} & \text{if } w \neq b \end{cases}$ is continuous. Thus $g = f^{-1}$ is differentiable at $b \in V$. Since $b$ was arbitrary, $g$ is differentiable on $V$ and so $f^{-1} = g$ holomorphic. □

**Theorem 16** (Riemann Mapping Theorem)**.** *Let $U$ be a nonempty open simply connected subset of $\mathbb{C}$ where $U \neq \mathbb{C}$. Then there exists a biholomorphic function $f : U \to B_1(0)$.*

*Proof.* Let $U \in \mathbb{C}$ be an open connected set such that $U \neq \emptyset$ and $U \neq \mathbb{C}$. Fix $z_0 \in U$ and let $\mathcal{F}$ be a family of holomorphic injective functions on $U$ such that for each $f \in \mathcal{F}$, $f(z_0) = 0$ and $|f(z)| < 1$ for all $z \in U$.

Since $U$ is a proper subset of $\mathbb{C}$ , $\mathbb{C}/U$ is not empty and so pick $a \in \mathbb{C}/U$. Since $U$ is simply connected we can define a single valued square root function $g(z) = \sqrt{(}z - a)$ on $U$ where the branch point $z = a$ is not it $U$. Furthermore, we have that $-g(z) \in g(U)$ for all $z \in U$, other wise for some $z$, there exists $w \in U$ such that $g(w) = -g(z)$. This would give us $w = (-g(z))^2 + a = g(z)^2 + a = z$, and so $g(z) = -g(z)$ meaning $g(z) = 0$ and so we get $z = a$ and so the contradiction $a \in U$.

Clearly $g : U \to \mathbb{C}$ is holomorphic and not constant, $U$ is open and (simply) connected. So by the open mapping theorem g(U) is open. Let $0 < r$ be such that $B_r(g(z_0)) \subset g(U)$. For each $w \in B_r(g(z_0))$ we have $| - w + g(z_0)| = |w - g(z_0)| < r$ where $-w \notin g(U)$ for each $w \in B_r(g(z_0))$ so $B_r(g(z_0)) \subset \mathbb{C}/g(U)$. From this we get that for each $z \in U$ $|g(z) + g(z_0)| = |g(z) - -g(z_0)| \geq r$ and so $2|g(z_0)| \geq r$.

Define the function $G : U \to \mathbb{C}$ by $G(z) = \frac{r}{5|g(z_0)|}(\frac{g(z)-g(z_0)}{g(z)+g(z_0)})$. Then $G$ is a composition of $g$ and the mobius transformation $z \mapsto \frac{z-g(z_0)}{z+g(z_0)}$. Since both $g$ and the mobius transformation $\frac{z-g(z_0)}{z+g(z_0)}$ are holomorphic and injective, and the composition of two holomorphic functions are holomorphic, and of two injective functions are injective, $G$ is both holomorphic and injective.

Furthermore, let $z \in U$ be arbitrary. We have $|G(z)| = \frac{r}{5|g(z_0)|}|g(z_0)||\frac{1}{g(z_0)} - \frac{2}{g(z)+g(z_0)}| \leq \frac{r}{5|g(z_0)|}|g(z_0)|(|\frac{1}{g(z_0)}| + |\frac{2}{g(z)+g(z_0)}|) \leq \frac{r}{5|g(z_0)|}|g(z_0)|(\frac{2}{r} + \frac{2}{r}) = \frac{r}{5|g(z_0)|}\frac{4|g(z_0)|}{r} = \frac{4}{5} < 1$. Since $z \in U$ was arbitrary, $|G(z)| < 1$ for all $z \in U$

.
Lastly, $G'(z) = \frac{r}{5|g(z_0)|}\frac{2g'(z)(z_0)}{(g(z)+g(z_0))^2}$
Thus $G \in \mathcal{F}$ and $\mathcal{F}$ is not empty.

Let $M = sup_{f \in \mathcal{F}}(|f'(z_0)|)$ (or $\infty$ if the supremum doesn't exist) and let $(f_n(z_0))$ be a sequence of complex numbers that converges to M and let $(f_n)$ be the corresponding sequence of functions from $\mathcal{F}$. Since $\mathcal{F}$ is uniformly bounded by 1, by Montel's theorem it is a normal family. Let $(f_{n_k})$ be a subsequence of $(f_n)$ that converges uniformly in every compact subsets of $U$ then $(f_{n_k})$ is locally uniformly convergent in $U$. Let f be the limit of $(f_{n_k})$. By theorem 7, $f$ is holomorphic on $U$ and so $f'(z_0)$ exists and is finite. By construction $f'(z_0) = sup_{f \in \mathcal{F}}(|f'(z_0)|) = M$ is finite.

Let $w_0 \in U$ be arbitrary and define the functions $g_k : U/\{w_0\} \to \mathbb{C}$ by $g_k(z) = f_{n_k}(z) - f_{n_k}(w_0)$ for all $n \in \mathbb{N}$ and every $z \in \mathbb{C}$, then $(g_k)$ converges locally uniformly to $f - f(w_0)$. Since $f_n$ is injective for each $n$, $g_n$ is injective for each $n$ and thus $g_n$ does not have any zeros in $U/\{w_0\}$ for all $n$. By Hurwitz's theorem $f - f(w_0)$ does not have any zeroes in $U/\{w_0\}$ and so $f(z) \neq f(w_0)$ for all $z \in U/\{w_0\}$. Since $w_0 \in U$ was arbitrary, $f$ is injective.

Since $(f_{n_k})$ is a sequence of holomorphic functions, locally uniformly convergent to $f$, by Theorem 7 $(f_{n_k})$, is holomorphic. Moreover, by construction $f'(z_0) = M$.

