Moduli of Vector Bundles over Algebraic Curves

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Abstract

We give an exposition of some topics pertaining to the moduli of vector bundles over algebraic curves. This includes a construction of the moduli space of stable bundles using geometric invariant theory, a proof of the Chern correspondence and a discussion of Donaldson's proof of the Narasimhan-Seshadri theorem.

ii

Acknowledgements

iv

Contents

Introduction

Classification problems are arguably the most important ones in geometry, and the fact that the objects which one is trying to classify are oftentimes identified canonically and bijectively with points in a geometric space has been known for the majority of the 20th century. For example, Chow and van der Waerden introduced the famous *Chow variety*, which may be thought of as the space of effective cycles of a given codimension and degree over some projective space, in their 1937 paper [7]. This is the idea of a *moduli space*, a space whose points are in natural bijection with the objects that one is trying to classify.

There are various reasons a moduli space may not exist, for example a *jump phenomenon* is, roughly speaking, when there are certain badly-behaved isolated objects. Such is the case for the moduli problem of vector bundles. However, in his ICM talk in 1962 ([29]), Mumford first announced the definition of a *stable* vector bundle on a given curve X over an algebraically closed field k of characteristic zero, and declared that the space of stable bundles of signature (n, d) (i.e. rank n and degree d) has the structure of a "natural" quasiprojective variety, say $V_{n,d}^s$. Since then, stable bundles have been studied extensively, for example in the case $k = \mathbb{C}$, Narasimhan and Seshadri defined a canonical bijection between the $\mathbb{C}$-points of $V_{n,0}^s$ and irreducible $U(n)$ representations of the fundamental group $\pi_1(X)$ ([31, Corollary 1]), and Seshadri later gave a complete and detailed construction of the moduli space $V_{n,d}^s$ as well as its compactification ([40]).

Seshadri's construction makes heavy use of Mumford's *geometric invariant theory* (or GIT), which at its core is a method for taking quotients in algebraic geometry, but finds fundamental applications in moduli theory. For example, it was used by Mumford ([28, §7.4]) to show that the coarse moduli space of (complete nonsingular) curves of genus g is quasiprojective, which contributed to him being awarded a Fields medal in 1974.

Since the introduction of the subject in 1965 (via [28]), GIT has been developed extensively, and surprising links, for example to symplectic geometry, have been found. As a concrete example, the Kempf-Ness theorem ([30, Theorem 8.3]) relates GIT quotients with symplectic quotients constructed through symplectic reduction. In similar fashion, Donaldson found an infinite-dimensional analogue of this ([8]) which gives a map between stable vector bundles and flat unitary connections, and this turns out to give an alternative proof of the theorem of Narasimhan and Seshadri.

These ideas have now been extended to a great degree, for example stability has been defined not just for coherent sheaves on a projective variety ([24]), but as far as for vector bundles on the Fargues-Fontaine curve ([11, II.2.4]), and the correspondences between stable bundles, representations of the fundamental group and flat connections have evolved into, for example, the Kobayashi-Hitchin correspondence ([48]) and the non-abelian Hodge correspondence ([43]).

This thesis aims to present selected topics in the theory of the moduli of stable bundles over a curve to the reader who is familiar with the basics of scheme theory. It is split into two parts, giving an algebraic and an analytic construction of this moduli space.

In Part I, we will be taking a solely algebraic look at this problem, working over an algebraically closed field k of characteristic zero and constructing the space algebraically. Chapter 1 introduces the basics of moduli spaces and includes two detailed examples to illustrate the strategies and difficulties one might encounter in this area of mathematics. One of these is a method of constructing the j-line for elliptic curves (which, to the author's knowledge, is not found in the literature) that additionally serves to motivate GIT, the content of Chapter 2. We will only be looking at the very basics of GIT in this thesis, sufficient for our applications, but this will include a proof of the Hilbert-Mumford criterion for SL_m which aims to combine the utility of the original abstract proof by Mumford ([28, pp. 53-54]) with the intuition of the more elementary proof found in ([27, pp. 216-218]) to give a less hands-on but still quite concrete and easy-to-understand proof. Finally, Chapter 3 constructs the moduli space as the stable locus of a projective GIT quotient of a Quot scheme, but in contrast to Seshadri's construction, we will be using the construction of Le Potier-Simpson ([36], [42]).

In Part II, we adopt an analytic approach and work over $\mathbb{C}$. We reinterpret the problem of classifying vector bundles of a given signature as classifying holomorphic structures on a given smooth vector bundle, and using the Chern correspondence, relate this to the space of unitary connections. We then relate stability to flatness of the corresponding connection, giving an overview of Donaldson's paper [8] before finally constructing the character variety of the fundamental group as a topological space, and using the correspondences to directly topologise the moduli space of stable bundles.

We will assume familiarity with the basic definitions and results of scheme theory, such as found in chapters II and III of Hartshorne [17]. Key resources for Part I include the lecture notes by Hoskins [20], which form the backbone of the entire part, the lecture notes by Le Potier [36], Mumford's book [28] and the standard references in algebraic geometry such as Hartshorne [17] and Vakil [49]. Part II uses Well's book [51], the book by Griffiths-Harris [13] and of course Donaldson's paper [8]. Several theses written by (former) students were also used by the author, their content inspiring several ideas used across this thesis. These theses are [26], [34], [35], [38] and [45].

List of Notation and Conventions

X, Y	Spaces (usually schemes or manifolds).
A, B	Rings (always commutative and with identity).
$A_{\deg d}, M_{\deg d}$	The degree d-component of a graded ring or module.
k	A field.
$X(A)$	The set of A-valued points (where A is a ring) of the scheme X.
Sch	The category of schemes.
Sch/k	The category of schemes over some field k.
FTSch/k	The category of schemes of finite type over some field k.
Var/k	The category of varieties over the field k.
Sets	The category of sets.
$\mathcal{O}_X, \mathcal{O}_Y, \mathcal{O}_S$	Structure sheaves for locally ringed spaces X, Y or S.
$\mathcal{E}, \mathcal{F}, \mathcal{G}$	Sheaves (usually coherent or locally free) on some scheme/manifold.
$\mathcal{E}_p$	The stalk of the sheaf $\mathcal{E}$ at p.
$\mathcal{O}_{X,p}$	The stalk of $\mathcal{O}_X$ at p.
$\mathcal{E} \subseteq \mathcal{F}$	$\mathcal{E}$ is a subsheaf of $\mathcal{F}$.
$\mathcal{E}^{\otimes r}$	The r-fold tensor product of $\mathcal{E}$.
$\mathcal{E}^r$	The r-fold direct sum of $\mathcal{E}$.
$\mathcal{H}om(\mathcal{E}, \mathcal{F}), \mathcal{E}nd(\mathcal{E})$	The **sheaf** of homomorphisms from $\mathcal{E}$ to $\mathcal{F}$ (of endomorphisms of $\mathcal{E}$).
$\mathrm{Hom}(\mathcal{E}, \mathcal{F}), \mathrm{End}(\mathcal{E})$	The **group** of homomorphisms from $\mathcal{E}$ to $\mathcal{F}$ (of endomorphisms of $\mathcal{E}$)
$\mathcal{E}^\vee$	The dual sheaf of the locally free sheaf $\mathcal{E}$. Equal to $\mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$.
$\Gamma(U, \mathcal{F})$	The space of sections over the open set U of the sheaf $\mathcal{F}$.
$\Gamma(\mathcal{F})$	The space of global sections of the sheaf $\mathcal{F}$.
$\mathcal{L}$	A line bundle/invertible sheaf.
$\mathcal{L}(D)$	The line bundle associated to a divisor D.
$\tilde{M}, M^{\sim}$	The quasicoherent sheaf defined by the module M.
$H^i(X, \mathcal{F}), H^i(\mathcal{F})$	The i-th cohomology of the sheaf $\mathcal{F}$ (on the space X).
$h^i(\mathcal{F})$	The dimension, as a vector space, of the cohomology $H^i(\mathcal{F})$.
$\chi(\mathcal{F})$	The Euler characteristic of the coherent sheaf $\mathcal{F}$.
$\mathrm{Spec}\, A$	The spectrum of the ring A.
$\mathrm{Proj}\, A$	The projective scheme associated to the graded ring A.
$\mathbb{A}^n$	Affine n-space over some field k that will be clear from context.
$\mathbb{P}^n$	Projective n-space over some field k that will be clear from context.
$\mathcal{O}_{\mathbb{P}^n}(m)$	The m-fold tensor product of the twisting sheaf of Serre on $\mathbb{P}^n$.
$\mathcal{O}(m), \mathcal{O}_X(m)$	The m-fold tensor product of some very ample line bundle (on X).
ω_X	The canonical bundle on X.

$\mathscr{P}$	The property of smoothness or holomorphicity.
E, F, G	Smooth vector bundles.
T_X	The real smooth tangent bundle on X.
$T_X^{1,0}, T_X^{0,1}$	The holomorphic and anti-holomorphic tangent bundles, respectively.
$\mathcal{T}_X$	The holomorphic tangent bundle on X.
Ω^p	The sheaf of smooth p-forms on some manifold.
$\Omega^{p,q}$	The sheaf of smooth (p,q)-forms on some manifold.
$\bar{\partial}$	The $(0,1)$-component of the exterior derivative.
∂	The $(1,0)$-component of the exterior derivative.
∇	A connection on some vector bundle.
ω_α	The local matrix of 1-forms associated to some connection.
Θ	The curvature of some connection.
$\bar{\partial}_E, \bar{\partial}_{\mathcal{E}}$	Dolbeault operators on some smooth bundle E.
$\partial_E, \partial_{\mathcal{E}}$	The $(1,0)$-component of a connection on some smooth bundle E.
$H_{\mathrm{DR}}^i(X)$	the i-th de Rham cohomology group of X.
vol	The volume form on some manifold.
$\pi_1(X)$	The fundamental group of some connected manifold X.
$\star$	The Hodge star (with respect to a given volume form).
$\mathrm{diag}(a_1, ..., a_n)$	The diagonal matrix with entries the a_i.
$\mathrm{diag}(a)$	The diagonal matrix with constant entry a of some rank clear from context.

In Part 1, our conventions will mostly follow Hartshorne [17], so in particular, rings will always be **commutative with identity** and ring homomorphisms will take the **identity to the identity**. A *variety* will mean an integral scheme, separated and of finite type over some base field k. A *curve* will mean a variety of dimension one. One difference between this thesis and Hartshorne is that **vector bundles will be identified with their sheaf of sections** (so for example, the tautological bundle on $\mathbb{P}^n$ will be identified with $\mathcal{O}_{\mathbb{P}^n}(-1)$). In particular, the correspondence between vector bundles and locally free sheaves will be **covariant**. We will completely use the terms "locally free sheaf" and "vector bundle" interchangeably under this identification. We will also adopt the convention that the product $X \times Y$ will always **mean the fibred product over** k, where k is some base field from context.

In Part II, our conventions will not follow any particular source. On a Riemann surface X, we will adopt the usual **orientation given by** $dx \wedge dy = i dz \wedge d\bar{z}$ for any holomorphic coordinate $z = x + iy$. *Conjugation* of $g \in G$ by $x \in G$ will mean $g^x := x^{-1}gx$, where G is some group. By a *path* in X, we will mean a continuous map $\gamma : [0,1] \to X$. Paths are usually piecewise smooth. To avoid confusion, we will avoid using the term "curve". A *loop* is a path that starts and ends at the same point. If γ_1, γ_2 are paths, then $\gamma_2 \cdot \gamma_1$ will mean γ_2 after γ_1. Complex inner products will be linear in the **first entry**.

Finally, **0 is not a natural number**.

Part I

Algebraic Theory

Chapter 1

Moduli Theory

Commonly, the solution to a classification problem in geometry (especially algebraic geometry) in which the variation between objects is "continuous" rather than discrete takes the form of a *moduli space* which, roughly speaking, is a geometric space whose points are in bijection with equivalence classes of the objects we are classifying. We begin with a very informal example, to illustrate the sort of thing we are looking for:

Example 1.0.1. Consider the set of all circles in $\mathbb{R}^2$, up to equality. A circle is uniquely determined by its centre and its radius, and hence the set of circles is in natural 1-1 correspondence with the set $M = \mathbb{R}^2 \times \mathbb{R}_{>0}$ (we are discounting circles of radius zero), with the $\mathbb{R}^2$-component representing its centre and the $\mathbb{R}_{>0}$-component representing its radius. Written explicitly, we have a bijection $M \to \{\text{circles in } \mathbb{R}^2\}$ given by

$$(a, b, r) \mapsto \{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 = r^2\},$$

and in particular we observe that continuous variation in M in some way is identified with continuous variation of circles.

Now that we have a rough idea of what we are looking for, we make our informal definition:

Informal Definition 1.0.2. A *naïve moduli problem* is a pair $(\mathcal{M}, \sim)$ where $\mathcal{M}$ is a collection of objects and $\sim$ is an equivalence relation on $\mathcal{M}$. We will often just denote the problem by $\mathcal{M}$, and we often assume without loss of generality (for example, by replacing $\mathcal{M}$ by $\mathcal{M}/\sim$) that the equivalence relation is equality. A *naïve moduli space* of $\mathcal{M}$ is a geometric space M equipped with a bijection $\eta : (\mathcal{M}/\sim) \to M$.