Suppose for the sake of contradiction that the restriction of the range of $f$ to $B_1(0)$, $f : U \to B_1(0)$ is not surjective. Let $w_0 \in B_1(0)$ be such that $f(z) \neq w_0$ for all $z \in U$. Again we can define a single valued, square root function $z \mapsto \sqrt{z}$ which is holomorphic in the simply connected set $U$.

Let $\phi : U \to B_1(0)$ be defined by $\phi(z) = \frac{f(z)-w_0}{1-\overline{w_0}f(z)}$ and the function $F : U \to B_1(0)$ by $F(z) = \sqrt{\phi}(z)$. Let $\gamma$ be a closed piecewise smooth curve where the initial point is some fixed $z_1$ in $U$ and the end point is $z$. Since $U$ is simply connected and $\phi(z) \neq 0$ for all $z \in U$ we can define the composition $log \circ \phi : U \to \mathbb{C}$ by $log \circ \phi(z) = \int_\gamma \frac{\phi'(w)}{\phi(w)}dw$ which is holomorphic on $U$.

Since $F(z) = e^{\frac{1}{2} log \circ \phi(z)}$ for all $z \in U$, F is holomorphic and Injective for all $z \in U$. Also, we have that the derivative of $F$ is

$$F'(z) = \frac{1}{2}\frac{\phi'(z)}{\sqrt{\phi(z)}} = \frac{f'(z_0)(1-|w_0|^2)}{2(1-\overline{w_0}f(z_0))^2}\sqrt{(}\frac{1-\overline{w_0}f(z_0)}{f(z_0)-w_0})$$

where since by construction $f(z_0) = 0$, we have at $z_0$

$$F'(z_0) = \frac{f'(z_0)}{2}(1-|w_0|^2)(\frac{1}{\sqrt{-w_0}})$$

and $F(z_0) = \sqrt{(}\frac{f(z_0)-w_0}{1-\overline{w_0}f(z_0)}) = \sqrt{(}-w_0)$.

Now we define the function $G : U \to \mathbb{C}$ by $G(z) = \frac{F(z)-F(z_0)}{1-\overline{F(z_0)}F(z)}$. We get that $|G'(z_0)| = \frac{|F'(z_0)|}{(1-|F(z_0)|)^2} = \frac{|f'(z_0)||1-|w_0|^2|}{2\sqrt{|w_0|}(1-|w_0|)^2} = \frac{1+|w_0|}{2\sqrt{|w_0|}}|f(z_0)|$.

Since $1 - 2\sqrt{|w_0|} + (\sqrt{|w_0|})^2 = (\sqrt{|w_0|} - 1)^2 > 0$, we have that $1 + |w_0| > 2\sqrt{|w_0|}$ and so $|G'(z_0)| = \frac{1+|w_0|}{2\sqrt{|w_0|}}|f(z_0)| > |f(z_0)| = M$ which contradicts the fact that $M = sup_{f\in\mathcal{F}}|f'(z_0)|$. Thus our function $f : U \to B_1(0)$ must be surjective.

The function $f : U \to B_1(0)$ is then holomorphic and bijective on U, and so by theorem 15 (Inverse Function Theorem), it's inverse is also holomorphic and thus $f$ is a biholomorphic map from $U$ to $B_1(0)$. $\square$

In some literatures, the Riemann mapping theorem is stated interms of the existence of a conformal mapping, conformal meaning angle preserving. We show that if $f$ is biholomorphic in $U$, $f$ is conformal in $U$.

**Definition 12.** *Let $U \subset \mathbb{C}$ be open, $c \in [a, b] \subset \mathbb{R}$, $z \in U$ and $\gamma_1, \gamma_2 : [a, b] \to U$ two smooth curves with $\gamma_1(c) = \gamma_2(c) = z$ and $\gamma_1'(c) \neq 0 \neq \gamma_2'(c)$. Then the* angle between $\gamma_1$ and $\gamma_2$ at $z$ *is the unique $\theta \in (-\pi, \pi]$ such that $\frac{\gamma_1(c)}{\gamma_2(c)} = \frac{\gamma_1(c)}{\gamma_2(c)}e^{i\theta}$.*

**Lemma 6.** *Let $U \subset \mathbb{C}$ be open, $z \in U$ and $\gamma_1, \gamma_2 : [a, b] \to U$ two smooth curves with $\gamma_1(c) = \gamma_2(c) = z$ and $\gamma_1'(c) \neq 0 \neq \gamma_2'(c)$. Let $f : U \to \mathbb{C}$ be a holomorphic function and suppose that $f'(z) \neq 0$. Then the angle between $\gamma_1$ and $\gamma_2$ at $z$ is equal to the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(z)$.*

*Proof.* We have $\frac{f\circ\gamma_1(c)}{f\circ\gamma_2(c)} = \frac{f'(z)\gamma_1(c)}{f'(z)\gamma_2(c)} = \frac{\gamma_1'(c)}{\gamma_2'(c)}$ which proves the lemma. $\square$

**Definition 13.** *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ a function. Then $f$ is called a* conformal map *if $f$ is holomorphic and $f'(z) = 0$ for all $z \in U$.*

**Theorem 17.** *Let $U$ be a nonempty open simply connected subset of $\mathbb{C}$ where $U \neq \mathbb{C}$. Then there exists a surjective conformal map $f : U \to B_1(0)$.*

*Proof.* By theorem 16 (Riemann), there exists a biholomorphic map $f$ from $U$ to $B_1(0)$ and by theorem 15, $f'(z) \neq 0$ for all $z \in U$. Thus $f : U \to \mathbb{C}$ is conformal. $\square$

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