Firstly, observe that "geometric space" is undefined in general, which is why this is an informal definition. But even if we insist that a geometric space is a (smooth/Riemannian/Kähler) manifold or a scheme (or a stack), this is still not a very useful notion to work with, since it is literally just a question of cardinality; indeed, a moduli problem $\mathcal{M}$ with cardinality $2^{\aleph_0}$ always has a moduli space, and in fact **any** manifold or variety of positive dimension over $\mathbb{C}$ could be one such moduli space! Hence we have to insist that η has to be "natural" in some way. We dedicate this chapter into formalising and studying this last condition in the context of algebraic geometry.

Remark 1.0.3. We will also encounter naïve moduli spaces which do not fall under this formalism; in fact this is the entire content of Part II. In these cases, our construction will be very ad-hoc: we will take $\mathcal{M}$ as a set, and endow it with a "natural" (topological/smooth/variety etc.) structure, and define our resulting naïve moduli space M as **the** moduli space of $\mathcal{M}$.

1.1   The Functor of Points

Let X be a scheme over a base scheme T. We make the following definition:

Definition 1.1.1. Let S be another T-scheme. An *S-valued point*, or simply *S-point* of X is a T-morphism $p : S \to X$. If S is affine, equal to $\operatorname{Spec} R$ then an S-valued point will be called an *R-valued point*. The set of R-valued points of X will be denoted $X(R)$.

This is perhaps a weird definition to make, since X already has an underlying topological space, so we already have the notion of a "point" of X. To make some sense of it, we consider the following examples:

Example 1.1.2. Let $T = \operatorname{Spec} A$ for some noetherian ring A, let R be an A-algebra, let $I = \langle f_1, ..., f_r \rangle$ be an ideal of $A[x_1, ..., x_n]$ (the noetherian hypothesis is just so that I can be finitely generated), and let $X = \operatorname{Spec} A[x_1, ..., x_n]/I$. Then

$$X(R) = \{ a = (a_1, ..., a_n) \in R^n \mid f_i(a) = 0 \text{ for all } 1 \le i \le r \}$$

(where, by abuse of notation, the equality above means "canonical identification"). Indeed, an element of $X(R)$ is just an A-algebra homomorphism $A[x_1, ..., x_n]/I \to R$, and this is equivalent to giving a tuple $a = (a_1, ..., a_n)$ such that $f_i(a) = 0$ for all i, with the homomorphism given by $x_i \mapsto a_i$.

Example 1.1.3. Retain the notation and hypothesis of the above example. In the special case $A = k$ for an algebraically closed field k and I is prime, the set $X(k)$ is just the maximal ideals of $k[x_1, ..., x_n]/I$, by Hilbert's Nullstellensatz; so in other words $X(k)$ is the object that is classically known as an "affine variety", as defined in [17, I, 1] (in this thesis, a *variety* will be an integral scheme, separated and of finite type over some algebraically closed field). In fact, given any scheme Y of finite type over k, one can show that the k-points and closed points of Y (as a locally ringed space) are in bijection, and we will be making use of this identification without further comment.

The set of S-valued points is (tautologically) the set $\operatorname{Hom}(S, X)$. As S varies, we get a contravariant functor $\operatorname{Hom}(-, X)$, known as the *functor of points* of X. We will study this functor, firstly showing that this determines X up to isomorphism. We work in an arbitrary category C, since our proofs are no more difficult, but the important case is when C is the category of schemes of finite type over a ground field k.

Proposition 1.1.4. *Let X and Y be objects in a category* C. *Suppose* $\alpha : \operatorname{Hom}(-, X) \to \operatorname{Hom}(-, Y)$ *is a natural isomorphism (that is, a natural transformation with an inverse). Then $X \cong Y$*

Proof. Consider the map $\alpha_X : \operatorname{Hom}(X, X) \to \operatorname{Hom}(X, Y)$. Then $\alpha_X(\mathrm{id})$ is a morphism from X to Y. Denote this morphism by f. Now applying α to the map $f : X \to Y$ we get the following commutative diagram:

$$
\begin{array}{ccc}
\operatorname{Hom}(Y, X) & \xrightarrow{\ f^* \ } & \operatorname{Hom}(X, X) \\
\downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle \alpha_X} \\
\operatorname{Hom}(Y, Y) & \xrightarrow{\ f^* \ } & \operatorname{Hom}(Y, X)
\end{array}
$$

where f^* is the map $u \mapsto u \circ f$. Consider $g := \alpha_Y^{-1}(\mathrm{id}) \in \mathrm{Hom}(Y, X)$. The commutativity of the diagram says that $g \circ f = \mathrm{id}$. Now reversing the roles of X and Y will show that $f \circ g = \mathrm{id}$ too, hence f is an isomorphism. $\qquad\square$

We now consider the functor category $\mathrm{Fun}(\mathsf{C}^{\mathrm{opp}}, \mathsf{Sets})$, whose objects are contravariant functors $\mathsf{C} \to \mathsf{Sets}$ and whose morphisms are natural transformations (this is also known as the *presheaf category* on C; recall that a *presheaf* is just a contravariant functor $\mathsf{C} \to \mathsf{Sets}$). There is a natural functor $\mathsf{C} \to \mathrm{Fun}(\mathsf{C}^{\mathrm{opp}}, \mathsf{Sets})$ which sends X to $\mathrm{Hom}(-, X)$ and $X \to Y$ to the obvious natural transformation $\mathrm{Hom}(-, X) \to \mathrm{Hom}(-, Y)$. This functor is known as the *Yoneda embedding*. The name is justified by our first corollary to the following proposition:

Proposition 1.1.5 (Yoneda's Lemma)**.** *Let* C *be a category, let* $F \in \mathrm{Fun}(\mathsf{C}^{\mathrm{opp}}, \mathsf{Sets})$ *be a contravariant functor from* C *into* Sets *and let* A *be an object in* C. *Then there is a canonical bijection between the set of natural transformations* $\mathrm{Hom}(-, A) \to F$ *and* $F(A)$ *given by* $\alpha \mapsto \alpha_A(\mathrm{id})$.

Proof. [20, p. 5]. $\qquad\square$

Before we state our corollaries, recall that a functor $F : \mathsf{C} \to \mathsf{C}'$ is *fully faithful* if for every pair of objects A, B in C, the induced map $\mathrm{Hom}_{\mathsf{C}}(F(A), F(B)) \to \mathrm{Hom}_{\mathsf{C}'}(A, B)$ is a bijection.

Corollary 1.1.6. *The Yoneda embedding is fully faithful.*

Proof. For any given objects A and B, take $F = \mathrm{Hom}(-, B)$ in the above proposition. Then there is a bijection between the set of natural transformations $\mathrm{Hom}(-, A) \to \mathrm{Hom}(-, B)$ and $\mathrm{Hom}(A, B)$, given by $\alpha \mapsto \alpha_A(\mathrm{id})$. Fix some α and write $f := \alpha_A(\mathrm{id})$. Applying the Yoneda embedding to $f : A \to B$ gives the natural transformation $f^* : \mathrm{Hom}(-, A) \to \mathrm{Hom}(-, B)$, where for an object X and $\varphi \in \mathrm{Hom}(X, A)$, we have $f_X^*(\varphi) = f \circ \varphi$. Taking $X = A$ and $\varphi = \mathrm{id}$, we observe

$$f_A^*(\mathrm{id}) = f \circ \mathrm{id} = f = \alpha_A(\mathrm{id})$$

By Yoneda's Lemma, this means $f^* = \alpha$. Conversely, given some $f : A \to B$, we see that $f_A^*(\mathrm{id}) = f$ hence the Yoneda embedding induces the bijection between $\mathrm{Hom}(A, B)$ and the set of natural transformations $\mathrm{Hom}(-, A) \to \mathrm{Hom}(-, B)$ described in Yoneda's Lemma. But the set of natural transformations $\mathrm{Hom}(-, A) \to \mathrm{Hom}(-, B)$ is exactly the set of morphisms from $\mathrm{Hom}(-, A)$ to $\mathrm{Hom}(-, B)$ in the functor category, and hence the embedding is fully faithful. $\qquad\square$

Remark 1.1.7. In fact, Proposition 1.1.4 follows easily from the above corollary too.

Corollary 1.1.8 (Cayley's Theorem)**.** *Let* G *be a group. Then* G *is isomorphic to a subgroup of* $\mathrm{Sym}(G)$.

Proof. We can interpret G as a groupoid G with one object, say x, and automorphism group equal to G. In other words, $\mathrm{Hom}_{\mathsf{G}}(x, x) = G$. Now $\mathrm{Hom}_{\mathsf{G}}(-, x)$ is the image of x via the Yoneda embedding in the functor category $\mathrm{Fun} = \mathrm{Fun}(\mathsf{G}^{\mathrm{opp}}, \mathsf{Sets})$, and by the above corollary $\mathrm{Aut}_{\mathrm{Fun}}(\mathrm{Hom}_{\mathsf{G}}(-, x)) \cong G$. Now each such natural isomorphism induces a bijection of sets $\mathrm{Hom}_{\mathsf{G}}(x, x) \to \mathrm{Hom}_{\mathsf{G}}(x, x)$, in other words an element of $\mathrm{Sym}(G)$, and this is clearly a group homomorphism. It is also injective, since by Yoneda's lemma, a natural transformation $\mathrm{Hom}(-, x) \to \mathrm{Hom}(-, x)$ is completely determined by its value on x. $\qquad\square$

Definition 1.1.9. A contravariant functor is *representable* if it is in the image of the Yoneda embedding. More precisely, a functor F is representable if there exists an object X such that $\mathrm{Hom}(-, X) \cong F$ as functors. If such an X exists, we say that X *represents* F.

Of course, a natural question to ask is whether every functor is representable. The answer is an emphatic **NO**, and as we will see in the next few sections, finding a (fine) moduli space is equivalent to finding a representative of a certain Set-valued contravariant functor from Sch. Such a representative is unique, by Proposition 1.1.4.

1.2   Moduli Problems and Spaces

We will begin by stating our definition:

Definition 1.2.1. Let Sch/k be the category of schemes over a ground field k, and let C be a subcategory of Sch/k (in this thesis, this will usually be the category FTSch/k of schemes of finite type over k, or the category Var/k of varieties over k). A *moduli problem* is a contravariant functor $\mathcal{M} : \mathsf{C} \to \mathsf{Sets}$. An element of $\mathcal{M}(S)$ is known as an (equivalence class of) *families over S* and $\mathcal{M}(S)$ is the *set of families* (up to equivalence) over X. For a morphism $f : T \to S$, the induced morphism $\mathcal{M}(f) : \mathcal{M}(S) \to \mathcal{M}(T)$ is known as the *pullback map*. If Y is in the image of $\mathcal{M}(f)$, then we say Y *is obtained by pullback through f*. In the case $T = \mathrm{Spec}\, k$ and f is a k-valued point which we will denote p, we will write $X_p \in \mathcal{M}(\mathrm{Spec}\, k)$ for $\mathcal{M}(p)(X)$, and we will call X_p the *fibre of X over p*. We will often call $\mathcal{M}(\mathrm{Spec}\, k)$ the *underlying naïve moduli problem*.

This is obviously a very general definition, but in practice our moduli problems will have a certain "flavour" to them. This is probably best illustrated by an example:

Example 1.2.2. Consider the problem of classifying 1-dimensional quotient spaces of k^{n+1}. We will carefully turn this into a moduli problem: firstly, our naïve moduli problem is simply the set of surjective linear maps $k^{n+1} \to k$, where two such maps are equivalent if and only if they have the same kernel, or equivalently, $\varphi \sim \psi$ if and only if there is a fixed $\lambda \in k^*$ such that $\varphi(v) = \lambda \psi(v)$ for all $v \in k^{n+1}$. This should be the set of families over $\mathrm{Spec}\, k$.

Now let S be a scheme over k. We will define a family over S to be a line bundle $\mathcal{L}$ equipped with a surjection $\mathcal{O}_S^{n+1} \to \mathcal{L}$. Two families are equivalent if and only if they have the same kernel. Now given a morphism $f : T \to S$ and a family $\mathcal{L}$ over S, we define the pullback to be $f^*(\mathcal{L})$, and it is not difficult to check that this satisfies the required conditions. Thus we define $\mathcal{M} : \mathsf{Sch}/k \to \mathsf{Sets}$ as

$$\mathcal{M}(S) := \{\mathcal{O}_S^{n+1} \to \mathcal{L}\}/\sim$$

and

$$\mathcal{M}(f : T \to S) := (\mathcal{L} \mapsto f^*(\mathcal{L})).$$

Now observe that if $S = \mathrm{Spec}\, k$, the trivial rank $n + 1$ vector bundle is exactly k^{n+1}, and hence the families over $\mathrm{Spec}\, k$ are exactly the 1-dimensional quotient spaces of k^n, which is what we expect.

This sets the stage for what most of our moduli problems look like. Firstly, we will state what our naïve moduli problem is. A family over S will commonly be a morphism $X \to S$ (in this case, $\mathcal{L}$ is a line bundle), sometimes equipped with some extra structure, such as a coherent sheaf (in this case, a surjection $\mathcal{O}_S^{n+1} \to \mathcal{L}$), satisfying some conditions (usually including some sort of flatness), and the fibres X_p in our definition are literally just the fibres of the morphism.

Remark 1.2.3. This is not the usual definition for a moduli problem. A moduli problem is usually defined as the data of a family over S for every S and a way to pull them back, to which we associate the functor

$$\mathcal{M}(S) := \{\text{families over } S\}/\sim$$

However, the author did not wish to define it this way, since using this definition it is not possible to answer the question "what is **not** a moduli problem?". As a consequence though, we gain a **lot** of things we call "moduli problems", which would otherwise not be the case.

We may now define our first formal notion of a moduli space, specifically a *coarse moduli space*:

Definition 1.2.4. Let $\mathcal{M} : \mathsf{C} \to \mathsf{Sets}$ be a moduli problem. A *coarse moduli space* for $\mathcal{M}$ is a scheme M in C equipped with a natural transformation $\eta : \mathcal{M} \to \mathrm{Hom}(-, M)$, known as a *moduli transformation* such that the following conditions hold:

(i) $\eta_{\mathrm{Spec}\,k} : \mathcal{M}(\mathrm{Spec}\,k) \to \mathrm{Hom}(\mathrm{Spec}\,k, M)$ is a bijection.

(ii) If N is another scheme and $\eta' : \mathcal{M} \to \mathrm{Hom}(-, N)$ another natural transformation, there is a unique morphism $e : M \to N$ such that

$$\eta'_S(X) = e \circ \eta_S(X)$$

for any scheme S in C and family X over S. Note that this means M is unique.

We unpack this definition a little. Firstly, condition (i) states that $\eta_{\mathrm{Spec}\,k}$ induces a bijection between the k-points of M and the underlying naïve moduli problem, which is as expected (one could consider $\eta_{\mathrm{Spec}\,k} : \mathcal{M}(\mathrm{Spec}\,k) \to M(k)$ the underlying naïve moduli space). The functorial condition is a little more interesting: let S be a scheme in C. For a k-point p, we have the following diagram:

$$
\begin{array}{ccc}
\mathcal{M}(S) & \xrightarrow{\;\;\eta\;\;} & \mathrm{Hom}(S, M) \\
\downarrow{\scriptstyle \mathcal{M}(p)} & & \downarrow{\scriptstyle p^*} \\
\mathcal{M}(\mathrm{Spec}\,k) & \xrightarrow{\;\;\eta\;\;} & \mathrm{Hom}(\mathrm{Spec}\,k, M)
\end{array}
$$

Let X be a family over S. Then we get a morphism $\eta_S(X) \in \mathrm{Hom}(S, M)$; call this f. The commutativity of the diagram tells us that the fibre X_p over $p \in S(k)$ is equal to the object in $\mathcal{M}(\mathrm{Spec}\,k)$ corresponding to the point $f(p) \in M(k)$. Finally, the universal property makes M initial with this property, fixing it up to isomorphism.

Of course, a natural question to ask is whether or not either of the two conditions above are obsolete. The answer is no, and we will see why in various contexts.

Example 1.2.5. Let us now show that $\mathbb{P}^n$ is the coarse moduli space of Example 1.2.2. Let S be a scheme, and let $\mathcal{O}_S^{n+1} \to \mathcal{L}$ be a family, so in other words the images of $e_i \in \Gamma(S, \mathcal{O}_S)$ generate $\mathcal{L}$. We define $\eta_S(\mathcal{L})$ as follows: let $U = \operatorname{Spec} A$ be a sufficiently small open affine subset of S, specifically one such that $\mathcal{L}|_U \cong \mathcal{O}_S|_U$. Then the images of e_i in A via $\mathcal{L}$ generate A. Now let $U_i = \operatorname{Spec} A[e_i^{-1}]$ be the corresponding open affine subset, and we define a morphism $U_i \to \mathbb{P}^n$ by composing the map $U_i \to \operatorname{Spec} k[x_0/x_i, ..., x_n/x_i] \cong \mathbb{A}^n$ associated to the ring homomorphism $x_j/x_i \mapsto e_j/e_i \in A$ with the inclusion $\mathbb{A}^n \to \mathbb{P}^n$. It is not hard to check that this glues, and since the e_i generate A, it follows that the U_i cover U. Hence we have a morphism $U \to \mathbb{P}^n$. Now covering X with these open affines, it is clear that this glues on overlaps, and hence we have a morphism $\eta_S(\mathcal{L}) : X \to \mathbb{P}^n$. The key property of $\eta_S(\mathcal{L})$ is that this is the unique morphism $X \to \mathbb{P}^n$ such that $\eta_S(\mathcal{L})^*(\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1))$ is equal to our family; or more precisely the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{O}_S^{n+1} & \xrightarrow{\ =\ } & \eta_S(\mathcal{L})^*(\mathcal{O}_{\mathbb{P}^n}^{n+1}) \\
& \searrow & \downarrow \\
& & \eta_S(\mathcal{L})^*(\mathcal{O}_{\mathbb{P}^n}(1))
\end{array}
$$

and the kernel of the diagonal morphism above is equal to $\ker(\mathcal{O}_S^{n+1} \to \mathcal{L})$. This is shown in the proof of [17, II Theorem 7.1].

We now have to check that $\eta_{\operatorname{Spec} k}$ is bijective. Firstly, we show that it is surjective; so let $p = [p_0 : ... : p_n]$ be a k-point, and assume without loss of generality $p_0 \neq 0$. Then p factors through the open affine subset $\operatorname{Spec} k[x_1/x_0, ..., x_n/x_0]$ as the dual of the k-algebra homomorphism $k[x_1/x_0, ..., x_n/x_0] \to k$ sending $x_i/x_0 \mapsto p_i/p_0$. Then it is clear from the definition of η in the above paragraph that the family $L : e_i \mapsto p_i/p_0$ satisfies $\eta(L) = p$. This proves that $\eta_{\operatorname{Spec} k}$ is surjective. To prove that it is injective, suppose two families $L : e_i \mapsto p_i$ and $L' : e_i \mapsto p_i'$ satisfy

$$
[p_0 : ... : p_n] = \eta_{\operatorname{Spec} k}(L) = \eta_{\operatorname{Spec} k}(L') = [p_0' : ... : p_n'].
$$

Then $p_i p_j' = p_i' p_j$ for all i, j, hence $\ker L = \ker L'$, so in particular L and L' are equivalent, as desired.

Finally, we show the universal property, starting with existence. Let N be another scheme over k and $\eta' : \mathcal{M} \to \operatorname{Hom}(-, N)$ another natural transformation. Applying to $\mathbb{P}^n$, we obtain a map $\eta' : \mathcal{M}(\mathbb{P}^n) \to \operatorname{Hom}(\mathbb{P}^n, N)$, and the natural family $\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1)$ induces a map $\eta'(\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1)) : \mathbb{P}^n \to N$. We claim $e := \eta'(\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1))$ satisfies the desired property. Indeed, let S be any scheme, and chasing $\mathcal{O}_S^{n+1} \to \mathcal{L}$ a family. Then $f = \eta(\mathcal{O}_S^{n+1} \to \mathcal{L}) : S \to \mathbb{P}^n$ satisfies $f^*(\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{O}_S^{n+1} \to \mathcal{L}$. The following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{M}(\mathbb{P}^n) & \xrightarrow{\ f^*\ } & \mathcal{M}(S) \\
\downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta'} \\
\operatorname{Hom}(\mathbb{P}^n, N) & \xrightarrow{\ f^*\ } & \operatorname{Hom}(S, N)
\end{array}
$$

and $\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1)$ in the above diagram, we find

$$\eta'(\mathcal{O}_S^{n+1} \to \mathcal{L}) = e \circ f = e \circ \eta(\mathcal{O}_S^{n+1} \to \mathcal{L}),$$

as desired. Uniqueness is immediate, since any such e satisfies $e \circ \mathrm{id} = \eta'(\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1))$.

We will now show that the second condition (the universal property) in the definition of a coarse moduli space is not obsolete. To this end, we take inspiration from the exercise on [16, p. 4] and consider the moduli problem of one-dimensional subspaces of k^2, (which, if we consider the quotient of such a one-dimensional subspace, is just the moduli problem above for $n = 1$), and suppose k is algebraically closed. As we saw above, the moduli space for this problem is $\mathbb{P}^1$. However, observe that there is a natural map from $\mathbb{P}^1 = \mathrm{Proj}\, k[x_0, x_1]$ to the cuspidal cubic $Y = \mathrm{Proj}\, k[x, y, z]/y^2 z = x^3$, dual to the ring homomorphism $x \mapsto x_0^2 x_1, y \mapsto x_0^3, z \mapsto x_1^3$ (on the level of k-points this is just $[p : q] \mapsto [p^2 q : p^3 : q^3]$). In particular, this map is bijective on k-points (in fact, a homeomorphism), and so composing the η above with $\mathbb{P}^1 \to Y$, we have a natural transformation $\mathcal{M} \to \mathrm{Hom}(-, Y)$ such that the map $\mathcal{M}(\mathrm{Spec}\, k) \to Y(k)$ is bijective. However, the cuspidal cubic is not the coarse moduli space, because $\mathbb{P}^1$ is, and coarse moduli spaces are unique up to isomorphism.

In fact, $\mathbb{P}^n$ satisfies a stronger condition above, in that every family is a unique pullback of $\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1)$. This is formalised as follows:

Definition 1.2.6. Let $\mathcal{M}$ be a moduli problem. A scheme M is a *fine moduli space* for $\mathcal{M}$ if M represents $\mathcal{M}$.

Note that such an M is unique, by Proposition 1.1.4. Also, as expected, a fine moduli space is also a coarse one:

Proposition 1.2.7. *If M is a fine moduli space for $\mathcal{M}$, then for any representation $\eta : \mathcal{M} \to \mathrm{Hom}(-, \mathcal{M})$, the pair (M, η) is a coarse moduli space.*

Proof. The first condition in the definition is satisfied automatically. Now let N be another scheme and $\eta' : \mathcal{M} \to \mathrm{Hom}(-, N)$ another natural transformation. Composing with η^{-1}, we get a natural transformation $\mathrm{Hom}(-, M) \to \mathrm{Hom}(-, N)$. Since the Yoneda embedding is fully faithful (Corollary 1.1.6), this is induced by a unique $e : M \to N$. $\square$

Definition 1.2.8. Let $\mathcal{M}$ be a moduli problem with coarse moduli space M and moduli transformation η. A *tautological family* is a family $X \in \mathcal{M}(M)$ such that for every $p \in M(k)$, we have $X_p = \eta_{\mathrm{Spec}\, k}^{-1}(p)$. If M is a fine moduli space, the family $\eta_M^{-1}(\mathrm{id})$ is known as the *universal family*.

Proposition 1.2.9. *Let $\mathfrak{X}$ denote the universal family. Then it satisfies the following universal property: if Y is a family over S then there exists a unique morphism $f : S \to M$ such that $Y = \mathcal{M}(f)(\mathfrak{X})$. In particular, taking $S = \mathrm{Spec}\, k$, we see that $\mathfrak{X}$ is tautological.*

Proof. Since the moduli transformation η is an isomorphism and hence η_S is bijective, we see that Y

corresponds uniquely to a morphism $f : S \to M$. The following diagram commutes:

$$
\begin{array}{ccc}
\operatorname{Hom}(M, M) & \xrightarrow{\ f^* \ } & \operatorname{Hom}(S, M) \\
\Big\downarrow {\scriptstyle \eta} & & \Big\downarrow {\scriptstyle \eta} \\
\mathcal{M}(M) & \longrightarrow & \mathcal{M}(S)
\end{array}
$$

The result then follows by chasing $\mathrm{id} \in \operatorname{Hom}(M, M)$ in the diagram. $\qquad\square$

In fact, it is not hard to show the converse is true: if $\mathfrak{X}$ is a family over a coarse moduli space M such that every family is pulled back from $\mathfrak{X}$ in a unique way, then M is fine and $\mathfrak{X}$ is universal.

Example 1.2.10. The family $\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1)$ is universal for Example 1.2.5, as every family is obtained uniquely by pullback from this family. In particular, $\mathbb{P}^n$ is fine.

Example 1.2.11. Consider the moduli problem of classifying curves isomorphic to $\mathbb{P}^1$, up to isomorphism. Of course, there is only one, namely $\mathbb{P}^1$ itself. We define a family of genus 0 curves over S to be a flat proper morphism $X \to S$ such that for any $p \in S(k)$ we have

$$
X_p := X \times_S \operatorname{Spec} k \cong \mathbb{P}^1
$$

and if $f : T \to S$ is a morphism and $X \to S$ is a family, we define $f^*(X) = X \times_S T$. This defines our moduli problem. By [18, Proposition 25.1] the coarse moduli space for this is just $\operatorname{Spec} k$, and it is easy to see that $\mathbb{P}^1 \to \operatorname{Spec} k$ is the tautological family. However, there is no universal family, since there exist nontrivial ruled surfaces ([17, V, 2]).

Remark 1.2.12. Note that proving $\operatorname{Spec} k$ is the moduli space in the above example is actually nontrivial. Indeed, it is obvious that the moduli space, if it exists, is a one-point scheme and is thus necessarily equal to the spectrum of a local Artinian ring. However, in order to prove it is reduced, one must make sure that every family over a local Artinian ring is trivial; this follows from [18, Lemma 25.2].

1.3 Examples

We will now study in detail two examples of moduli problems and spaces. Both will illustrate interesting phenomena, and discuss concepts which will be used as motivation later on in the thesis.

1.3.1 Conics in $\mathbb{P}^2$

Our first example is the problem of conics in $\mathbb{P}^2 = \operatorname{Proj} k[x, y, z]$. In particular, we are considering not just the conic, but the embedding in $\mathbb{P}^2$ as well (some texts will refer to a moduli space parameterising objects equipped with an embedding a *parameter space*, but we will not make that distinction here). We will work with schemes of finite type over k, and further we make the assumption that k is algebraically closed. To formalise:

Definition 1.3.1. A *conic* is a closed subscheme of $\mathbb{P}^2$, cut out by a homogeneous polynomial of degree 2 (in particular, we are allowing degenerate conics). Two conics are *equivalent* if and only if they are equal as subschemes of $\mathbb{P}^2$. Now let S be a scheme of finite type over k. We define a *family of conics* over S to be a closed subscheme $X \subseteq \mathbb{P}^2 \times S$, flat over S via the projection, whose scheme-theoretic fibres at k-points are conics in $\mathbb{P}^2$. Two families are *equivalent* if they are equal as subschemes of $\mathbb{P}^2 \times S$. Now let $f : T \to S$ be a morphism and let $X \to S$ be a family over S. We define the *pullback* of X along f is the fibred product of the following diagram:

$$
\begin{array}{ccc}
f^*(X) := X \times_S T & \longrightarrow & X \\
\downarrow & & \downarrow \\
T & \xrightarrow{\ \ f\ \ } & S
\end{array}
$$

The *moduli problem of conics in $\mathbb{P}^2$* is the functor $\mathcal{M} : \mathsf{FTSch}/k \to \mathsf{Sets}$ defined by

$$\mathcal{M}(\mathcal{S}) = \{\text{families over } \mathcal{S}\}$$

and $\mathcal{M}$ maps a morphism $f : T \to S$ to $f^* : \mathcal{M}(S) \to \mathcal{M}(T)$.

The key theorem of this section is:

Theorem 1.3.2. *The scheme* $\mathbb{P}^5 = \mathrm{Proj}[a_0, ..., a_5]$ *is a fine moduli space for the above moduli problem, and the family* $\mathfrak{X} \subseteq \mathbb{P}^2 \times \mathbb{P}^5$ *cut out by the polynomial*

$$a_0 x^2 + a_1 xy + a_2 y^2 + a_3 yz + a_4 z^2 + a_5 zx$$

is the universal family.

Following the approach outlined in the exercise [18, Ex. 1.1], we will prove this after some lemmas. This approach works for general degree d curves and $\mathbb{P}^{\binom{d+2}{2}-1}$ in place of $\mathbb{P}^5$, but for concreteness we will work with $d = 2$. To begin, we have the following:

Lemma 1.3.3. *Let S be a scheme of finite type over k, and let $X \subseteq \mathbb{P}^2 \times S$ be a family over S. Then S can be covered by open affines $\{U = \mathrm{Spec}\, A\}$ such that the restricted family $X|_U \subseteq \mathbb{P}^2_A :=$ $\mathrm{Proj}\, A[x, y, z]$ is cut out by a single homogeneous polynomial (that is, the homogeneous ideal corresponding to $X|_U$ as a closed subscheme of $\mathbb{P}^2_A$ is principal), necessarily of degree 2.*

Proof. Write $\mathcal{I}$ for the sheaf of ideals of X. We have the following short exact sequence of sheaves on $\mathbb{P}^2 \times S$:

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^2 \times S} \to \mathcal{O}_X \to 0, \tag{1.1}$$

where $\mathcal{O}_X$ is considered an $\mathcal{O}_{\mathbb{P}^2 \times S}$-module. Let p be a k-point of S and let $\mathcal{O}_{S,p}$ be the local ring of p, with maximal ideal $\mathfrak{m}_p$. Pulling back (1.1) along the morphism $\mathrm{Spec}\, \mathcal{O}_{S,p} \to S$ and taking the associated graded objects (see [17, p. 118]), we have the following sequence of graded $\mathcal{O}_{S,p}[x, y, z]$-modules

$$0 \to I_p \to \mathcal{O}_{S,p}[x, y, z] \to \Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_{S,p} \to 0, \tag{1.2}$$

which is exact because $\operatorname{Spec}\mathcal{O}_{S,p} \to S$ is just localisation on the level of rings. Now since X is flat over S, it follows $\Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_{S,p}$ is flat over $\mathcal{O}_{S,p}$; in particular, we have

$$\operatorname{Tor}_1^{\mathcal{O}_{S,p}}(\Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_{S,p}, \mathcal{O}_{S,p}/\mathfrak{m}_p) = 0,$$

and so annihilating $\mathfrak{m}_p$, the following sequence of graded $\mathcal{O}_{S,p}/\mathfrak{m}_p[x,y,z] \cong k[x,y,z]$-modules is also exact:

$$0 \to I_p \otimes k \to k[x,y,z] \to \Gamma_*(\mathcal{O}_{X_p}) \to 0,$$

where X_p is the fibre over p. Since $I_p \otimes k$ is generated by its degree 2 component, I_p must be too, and since S is of finite type over k, it follows that the local ring $\mathcal{O}_{S,p}$ is noetherian, and hence I_p is finitely generated as an $\mathcal{O}_{S,p}[x,y,z]$-module, and hence $I_{p,\deg 2}$ is finitely generated as a $\mathcal{O}_{S,p}$-module. Finally, since the $\mathcal{O}_{S,p}/\mathfrak{m}_p$-module $I_{p,\deg 2} \otimes k = I_{p,\deg 2}/\mathfrak{m}_p I_{p,\deg 2}$ is generated by a single element, it follows by Nakayama's lemma that $I_{p,\deg 2}$, as an $\mathcal{O}_{S,p}$-module is also generated by a single polynomial of degree 2.

Now let $U' = \operatorname{Spec} A' \subseteq S$ be an open affine subset. Since S is of finite type over k, its closed points are dense, so we may assume U' contains p, so that $\mathfrak{m}_p$ may be considered a maximal ideal of A' and $\mathcal{O}_{S,p} = A'_{\mathfrak{m}_p}$. Then, analogous to (1.2), we have the following short exact sequence of graded $A'[x,y,z]$-modules:

$$0 \to I_{A'} \to A'[x,y,z] \to \Gamma_*(\mathcal{O}_{X|_{U'}}) \to 0, \tag{1.3}$$

Now as established, I_p is generated by a polynomial of the form

$$f = \frac{s_0}{t_0}x^2 + \frac{s_1}{t_1}xy + \dots + \frac{s_5}{t_5}zx,$$

where $s_i, t_i \in A'$ and $\prod t_i \notin \mathfrak{m}_p$, and hence by the universal property of localisation, the pullback of (1.2) from (1.3) along $A' \to \mathcal{O}_{S,p}$ factors uniquely through $A' \to A = A'[\prod t_i^{-1}]$ and it is clear that I_A is generated by f, as desired. $\qquad\square$

Now that we know that any family is locally cut out by a single polynomial, the plan of attack is clear: we map the coordinates of $\mathbb{P}^5$ to the coefficients of our polynomial. In order for this to be possible, we present the next result:

Lemma 1.3.4. *Let A be a finitely generated k-algebra, and suppose $X \subseteq \mathbb{P}_A^2$ is a flat family over A cut out by $f = s_0 x^2 + \dots + s_5 zx \in A[x,y,z]_{\deg 2}$. Then $s_0, \dots, s_5$ generate A. Conversely, given $s_0, \dots, s_5$ that generate A, the subscheme $X \subseteq \mathbb{P}_A^2$ cut out by f is a flat family (and hence $\mathfrak{X}$ is flat over $\mathbb{P}^5$).*

Proof. Observe that for every $d \geq 0$ the A-module $(A[x,y,z]/f)_{\deg d}$ must be flat. In particular, the map

$$I \otimes (A[x,y,z]/f)_{\deg 2} \to (A[x,y,z]/f)_{\deg 2}$$

is injective, where $I = \langle s_0, \dots, s_n \rangle$. This means $s_0 \otimes x^2 + \dots + s_5 \otimes zx = 0$ in $I \otimes (A[x,y,z]/f)_{\deg 2}$, or in other words there is some $\lambda \in I$ such that $(1-\lambda)f = 0$ in $A[x,y,z]$. Now if $I \neq A$, then it is contained in some maximal ideal, say $\mathfrak{m}$, and localising A at $\mathfrak{m}$, we deduce $(1-\lambda)f = 0$ in $A_{\mathfrak{m}}$. But $1-\lambda$ is a unit in $A_{\mathfrak{m}}$ and hence $f = 0$ in $A_{\mathfrak{m}}$, which is absurd. Hence $I = A$. The converse is clear. $\quad\square$

In particular, the $s_i \in A$ may be considered as global sections of $\operatorname{Spec} A$, which generate the structure sheaf, and hence by Example 1.2.5, this corresponds uniquely to a morphism $\operatorname{Spec} A \to \mathbb{P}^5$ such that $\mathcal{O}_{\mathbb{P}^5}(1)$ pulls back to $\mathcal{O}_{\operatorname{Spec} A}$, and $a_i \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$ pull back to s_i. It is then clear that the family $\mathfrak{X}$ pulls back to the family $\operatorname{Proj} A[x, y, z]/\langle f \rangle$.

Proof of Theorem 1.3.2. Let S be a scheme, and $X \subseteq \mathbb{P}^2 \times S$ a family. Cover X with open affine subsets $U_i = \operatorname{Spec} A_i$ such that $X|_{U_i} := X \times_S U_i \subseteq \mathbb{P}^2_{A_i}$ is cut out by a single polynomial. Then by our previous discussions, we have a unique collection $\varphi_i : U_i \to \mathbb{P}^n$ such that $\varphi_i^*(\mathfrak{X}) = X|_{U_i}$, and hence it suffices to show that these glue. To this end, suppose U_i and U_j are two open sets as above, and suppose $X|_{U_i}$ and $X|_{U_j}$ are defined by f_i and f_j. Then the restriction of $X|_{U_i}$ and $X|_{U_j}$ to $U_i \cap U_j$ agree; in other words they are the same subscheme of $\mathbb{P}^2 \times U_{ij}$. But that means the images of f_i and f_j agree in $A_{ijk}[x, y, z]$ for any affine open $U_{ijk} = \operatorname{Spec} A_{ijk} \subseteq U_i \cap U_j$, and hence φ_i and φ_j also agree, which means the φ_i glue, as desired. $\qquad\square$

Remark 1.3.5. Note that we are crucially not defining conics up to abstract isomorphism (or even up to projective automorphisms); indeed consider the following family over $\mathbb{A}^1$:

$$X = \operatorname{Proj} k[t, x, y, z]/\langle tyz - x^2 \rangle \to \mathbb{A}^1 = \operatorname{Spec} k[t],$$

where $k[t, x, y, z]$ is graded in x, y, z (in other words, t is degree 0). Flatness, which is equivalent to torsion-freeness, is obvious. For every nonzero $\lambda \in \mathbb{A}^1(k)$, the fibre X_λ is a nondegenerate parabola defined by $\lambda yz = x^2$, and in particular is isomorphic to $\mathbb{P}^1$ via the 2-uple embedding followed by scaling. However, the fibre X_0 is the degenerate conic defined by $x^2 = 0$, which is clearly not isomorphic to the $\mathbb{P}^1$ (indeed, the former is not reduced but the latter is). This is an example of a *jump phenomenon*, which is an obstruction to the existence of a moduli space: if a coarse moduli space M exists, there would be a morphism $\mathbb{A}^1 \to M$ which maps each nonzero k-point of $\mathbb{A}^1$ to some $s \in M(k)$, but maps 0 to some $s' \neq s$. In particular, the preimage of the closed point s would be a dense and proper subset, which is clearly impossible, and hence no coarse moduli space exists.

This is now a good time to show that the first condition in the definition of a coarse moduli space is also not obsolete, which we take from the exercise [16, Ex. 1.7]. Consider the moduli problem of reduced conics in $\mathbb{P}^2$, **up to isomorphism**, and make the further assumption $\operatorname{char} k \neq 2$. As we know from the above remark, the family $\operatorname{Proj} k[t, x, y, z]/\langle xy - tz^2 \rangle$ over $\operatorname{Spec} k[t]$ exhibits a jump phenomenon, and so there is no coarse moduli space. However, we claim that $M = \operatorname{Spec} k$, with the natural transformation η sending a k-scheme S to the morphism $S \to \operatorname{Spec} k$ is a natural transformation which satisfies property (ii) of the definition of a coarse moduli space. However, $\eta_{\operatorname{Spec} k}$ is not injective, since the nondegenerate conic and the union of two lines are both reduced conics (in fact, the only two), but $\operatorname{Hom}(\operatorname{Spec} k, \operatorname{Spec} k) = \{\operatorname{id}\}$. Since any scheme with property (ii) is unique, this gives another proof that this moduli problem has no coarse moduli space.

Let N be a scheme equipped with a natural transformation η' of our moduli problem into $\operatorname{Hom}(-, N)$. We want to show that there exists a unique $e : \operatorname{Spec} k \to N$ such that $\eta'(S) = e \circ \eta(S)$ for any relevant scheme S. Uniqueness is obvious, indeed, any such e must also satisfy this property for families of nondegenerate conics, and since a family of nondegenerate conics is a family of nonsingular complete rational curves, uniqueness is guaranteed by Example 1.2.11. We now prove existence, that is, this e above does satisfy the required property. Let S be a scheme and let $X \subseteq \mathbb{P}^2 \times S$ be a family. As

above, let $U = \operatorname{Spec} A$ be a sufficiently small affine open subset of S, so that $X|_U$ is cut out by a single polynomial $f = s_0 x^2 + \ldots + s_5 zx \in A[x, y, z]_{\deg 2}$ (the difference is here we only care about X up to isomorphism). Firstly, observe that the degenerate locus of U (that is, the locus where very fibre is degenerate) is closed; indeed it is defined by the vanishing of the determinant of the following matrix:

$$\begin{pmatrix} 2s_0 & s_1 & s_5 \\ s_1 & 2s_2 & s_3 \\ s_5 & s_3 & 2s_4 \end{pmatrix}.$$

In particular, the nondegenerate locus of U, call it $U^\flat$ is open and (applying this to every irreducible component) dense, and taking the union across all such U, it follows that the nondegenerate locus of S, say $S^\flat$ is dense. Now by Example 1.2.11, we have $\eta'(X|_{S^\flat}) = e \circ \eta(X|_{S^\flat})$, and since $S^\flat$ is dense and the image of $S^\flat$ in N is a closed point, and moreover $\eta'(X|_{S^\flat}) = \eta'(X)|_{S^\flat}$ it follows that $\eta'(X) = e \circ \eta(X)$ too, as desired.

We conclude with a remark. Observe that a family of conics (in the original problem of) may alternatively be thought of as a family of quotients (or equivalently subsheaves) of $\mathcal{O}_{\mathbb{P}^2}$. Indeed, a conic may be identified with its coherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}}$. What distinguishes conics (or indeed degree d curves for any $d > 0$), is their *Hilbert polynomial*, a concept which will be discussed in Chapter 3. In general, given any projective variety X, a coherent sheaf $\mathcal{F}$ on X and a numerical polynomial $P \in \mathbb{Q}[z]$ (that is, $P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$), there exists a fine moduli space, known as the *Quot scheme* of $\mathcal{F}$, often denoted $\operatorname{Quot}_X^P(\mathcal{F})$, parameterising quotients of $\mathcal{F}$ with Hilbert polynomial P. If $X = \mathbb{P}^n$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}$, then the Quot scheme is called a *Hilbert scheme*; and in particular we have proven that the Hilbert scheme of $\mathbb{P}^2$ with Hilbert polynomial $P = 2z + 1 \in \mathbb{Q}[z]$ is $\mathbb{P}^5$.

1.3.2   Elliptic Curves

Recall our general method in the previous example: we found a candidate space and a candidate universal family $\mathfrak{X} \to M$, showed that for a general $X \to S$, there is locally a unique morphism for an open subscheme $U \subseteq S$ such that $X \times_S U$ is pulled back from $\mathfrak{X}$, and finally we showed that these glue. This approach is illustrative of a typical approach for constructing moduli spaces, with some simplifications of course. The first and most obvious is that a moduli space need not be fine, and thus finding a candidate universal family is not always possible. Instead, what often happens is that we look for a *locally versal* family, that is, an overparameterised (i.e. there are repeated elements) family $X_0 \to T$ with the property that for a general $X \to S$, there is an open cover $\{U_i\}$ of S and for each U_i a (not necessarily unique, hence "versal" as opposed to "universal") morphism $\varphi_i : U_i \to T$ such that $X \times_S U_i$ is the pullback of X_0 via φ_i. One then needs to find a way to contract the isomorphic fibres of T, which then defines a coarse moduli space. We will illustrate this technique now, in the context of elliptic curves. Throughout this section, we fix an algebraically closed ground field k of characteristic neither 2 nor 3.

Definition 1.3.6. An *elliptic curve* is a complete nonsingular curve X over k of genus 1, equipped with a distinguished point $p_0 \in X(k)$. A *family of elliptic curves* over a scheme S of finite type

over k is a scheme X equipped with a flat morphism $X \to S$ and a section $s : S \to X$ such that for any $p \in S(k)$, the fibre X_p is genus 1 curve, which is an elliptic curve with distinguished point $p^*s : \operatorname{Spec} k \to X_p$. Two families over S are equivalent if they are isomorphic as S-schemes. It is clear how families pull back along morphisms of finite type, and so we have the moduli problem of elliptic curves, which we will denote $\mathcal{M}_{1,1} : \mathsf{FTSch}/k \to \mathsf{Sets}$.

A detailed study of elliptic curves will take us too far afield, so we will focus solely on the study of their moduli space, and for that all we need to know is that for any family $(X \to S, s)$, there exists an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ of S such that $X|_{U_i}$ can be embedded inside $\mathbb{P}^2_{A_i}$ with an equation of the form $y^2 z = x^3 + axz^2 + bz^3$ (called a *Weierstrass cubic*) for $a, b \in A_i$, with $\Delta := 4a^3 + 27b^2$ a unit in A_i (Δ is called the *discriminant*), and s is the constant section $[0 : 1 : 0] \in X|_{U_i} \subseteq \mathbb{P}^2_{A_i}$ (see p. 47]Script). Conversely, any Weierstrass cubic with an invertible discriminant is a family of elliptic curves. In particular, the family over $R := k[a, b, \Delta^{-1}]$ defined by the above equation, call it X_0, is a locally versal family.

The next question to ask is when are two curves in Weierstrass form equivalent. It turns out that after an elementary (but tedious) calculation, two families $\operatorname{Proj} A[x, y, z]/\langle y^2 z - (x^3 + p_1 x z^2 + q_1 z^3)\rangle$ and $\operatorname{Proj} A[x, y, z]/\langle y^2 z - (x^3 + p_2 x z^2 + q_2 z^3)\rangle$ are isomorphic if and only if there is some invertible $u \in A^*$ such that $p_1 = u^4 p_2$ and $q_1 = u^6 q_2$ ([41, III, Table 1.2] presents this for individual curves, but the calculation could easily be adapted for families), with an isomorphism of the form $x \mapsto u^2 x, y \mapsto u^3 y$ (of course, one direction of this is easy, the hard part is showing that any isomorphism is of this form). In particular, there is a k^* action on $\operatorname{Spec} R$ via automorphisms, dual to the ring homomorphism

$$u \cdot a = u^4 a, \ u \cdot b = u^6 b,$$

such that the the orbit of a k-point p consists exactly the points q whose fibre is isomorphic to the fibre at p.

Lemma 1.3.7. *Let $\eta' : \mathcal{M}_{1,1} \to \operatorname{Hom}(-, N)$ be a natural transformation. Then*

$$\eta'(X_0) : \operatorname{Spec} R \to N$$

is k^*-invariant.

Proof. This follows since the k^* action on $\operatorname{Spec} R$ lifts to X_0, and moreover X_0 and $u \cdot X_0$ are equivalent for all $u \in k^*$. □

Our plan of attack is thus clear: we find a scheme M equipped with a natural transformation $\eta : \mathcal{M}_{1,1} \to \operatorname{Hom}(-, M)$ such that $\eta(X_0)$ is initial with respect k^*-invariant morphisms in some sense. Since any family $X \to S$ induces a local morphism to $\operatorname{Spec} R$, one should expect that such an (M, η) satisfies property (ii) in the definition of a coarse moduli space, and one can then hope that property (i) is satisfied.

In our quest to find such an M, we make the following definition:

Definition 1.3.8. Let $X = \operatorname{Proj} k[x, y, z]/\langle y^2 z - (x^3 + pxz^2 + qz^3)\rangle$ be an elliptic curve. The *j-invariant* of X is the quantity

$$j = 1728 \frac{4p^3}{4p^3 + 27q^2}.$$

Note that this only depends on the isomorphism class of X.

Lemma 1.3.9. *The map $X \mapsto j$ is a bijection between isomorphism classes of elliptic curves and $k = \mathbb{A}^1(k)$.*

Proof. We follow the proof in [41, pp. 51-52]. Let X and Y be elliptic curves given by the respective equations $y^2 z = x^3 + p_1 x z^2 + q_1 z^3$ and $y^2 z = x^3 + p_2 x z^2 + q_2 z^3$ and suppose they have the same j-invariant, that is,

$$4p_1^3(4p_2^3 + 27q_2^2) = 4p_2^3(4p_1^3 + 27q_1^2).$$

Then rearranging we find

$$p_1^3 q_2^2 = p_2^3 q_1^2.$$

Now if $p_1 = 0$, whence $j = 0, q_1 q_2 \neq 0, p_2 = 0$, we find that taking $u = (q_1/q_2)^{1/6}$ (any 6-th root will do) we have

$$0 = p_1 = u^4 p_2 = 0, \ q_1 = (\frac{q_1}{q_2})q_2 = u^6 q_2$$

as desired. Otherwise, $p_1 \neq 0$, hence $j \neq 0, p_2 \neq 0$, we have

$$q_1^2 = (\frac{p_1}{p_2})^3 q_2^2,$$

and thus one of the square roots of $(p_1/p_2)^3$, call it $(p_1/p_2)^{3/2}$, satisfies

$$q_1 = (\frac{p_1}{p_2})^{(3/2)} q_2.$$

Now take u to be any 6-th root of $(p_1/p_2)^{3/2}$ (and hence u is a 4-th root of p_1/p_2), and thus we have

$$q_1 = u^6 q_2 \ , p_1 = \frac{p_1}{p_2} p_2 = u^4 p_2,$$

as desired. This proves injectivity.

To prove surjectivity, let $j \in k$ be given. If $j = 0$, then $p = 0, q = 1$ will do. Otherwise, take $p = 1$, and since k is algebraically closed, there will be a solution for q. $\square$

Lemma 1.3.10. *Let A be a ring, let k^* act on R dually to the action on $\operatorname{Spec} R$, and let $\varphi : A \to R$ a k^*-invariant ring homomorphism. Then φ factors uniquely through through the inclusion $k[j] \to R$ given by*

$$j \mapsto 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Proof. Uniqueness is obvious, since $k[j] \to R$ is injective. To prove existence, grade the ring $R = k[a, b, \Delta^{-1}]$ with $a \in R_{\deg 2}$, $b \in R_{\deg 3}$, and hence observe $\Delta = 4a^3 + 27b^2 \in R_{\deg 6}$. Now observe that u acts on the degree d component of R by $u \cdot f = u^{2d} f$. In particular, an element $f \in R$ is fixed by the k^* action (which preserves the grading) if and only if f is degree 0, and hence it suffices to

identify $k[j]$ (or more precisely the image of the inclusion of $k[j]$) with $R_{\deg 0}$. To this end, firstly observe that clearly $k[j] \subseteq R_{\deg 0}$. For the other inclusion, suppose $f \in R_{\deg 0}$. Then we may write

$$f = \sum_{i \geq 0} \sum_{2m+3n=6i} c_{mn} \frac{a^m b^n}{\Delta^i},$$

where $c_{m,n} \in k$, and all but finitely many vanish. Now since $2m + 3n = 6i$, it follows that n is even and m is a multiple of 3, hence we can write $n = 2n'$, $m = 3n'$. Thus

$$f = \sum_{i \geq 0} \sum_{m'+n'=i} c_{m'n'} \frac{a^{3m'} b^{2n'}}{\Delta^i} = \sum_{i \geq 0} \sum_{m'+n'=i} c_{m'n'} (\frac{a^3}{\Delta})^{m'} (\frac{b^2}{\Delta})^{n'} = \sum_{i \geq 0} \sum_{m'+n'=i} c_{m'n'} (\frac{a^3}{\Delta})^{m'} (\frac{-4a^3}{27\Delta})^{n'}$$

and observe

$$(\frac{a^3}{\Delta})^{m'} (\frac{-4a^3}{27\Delta})^{n'} = \frac{(-1)^{n'}}{1728^i \times 4^{m'} \times 27^{n'}} j^i,$$

and so defining

$$c_i := \frac{1}{1728^i} \sum_{m'+n'=i} c_{m'n'} \frac{(-1)^{n'}}{4^{m'} \times 27^{n'}}$$

we find

$$f = \sum_{i \geq 0} c_i j^i \in k[j]$$

as desired. $\qquad \square$

Corollary 1.3.11. *More generally, if $S \subseteq k[j]$ is a multiplicative subset, and $\varphi : A \to S^{-1}R$ is a k^*-invariant homomorphism, then φ factors through $S^{-1}k[j] \to S^{-1}R$.*

Finally, we may construct our moduli space. Let $X \to S$ be a family of elliptic curves. Cover S with open affine subsets $U_i = \operatorname{Spec} A_i$ such that

$$X|_{U_i} = \operatorname{Proj} A_i[x,y,z]/\langle y^2 z - (x^3 + a_i x + b_i)\rangle$$

for $a_i, b_i \in A_i$. Then we have a morphism $f_i : \operatorname{Spec} A_i \to \operatorname{Spec} R = \operatorname{Spec} k[a,b,\Delta^{-1}]$ for each i dual to the homomorphism of rings by $a \mapsto a_i, b \mapsto b_i$ such that $X|_{U_i}$ is the pullback of X_0 along this map. Composing this with $\pi : \operatorname{Spec} R \to \operatorname{Spec} k[j]$, we have a map $\operatorname{Spec} A_i \to \operatorname{Spec} k[j]$.

Lemma 1.3.12. *The f_i glue into a morphism $\eta_S(X) : S \to \operatorname{Spec} k[j]$ which does not depend on our choice of cover $\{U_i\}$. Moreover, the map $(X \to S) \mapsto \eta_S(X)$ is a natural transformation $\eta : \mathcal{M}_{1,1} \to \operatorname{Hom}(-, \operatorname{Spec} k[j])$.*

Proof. Let U_i, U_j as above be given. Then the restriction of $X|_{U_i}$ and $X|_{U_j}$ to any affine open subset $U_{ijk} = \operatorname{Spec} A_{ijk} \subseteq U_i \cap U_j$ agree, and so there is some unit $u \in A_{ijk}^*$ such that

$$a_i = u^4 a_j, \ b_i = u^6 b_j$$

in A_{ijk}. In particular, if $\varphi_i : k[j] \to A_{ijk}$ is the dual homomorphism to $f_i|_{\operatorname{Spec} A_{ijk}}$ and similarly with φ_j, we find

$$\varphi_i(j) = \frac{4a_i^3}{4a_i^3 + 27b_i^2} = \frac{4a_j^3 u^{12}}{(4a_j^3 + 27b_j^2)u^{12}} = \frac{4a_j^3}{4a_j^3 + 27b_j^2} = \varphi_j(j),$$

as desired. Since we can cover the overlap with such U_{ijk}, and since the U_i cover S, we have a morphism $S \to \operatorname{Spec} k[j]$. Moreover, by the same argument, if we choose a different cover then we get the same morphism, because the Weierstrass cubics will differ between the covers by a unit as above. The fact that this induces a natural transformation is just a lot of obvious checking. $\qquad\square$
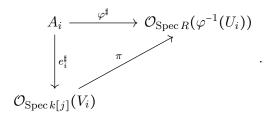
And finally, we have:

Theorem 1.3.13. *The pair* $(\operatorname{Spec} k[j], \eta)$ *is a coarse moduli space for* $\mathcal{M}_{1,1}$.

Proof. We need to check that the two conditions in the definition of a coarse moduli space are satisfied. Condition (i) is just Lemma 1.3.9, and hence we just have to check that η satisfies the required universal property. So let $\eta' : \mathcal{M}_{1,1} \to \operatorname{Hom}(-, N)$ be another natural transformation. By Lemma 1.3.7, the map $\eta'(X_0) : \operatorname{Spec} R \to N$ is k^*-invariant, and so the idea is to now apply Lemma 1.3.10 and Corollary 1.3.11 to show that $\eta'(X_0)$ factors through $\operatorname{Spec} R \to \operatorname{Spec} k[j]$, and that this factorisation is functorial and unique.

First, we claim that any k^*-invariant morphism $\varphi : \operatorname{Spec} R \to N$ factors uniquely through $\pi : \operatorname{Spec} R \to \operatorname{Spec} k[j]$. To see this, we will first find a suitable open cover of $\operatorname{Spec} k[j]$ on which we locally define the map. Let $\{U_i = \operatorname{Spec} A_i\}$ be an open affine cover for N. Then their preimages $\{\varphi^{-1}(U_i)\}$ cover $\operatorname{Spec} R$, and moreover are k^*-invariant. In particular, if $I_i \subseteq R$ is the ideal of the closed subset $\operatorname{Spec} R \setminus \varphi^{-1}(U_i)$, then I_i must also be k^*-invariant, and moreover $\sum I_i = R$. We claim $\sum(I_i \cap k[j]) = k[j]$. To see this, let $E : R \to R$ denote the $k[j]$-module homomorphism sending $\sum_{d \in \mathbb{Z}} f_d$ for homogeneous f_d (via the grading in the proof of Lemma 1.3.10) of degree d to $f_0 \in k[j]$ (this is known as the *Reynolds operator* of this action). Since $\sum I_i = R$, we have $1 = \sum f_i$ for $f_i \in I_i$, and all but finitely many f_i are zero, and hence

$$1 = E(1) = E\left(\sum f_i\right) = \sum E(f_i) \in \sum(I_i \cap k[j])$$

as claimed. In particular, if $V_i = \operatorname{Spec} k[j] \setminus V(I_i \cap k[j])$, then it follows that the V_i cover $\operatorname{Spec} k[j]$.

Now observe that $\pi^{-1}(V_i) = \varphi^{-1}(U_i)$, and since $\varphi'(X_0)$ is k^*-invariant, by Corollary 1.3.11 (since every open subset of $\operatorname{Spec} k[j]$ is a distinguished open affine subset) there exists a unique map $e_i^\sharp : A_i = \mathcal{O}_N(U_i) \to \mathcal{O}_{\operatorname{Spec} k[j]}(V_i)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A_i & \xrightarrow{\ \varphi^\sharp\ } & \mathcal{O}_{\operatorname{Spec} R}(\varphi^{-1}(U_i)) \\[2mm]
{\scriptstyle e_i^\sharp}\big\downarrow & {\scriptstyle \pi}\nearrow & \\[2mm]
\mathcal{O}_{\operatorname{Spec} k[j]}(V_i) & &
\end{array}
\qquad .
$$

Thus dual to each $e_i^\sharp$ is a morphism of affine schemes $e_i : V_i \to U_i$. It is clear they glue, and hence we have a morphism $e : \operatorname{Spec} k[j] \to N$ which commutes with φ. This proves existence. Uniqueness follows from the uniqueness of the $e_i^\sharp$. This proves the claim.

Now as mentioned, $\eta'(X_0)$ is k^*-invariant, and as above we have a morphism $e : \operatorname{Spec} k[j] \to N$. We now show that e is the unique morphism such that $\eta' = e \circ \eta$. Uniqueness follows from the claim above, since any such e must be compatible with the k^*-invariant $\eta'(X_0)$. To show that e does satisfy the required property, let $X \to S$ be a family over S. Then we may cover S with open affine subsets $U_i = \operatorname{Spec} A_i$ such that $X|_{U_i}$ is the pullback of X_0 via some $f_i : U_i \to \operatorname{Spec} R$. The following diagram commutes:

$$
\begin{array}{ccc}
& \operatorname{Spec} R & \\
f_i \nearrow & \Big\downarrow \pi & \searrow \eta'(X_0) \\
U_i & & N \\
\searrow \eta(X|_{U_i}) & & \nearrow e \\
& \operatorname{Spec} k[j] &
\end{array}
$$

and by the previous lemma, the $\eta(X|_{U_i})$ glue to a global morphism $\eta(X) : S \to \operatorname{Spec} k[j]$, and hence we have $\eta'(X) = e \circ \eta(X)$, as desired. $\qquad \square$

Finally, to kill any false hope that may have brewed, we have the following result:

Proposition 1.3.14. *The moduli problem of elliptic curves does not have a fine moduli space.*

Proof. Let $A = k[t, t^{-1}]$ and consider the two families $X = \operatorname{Proj} A[x, y, z]/\langle y^2 z - (x^3 + t z^3) \rangle$ and $Y = \operatorname{Proj} A[x, y, z]/\langle y^2 z - (x^3 + z^3) \rangle$ over $\operatorname{Spec} A$. Then j is constant, and equal to 1728 in both families, and hence if $\operatorname{Spec} k[j]$ is a fine moduli space with universal family $\mathfrak{X}$, then both families should be isomorphic to $(j \mapsto 1728)^*(\mathfrak{X})$. But these families are not isomorphic, otherwise there would be some invertible $u \in k[t, t^{-1}]$ satisfying $t = u^6$, which is not possible. Hence $\operatorname{Spec} k[j]$ is not a fine moduli space, and since any fine moduli space is a coarse moduli space, and coarse moduli spaces are unique, there is no fine moduli space. $\qquad \square$

We conclude this chapter with a remark. Observe that a key step in this proof is showing that $\pi : \operatorname{Spec} R \to \operatorname{Spec} k[j]$ is the initial k^*-invariant morphism from $\operatorname{Spec} R$. In the language to be developed, this is a *categorical quotient*, and more generally, one can show that the categorical quotient of the action of a group acting on the base of a locally versal family parameterising isomorphic fibres will satisfy property (ii) in the definition of a coarse moduli space. In our case, it just so happened that an orbit space exists, and hence property (i) is also satisfied, but this is not always the case; indeed in the language to be developed, this is a consequence of the fact that every k-point in $\operatorname{Spec} R$ is *stable*. These concepts, as well as the general methodology for taking quotients in algebraic geometry, forms the basis of the subject of *geometric invariant theory*, which is the topic of the next chapter.

Chapter 2

Geometric Invariant Theory

We begin with a motivating example:

Example 2.0.1. Let k be an algebraically closed field and let k^* act on k^2 by $\lambda \cdot (p, q) := (\lambda^{-1}p, \lambda q)$. The orbits consist of the axes without the origin, the origin and for every $t \in k^*$ the curve $xy = t$. If we consider the orbit space, it resembles k, indeed each nonzero t will represent the curve $xy = t$; however where the origin should be, we find three orbits, one of which is closed and two of which has closure equal to the union of the three orbits. But if the orbit space of $k^2 = \mathbb{A}^2(k)$ was given the structure of a scheme (i.e. we have a morphism $\mathbb{A}^2 \to Y$ and the map of k-valued points is a bijection between orbits of k^2 and $Y(k)$), then it cannot be separated over k, and hence no variety (integral scheme, separated and of finite type over k) could be an orbit space.

But observe that if we consider the action restricted to $k^2 \smallsetminus \{(0,0)\}$, the orbit space is canonically identified with $k \smallsetminus \{0\}$, which are the k-valued points of the variety $\operatorname{Spec} k[t, t^{-1}]$.

This raises the question, what is the "best" approximate quotient of an action and what sort of properties does it have? And can we always throw away "bad" orbits, like the restriction of k^2 to $k^2 \smallsetminus \{(0,0)\}$, so that an orbit space does exist? Geometric invariant theory allows us to answer these questions under certain circumstances.

Being a vast and difficult subject, however, we do not have the time to develop the subject in detail. Our focus will be on the action of affine algebraic groups (to be defined) on quasi-projective varieties over k. For a full account, see [28]. Our exposition roughly follows the one found in [20].
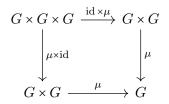
2.1 Algebraic Groups and Actions

We begin by formalising the notion of a *group action*, since the notion of an abstract group acting on a set is not sufficient for our purposes. Indeed, a morphism of schemes is not determined by where it sends its points (for example, there are two endomorphisms of the one-point scheme $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$), and even if we define an action of some G on some X as a group homomorphism into $\operatorname{Aut} X$, it is possible for G to have some scheme-theoretic structure (as in differential geometry where there is a concept of a Lie group), which will not be picked up if we treat G as a collection of points. Hence we need the stronger notion of an *algebraic group* acting on a scheme. We will begin with a very elementary approach to the subject, laying out all the formalisms, and doing calculations and examples extremely naïvely and from scratch.
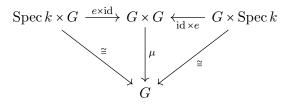
Definition 2.1.1. An *algebraic group* over k, also known as a *group scheme* over k is a scheme G over k equipped with a k-point $e : \operatorname{Spec} k \to G$, known as the *identity element* and morphisms

$\mu : G \times G \to G$ and $\iota : G \to G$ known as *multiplication* and *inversion* respectively such that the following three diagrams commute:
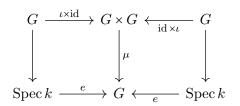
(i) (Associativity)

$$
\begin{array}{ccc}
G \times G \times G & \xrightarrow{\mathrm{id} \times \mu} & G \times G \\
\downarrow{\scriptstyle \mu \times \mathrm{id}} & & \downarrow{\scriptstyle \mu} \\
G \times G & \xrightarrow{\quad \mu \quad} & G
\end{array}
$$

(ii) (Identity)

$$
\begin{array}{ccccc}
\mathrm{Spec}\, k \times G & \xrightarrow{e \times \mathrm{id}} & G \times G & \xleftarrow{\mathrm{id} \times e} & G \times \mathrm{Spec}\, k \\
& \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \cong} & \\
& & G & &
\end{array}
$$

(iii) (Inverse)

$$
\begin{array}{ccccc}
G & \xrightarrow{\iota \times \mathrm{id}} & G \times G & \xleftarrow{\mathrm{id} \times \iota} & G \\
\downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow \\
\mathrm{Spec}\, k & \xrightarrow{\quad e \quad} & G & \xleftarrow{\quad e \quad} & \mathrm{Spec}\, k
\end{array}
$$

An algebraic group G is *affine* if G is an affine scheme.

Remark 2.1.2. There are subtle variations in the definition of the terms "algebraic group" and "group" scheme from one source to another (indeed, they are usually different!). For example, algebraic groups are sometimes required to be of finite type over k, the definition of a group scheme is usually a "group-valued functor represented by a scheme" (which we will make sense of very shortly), and sometimes algebraic groups are required to be varieties. However, since we will only ever deal with algebraic groups which are affine varieties, these subleties do not matter in this thesis.

It is important to emphasise that while an algebraic group does have the structure of a group, in this thesis they will **not** be considered groups; instead they will be thought of as **group-valued functors**, since their functor of points factors through the category of groups. Indeed, for any scheme S, the set $\mathrm{Hom}(S, G)$ has a group structure, with group operation

$$(f : S \to G) \cdot (g : S \to G) := (\mu \circ (f, g) \circ \Delta : S \to G)$$

where Δ is the diagonal map. The identity morphism is the composition $S \to \mathrm{Spec}\, k \xrightarrow{e} G$ where the first map is the unique morphism $S \to \mathrm{Spec}\, k$, and the inverse of $f : S \to G$ is simply $\iota \circ f$. In particular, using Yoneda's lemma, one may show that all of the statements deduced from the usual group axioms (uniqueness of identity and inverses, compatibility of inversion with homomorphisms, etc.) are valid for algebraic groups.

Now observe that since G has a scheme structure, the group operations induce co-operations of k-algebras. In fact, if G is affine, we may completely work with k-algebras, since there is an equivalence of categories. In this case, we will call $\Gamma(G, \mathcal{O}_G)$ the *associated co-group*.

Example 2.1.3. Let $G = \operatorname{Spec} k[t, t^{-1}]$, whose k-points are of course canonically identified with k^*. Now k^* obviously has a canonical multiplication which makes it a group; we will now extend this to an algebraic group structure on G. Note that since a morphism of varieties is detemined by its k-points ([17, II Proposition 2.6]), this extension is unique, if it exists. Define the co-multiplication

$$\mu^\sharp : k[t, t^{-1}] \to k[t, t^{-1}] \otimes k[t, t^{-1}]$$
$$t \mapsto t \otimes t$$

Using the identification $k[t, t^{-1}] \otimes k[t, t^{-1}] \cong k[t_1^{\pm 1}, t_2^{\pm 1}]$, we may write this in the more familiar way as $t \mapsto t_1 t_2$. Taking Spec, we get a morphism $G \times G \to G$, and we observe the induced map of k-points is simply $(p, q) \mapsto pq$ as expected. Then the co-identity is the k-algebra map $k[t, t^{-1}] \to k$ given by $t \mapsto 1$, and co-inversion is the endomorphism $t \mapsto t^{-1}$. We check the group axioms by checking the co-axioms on the co-group: note that

$$(\operatorname{id} \otimes \mu^\sharp) \circ \mu^\sharp(t) = t \otimes t \otimes t = (\mu^\sharp \otimes \operatorname{id}) \circ \mu^\sharp(t)$$

and extending algebraically this proves associativity. To check identity, we observe

$$(e^\sharp \otimes \operatorname{id}) \circ \mu^\sharp(t) = 1 \otimes t \cong t \cong t \otimes 1 = (\operatorname{id} \otimes e^\sharp) \circ \mu^*(t)$$

as desired, where, by abuse of notation, $\cong$ denotes the image of $1 \otimes t$ under the isomorphisms $k \otimes k[t^{\pm 1}] \cong k[t^{\pm 1}] \cong k[t^{\pm 1}] \otimes k$. Finally, we check inversion:

$$(\iota^\sharp \otimes \operatorname{id}) \circ \mu^\sharp(t) = 1 = e^\sharp(t)$$

and similarly in the other direction. Thus we have our expected group axioms.

Henceforth, we will denote $\operatorname{Spec} k[t, t^{-1}]$ by $\mathbb{G}_m$ (note that m stands for multiplication, not for any specific number).

Before stating our next example, a few comments about notation are in order. We will use $\operatorname{GL}_n(R)$ to denote the (abstract) group of invertible n-by-n matrices over R, and we will use V to describe the vector space k^n. In particular, these are **not** schemes. We will use $\mathbb{A}^n$ to describe the variety/scheme $\operatorname{Spec} k[x_1, ..., x_n]$, and we will use GL_n or GL_V to describe the variety/scheme, which is defined below.

Example 2.1.4. Observe $\operatorname{GL}_n(k)$ may be identified as the k-points of the affine scheme $\operatorname{Spec} k[x_{ij}, 1 \le i, j \le n, \det(x_{ij})^{-1}]$, which we will denote GL_n. Co-multiplication is given by

$$\mu^\sharp(x_{ij}) := \sum_{k=1}^n x_{ik} \otimes x_{kj}$$

or, once again, making the identification

$$k[x_{ij}, 1 \le i, j \le n, \det(x_{ij})^{-1}] \otimes k[x_{ij}, 1 \le i, j \le n, \det(x_{ij})^{-1}] \cong k[x_{ij}, y_{ij}, \det(x_{ij})^{-1}, \det(y_{ij})^{-1}]$$

this is just the familiar

$$\mu^{\sharp}(x_{ij}) = \sum_{k=1}^{n} x_{ik} y_{kj}$$

Similarly, the co-identity is given by $e^{\sharp}(x_{ij}) := \delta_{ij}$. The co-inversion is a little difficult to write explicitly, but $\iota^{\sharp}(x_{ij})$ is the i,j-th entry of the n-by-n matrix $(x_{k\ell})_{1\leq k,\ell\leq n}$, which may be shown to be algebraic. Once again, we can check the group axioms for this by checking the co-group co-axioms:

$$(\mu^{\sharp} \otimes \mathrm{id})(\mu^{\sharp}(x_{ij})) = (\mu^{\sharp} \otimes \mathrm{id})(\sum_{k} x_{ik} \otimes x_{kj})$$
$$= \sum_{k} \sum_{\ell} x_{i\ell} \otimes x_{\ell k} \otimes x_{kj}$$
$$= \sum_{k} \sum_{\ell} x_{ik} \otimes x_{k\ell} \otimes x_{\ell j}$$
$$= (\mathrm{id} \otimes \mu^{\sharp})(\sum_{k} x_{ik} \otimes x_{kj})$$
$$= (\mathrm{id} \otimes \mu^{\sharp})(\mu^{\sharp}(x_{ij}))$$

proves associativity and

$$(e^{\sharp} \otimes \mathrm{id})(\mu^{\sharp}(x_{ij})) = (e^{\sharp} \otimes \mathrm{id})(\sum_{k} x_{ik} \otimes x_{kj}) = \sum_{k} \delta_{ik} \otimes x_{kj} = \sum_{i} 1 \otimes x_{ij} \cong x_{ij}$$

and similarly proves identity. We will not prove inversion, but it follows from the properties of multiplying matrices. SL_n and PGL_n are defined similarly.

Observe that the group of invertible n-by-n matrices is also identified with the group of R-valued points of GL_n, hence both interpretations of the notation $\mathrm{GL}_n(R)$ agree.

Example 2.1.5. Let G be a finite group. We will endow G with a natural algebraic group structure. Write $n := |G|$. By Corollary 1.1.8, we may embed G into S_n, the symmetric group on n letters, and S_n, in turn, embeds into $\mathrm{GL}_n(k)$ via permutation matrices. We therefore may interpret G as a discrete, closed subscheme of GL_n, and the group axioms inherit from the group axioms in GL_n.

Next we will discuss algebraic group actions.

Definition 2.1.6. Let G be an algebraic group and X a scheme over k. An *action* of G on X is a morphism $\sigma : G \times X \to X$ such that the following diagrams commute:
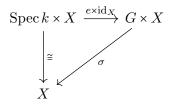
(i) (Associativity)

$$
\begin{array}{ccc}
G \times G \times X & \xrightarrow{\mathrm{id}_G \times \sigma} & G \times X \\
\downarrow{\scriptstyle \mu \times \mathrm{id}_X} & & \downarrow{\scriptstyle \sigma} \\
G \times X & \xrightarrow{\quad \sigma \quad} & X
\end{array}
$$

(ii) (Identity)

$$\operatorname{Spec} k \times X \xrightarrow{e \times \operatorname{id}_X} G \times X$$

with the diagonal map $\cong$ down to X and σ from $G \times X$ to X.

Example 2.1.7. The most straightforward example is G acting on itself via the map μ. Associativity and identity follow directly from the corresponding group axioms.

Let G act on X. We observe a few things: passing to k-points, the group $G(k)$ acts (as an abstract group, in the usual sense) on $X(k)$. Any k-point $g : \operatorname{Spec} k \to G$ induces an automorphism of X given by $\sigma(g \times \operatorname{id})$, which we will denote σ_g (on the level of $X(k)$, this is simply multiplication by g). Similarly, any k-point $p : \operatorname{Spec} k \to X$ induces a morphism $G \to X$.

Now assuming G and X are both affine, equal to $\operatorname{Spec} R$ and $\operatorname{Spec} A$ respectively, the action induces a co-action homomorphism of k-algebras $\sigma^\sharp A \to R \otimes A$. However, the group $G(k)$ also has an induced action (in the usual sense) on A via automorphisms: indeed, an element g of $G(k)$ is dual to a ring homomorphism $g^\sharp : R \to k$, and thus composing with the co-action homomorphism we have an automorphism of A:

$$A \to R \otimes A \to k \otimes A \cong A.$$

However, since the scheme-ring duality is contravariant, the composition works in the opposite direction. Hence the action is given by

$$g \cdot f = (g^\sharp)^{-1} \circ \sigma^\sharp.$$

It is easy to check that on k-points, the action is given by

$$g \cdot f(p) = f(g^{-1} \cdot p).$$

Example 2.1.8. Consider the action described in Example 2.0.1. We will extend this to an algebraic group action of $\mathbb{G}_m$ on $\mathbb{A}^2$. The associated coordinate rings are $k[t^{\pm 1}]$ and $k[x, y]$. We now define the co-action $\sigma^\sharp : k[x, y] \to k[t^{\pm 1}] \otimes k[x, y]$ by $x \mapsto t^{-1} \otimes x$ and $y \mapsto t \otimes y$; it is not hard to check the axioms and to show that this induces the action in the aforementioned example. Now we compute the action of $\mathbb{G}_m(k)$ on $k[x, y]$: let $\lambda \in k^* \cong \mathbb{G}_m(k)$ be given. This induces the map $k[t^{\pm 1}] \to k$ defined by $t \mapsto \lambda$, hence we have

$$\sigma_\lambda^\sharp(x) = \lambda^{-1} \otimes x = 1 \otimes (\lambda^{-1} x) \cong \lambda^{-1} x$$

and

$$\sigma_\lambda^\sharp(y) = \lambda \otimes y = 1 \otimes \lambda y \cong \lambda y$$

which is easily seen to be a group action.

Example 2.1.9. We consider the natural action of $\operatorname{GL}_n(k)$ (the group of invertible n-by-n matrices) on $V = k^n$. We will lift this to an algebraic group action of GL_n on $\mathbb{A}^n$. The coordinate rings are $k[x_{ij}, \det(x_{ij})^{-1}]$ and $k[v_1, ..., v_n]$. We define the co-action

$$\sigma^\sharp : k[v_1, ..., v_n] \to k[x_{ij}, \det(x_{ij})^{-1}] \otimes k[v_1, ..., v_n]$$

by

$$\sigma^\sharp(v_i) := \sum_{j=1}^n x_{ij} \otimes v_j$$

and for a k-point $g = (g_{ij}) \in \mathrm{GL}_n(k)$ (which is induced by $x_{ij} \mapsto g_{ij}$), we have the following automorphism $\sigma_g^\sharp$ of $k[v_1, ..., v_n]$:

$$\sigma_g^\sharp(v_i) = \sum_j g_{ij} \otimes v_j = \sum_j 1 \otimes g_{ij}v_j \cong \sum_j g_{ij}v_j$$

and in order for it to be a group action, we have to take the inverse of this map.

Definition 2.1.10. Let G and H be algebraic groups. A *homomorphism* of algebraic groups is a morphism of schemes $f : G \to H$ such that the following diagram commutes:

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu_G} & G \\ \downarrow{\scriptstyle f \times f} & & \downarrow{\scriptstyle f} \\ H \times H & \xrightarrow{\mu_H} & H \end{array} \qquad (2.1)$$

A homomorphism $\rho : G \to \mathrm{GL}_n$ will be called a *representation*. Composing this with the action on $\mathbb{A}^n$, we see that ρ induces an action of G on $\mathbb{A}^n$ and hence of $G(k)$ on $V = k^n$.

Example 2.1.11. Let G, H be any affine algebraic groups. Then the *trivial group homomorphism* $G \to H$ is the morphism of schemes

$$G \longrightarrow \mathrm{Spec}\, k \xrightarrow{e_H} H.$$

Clearly (2.1) commutes.

Example 2.1.12. Consider the representation $\rho : \mathbb{G}_m \to \mathrm{GL}_2$ induced by

$$\rho^*(x_{ij}) = \delta_{ij} t^{2i-3}.$$

Composing this with the natural action of GL_2 on $\mathbb{A}^2$ described in Example 2.1.9 we obtain the action in Example 2.1.8.

Note that the image of the induced morphism $k^* \to \mathrm{GL}_2(k)$ in the above representation is contained in the subgroup of diagonal matrices. This is no coincidence, as we will see below. This result is hugely important, and as we will soon see, representations of $\mathbb{G}_m$ play a critical role in our further discussions (for example, in our analysis of stability). For now, we end with the following theorem:

Theorem 2.1.13. *Let $\rho : \mathbb{G}_m \to \mathrm{GL}_n$ be a representation. Then there is a decomposition*

$$k^n =: V = \bigoplus_{m \in \mathbb{Z}} V_m$$

where

$$V_m := \{v \in V \mid \lambda \cdot v = \lambda^m v, \ \forall \lambda \in k^*\}$$

*is known as the m-**th weight space** of the action.*

Proof. [50, Section 4.6] $\square$

2.2 Reductive Groups and the Affine GIT Quotient

In this section, we will conduct a study on reductive groups and their action on affine schemes, and in the sequel we will be exclusively looking at the action of reductive algebraic groups. The reason for this is that the reductive hypothesis gives us many useful finiteness conditions. We will begin by studying the algebraic properties of the action of a reductive algebraic group on an affine scheme, and then translate these into geometric properties. First, the definition:

Definition 2.2.1. A group G is *(linearly) reductive* over k if, for every representation $\rho : G \to \mathrm{GL}_n(k)$, we can decompose $V = k^n$ into irreducible subrepresentations (i.e. G-invariant subspaces). An algebraic group is reductive if its group of k-valued points is.

Example 2.2.2. If G is a finite group, then it is reductive if $\mathrm{char}\, k$ does not divide $|G|$, by Maschke's Theorem.

Example 2.2.3. As we saw in Theorem 2.1.13, $\mathbb{G}_m$ is reductive. Indeed, given a representation $\rho : \mathbb{G}_m \to \mathrm{GL}_V$, we have a weight space decomposition $V = \bigoplus V_i$ and any subspace of each V_i is a subrepresentation.

Example 2.2.4. In characteristic zero, it is well known that many "familiar" algebraic groups (such as $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n$) are reductive.

Now let G be a reductive group with a representation $G \to \mathrm{GL}_V(k)$, where V is finite-dimensional, and let $V^G \subseteq V$ be the subspace where G acts trivially; it is clear that this is a G-invariant subspace. Then by reductivity, we have a decomposition $V = V^G \oplus W$ for some subrepresentation W.

Lemma 2.2.5. *The W above is unique.*

Proof. Suppose $V = V^G \oplus W = V^G \oplus W'$. By reductivity, we may decompose W as $W = \bigoplus W_i$, where each W_i is an irreducible subrepresentation. Let $w \in W_i$, so that $w = v_0 + w'$ for some $w' \in W'$. Since $w, w' \notin V^G$, it follows that there is some $g \in G$ which acts nontrivially on w; thus $g(w) - w \neq 0$. But $g(w) = v_0 + g(w')$ and hence

$$g(w) - w = g(w') - w' \in W_i \cap W' \smallsetminus \{0\}.$$

Since W_i is irreducible, it follows that $W_i \subseteq W'$. Since this holds for each W_i, it follows $W \subseteq W'$, and reversing their roles, we obtain $W' \subseteq W$ as well, as desired. $\square$

Definition 2.2.6. Let G be an (abstract) group and let $\rho : G \to \mathrm{GL}_V(k)$ be a representation. The *Reynolds operator* of ρ, denoted E_ρ, is defined to be the projection onto V^G. By the above lemma, this is well-defined. If G is an algebraic group, and $\rho : G \to \mathrm{GL}_V$ is a representation, then the Reynolds operator of ρ is the Reynolds operator of $\rho(k) : G(k) \to \mathrm{GL}_V(k)$.

We saw an example of a "Reynolds operator" in the proof of Theorem 1.3.13, where we projected onto our ring of invariant elements. This type of argument is very useful for showing something is invariant, since we need only exhibit it in the form $E_\rho(v)$, and so we would like to adapt this concept into infinite dimensions in certain circumstances; in particular for the induced action on the coordinate

ring of an affine scheme. To this end, let $G = \operatorname{Spec} R$ be an affine algebraic group acting on an affine scheme $X = \operatorname{Spec} A$ over k. Then as mentioned, there is an induced action of $G(k)$ as an abstract group on A via k-algebra automorphisms, which must a-priori be k-linear. In particular, we have the following:

Lemma 2.2.7. *Every element of A is contained in a finite-dimensional $G(k)$-invariant subspace. In particular, we have*

$$A = \varinjlim V_r$$

where the limit is taken across all $G(k)$-invariant subspaces of A.

Proof. Following [28, p. 26], recall that the action induced by $g \in G(k)$ is given by composing the co-action homomorphism $\sigma^\sharp : A \to R \otimes A$ with the dual k-algebra homomorphism $g^\sharp \circ \iota^\sharp : R \to k$ induced by the inverse of g. Now fix some $f \in A$, let R^* denote the vector space $\operatorname{Hom}(R, k)$ (which is essentially just $G(k)$), and let V be the image of the map $\alpha : R^* \to A$ sending u to $(u \otimes \operatorname{id}_A)(\sigma^\sharp(f))$. We claim V satisfies our desired properties. To see this, observe firstly that by the identity axiom of group actions, the co-identity homomorphism $e^\sharp \in R^*$ satisfies $\alpha(e^\sharp) = f$, and so $f \in V$. To see that this is finite dimensional, simply observe that if $\sigma^\sharp(f) = \sum h_i \otimes f_i$ then $V \subseteq \operatorname{span}\{f_i\}$. Finally, to see that V is $G(k)$-invariant, observe that by co-associativity, we have

$$\sigma^\sharp(\alpha(u)) = (u \otimes \sigma^\sharp)(\sigma^\sharp(f)) = ((u \otimes \operatorname{id}_R \otimes \operatorname{id}_A) \circ (\mu^\sharp \otimes \operatorname{id}_A) \circ \sigma^\sharp)(f),$$

hence for any $u' \in R^*$, we have

$$u' \cdot \alpha(u) = (u \otimes u' \otimes \operatorname{id}_A) \circ (\mu^\sharp \otimes \operatorname{id}_A)(\sigma^\sharp(f)) = (((u \otimes u') \circ \mu^\sharp \otimes \operatorname{id}_A) \circ \sigma^\sharp)(f) = \alpha((\mu^\sharp)^*(u \otimes u'))$$

where $(\mu^\sharp)^*(u \otimes u') \in R^*$ is the map $h \mapsto (u \otimes u')(\mu(h))$, as desired. $\square$

Corollary 2.2.8. *Every finite dimensional subspace is contained in a finite dimensional $G(k)$-invariant subspace.*

Proof. Apply the above argument to every element in a basis. $\square$

Definition 2.2.9. Let G be an affine reductive algebraic group acting on an affine scheme $X = \operatorname{Spec} A$. The *Reynolds operator* of this action is the k-linear map $E : A = \varinjlim V_r \to A$ induced by the Reynolds operator on each V_r. By the universal property of direct limits and the above lemma, it is well-defined. The *ring of invariants*, denoted A^G is the image of E.

Proposition 2.2.10 (Reynolds identity)**.** *Let $a \in A^G$ and $r \in A$. Then we have*

$$E(ar) = aE(r).$$

In particular, the Reynolds operator is an A^G-module homomorphism.

Proof. Firstly, let V be an irreducible finite dimensional subrepresentation. Then aV is also irreducible, and $v \mapsto av$ is a $k[G(k)]$-module homomorphism (where $k[G(k)]$ is the group algebra), and thus by Schur's lemma, is either trivial or an isomorphism.

Now let V be a finite-dimensional subrepresentation of A containing $a, r, E(ar)$, and $E(r)$, and let $V = \bigoplus_{i=0}^{m} V_i$ be an irreducible decomposition, with V_0 the subrepresentation of invariants. Then we may write $r = \sum r_i$, and hence

$$E(ar) = E(a \sum r_i) = ar_0 + \sum_{i \neq 0} E(ar_i).$$

Now for each $i \neq 0$, either $aV_i = 0$ or $aV_i \cong V_i$ in which case $v_i \mapsto av_i$ is an isomorphism. In the former case, ar_i is already zero, in the latter case, there is some g such that $g(r_i) - r_i \neq 0$ and hence

$$g(ar_i) - ar_i = ag(r_i) - ar_i = a(g(r_i) - r_i) \neq 0,$$

whence $E(ar_i) = 0$, and hence

$$E(ar) = ar_0 = aE(r)$$

as desired. $\qquad\square$

Corollary 2.2.11. *Let G be a reductive algebraic group acting on an affine scheme $X = \operatorname{Spec} A$.*

(i) *If B is an A^G-algebra, then there is an induced action on $B' := A \otimes_{A^G} B$, and B is the ring of invariants of B', with the induced action.*

(ii) *If $\{I_i\}$ is a collection of invariant ideals of A, then*

$$\left(\sum I_i\right) \cap A^G = \sum(I_i \cap A^G).$$

(iii) *If I is an invariant ideal of A, then the ring of invariants of A/I is $A^G/(A^G \cap I)$.*

Proof. We follow [28, pp. 28-29]. Firstly, note by the existence of the Reynolds operator E on A as an A^G-module homomorphism, we have the decomposition $A = A^G \oplus A'$ as A^G-modules, where $A' = \ker E$. Hence

$$B' = (A^G \oplus A') \otimes_{A^G} B = B \oplus A' \otimes_{A^G} B,$$

which means B is a subring of B' and so the statement makes sense. Now observe that E induces a B-module homomorphism $E' : B' \to B'$ sending $\sum(a_i + r_i) \otimes b_i$, where $a_i \in A^G, r_i \in A'$, to $\sum a_i b_i$. The action of $G(k)$ on B' is given by $g \cdot \sum a_i \otimes b_i = \sum g(a_i) \otimes b_i$. It is not hard to check that E' is the Reynolds operator of the induced action, and that $\operatorname{im} E' = B$. This proves (i).

To prove (ii), firstly note that clearly $\sum(I_i \cap A^G) \subseteq (\sum I_i) \cap A^G$. To prove the other inclusion, note that if $\sum f_i \in (\sum I_i) \cap A^G$, then

$$\sum f_i = E(\sum f_i) = \sum E(f_i).$$

It therefore suffices to prove that $E(I_i) \subseteq I_i$. To this end, let $f \in I_i$. Then f is contained in some finite-dimensional subrepresentation V, and $V \cap I_i$ is finite-dimensional and $G(k)$-invariant. Thus by reductivity, we may decompose $V \cap I_i$ into subrepresentations $V \cap I_i = (V \cap I_i)^G \oplus W$, and hence

$$E(f) = f - \pi_W(f) \in I_i$$

as desired.

Finally, to prove (iii), observe that the action on A/I is given by $g \cdot \bar{f} := \overline{(g \cdot f)}$, where $\bar{f} := f$ mod I. It is clear that the Reynolds operator on A/I is given by

$$\bar{E}(\sum_{i=0}^{n} \bar{a}_i) := \overline{E(\sum_{i=0}^{n} \bar{a}_i)} = \bar{a}_0,$$

where $a_0 \in A^G$, and from this the result is immediate. $\square$

Henceforth, we will assume $\operatorname{char} k = 0$. Arguably the most important result about reductive groups acting on affine schemes is the following:

Theorem 2.2.12. *Let G be a reductive algebraic group acting on an affine scheme $X = \operatorname{Spec} A$ of finite type over k, so that $G(k)$ acts on A via automorphisms. Then the ring of invariants A^G is a finitely generated k-algebra.*

Proof. Following [22, pp. 92-93], firstly, we claim A^G is noetherian. To this end, let $I_1 \subseteq I_2 \subseteq ...$ be a chain of ideals in A^G. Then $I_1 \otimes_{A^G} A \subseteq I_2 \otimes_{A^G} \subseteq ...$ is an increasing chain of ideals in A, which terminates since A is noetherian. Thus there is some $n_0 \in \mathbb{N}$ such that $I_i \otimes_{A^G} A = I_{i+1} \otimes_{A^G} A$ for all $i \geq n_0$. But observe that for any ideal $I \subseteq A^G$, we have

$$I = \sum_{f \in I} f A^G = \sum_{f \in I} (fA \cap A^G) = (\sum_{f \in I} fA) \cap A^G = (I \otimes_{A^G} A) \cap A^G$$

and so

$$I_i = (I_i \otimes_{A^G} A) \cap A^G = (I_{i+1} \otimes_{A^G} A) \cap A^G = I_{i+1}$$

for all $i \geq n_0$ too, proving the claim.

Now we first prove this result in the case $X = \mathbb{A}^n = \operatorname{Spec} k[x_1, ..., x_n]$. Since $G(k)$ acts via automorphisms, the action must preserve the graded pieces of the natural grading on $A = k[x_1, ..., x_n]$, and thus A^G will also be graded. It is clear that the irrelevant ideal $A^G_+ = A^G \cap A_+$ generates A^G, and by the above claim, A^G_+ is finitely generated, proving the result for $\mathbb{A}^n$.

Finally, we prove the general case, following [28, p. 29]. Let $f_1, ..., f_n$ be generators of A as a k-algebra, let V be a finite-dimensional $G(k)$-invariant subspace containing them, and let $R := \operatorname{Sym}(V)$ be the symmetric algebra on V, which is a polynomial ring. Then the action on A induces an action on R and there is a natural surjective equivariant map $\varphi : R \to A$. Clearly $\ker \varphi$ is invariant, and thus by Corollary 2.2.11 (iii), we have $A^G = R^G/(R^G \cap \ker \varphi)$. Since the action on R preserves its degree 1 component, it follows that this is linear and hence algebraic (i.e. G acts on $\operatorname{Spec} R$ as an algebraic group on a scheme), thus by the previous paragraph R^G is finitely generated. The result then follows from the observation that if $\{v_1, ..., v_m\}$ generate R^G, then $\{\varphi(v_1), ..., \varphi(v_m)\}$ generate A^G. $\square$

We are now in a position to construct the quotient:

Definition 2.2.13. Let X be an affine variety, and G a reductive algebraic group acting on X. The *affine GIT quotient* is the map $\varphi^G : X \to \operatorname{Spec} A^G$ induced by the inclusion $A^G \to A$. We denote $\operatorname{Spec} A^G$ by $X /\!\!/ G$.

Observe that this map is G-invariant, in the sense that the following diagram commutes:

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\ \sigma\ } & X \\
\downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \varphi^G} \\
X & \xrightarrow{\ \varphi^G\ } & X /\!/ G
\end{array}
$$

Indeed, one can check this on the level of rings, and this follows since A^G are exactly the elements $f \in A$ such that $\sigma^\sharp(f) = 1 \otimes f$. In fact, the affine GIT quotient satisfies a stronger property, which we will formalise:

Definition 2.2.14. Let G be an algebraic group acting on a scheme X of finite type over k. An FTSch/k-*categorical quotient* of this action is a G-invariant k-morphism of finite type $\varphi : X \to Y$ such that if $\varphi' : X \to Z$ is another such G-invariant morphism then there exists a unique $f : Y \to Z$ such $\varphi' = f \circ \varphi$.

As the reader may have guessed, the affine GIT quotient is an FTSch/k-categorical quotient, which we will now prove in conjunction with other important properties:

Theorem 2.2.15. *The affine GIT quotient satisfies the following:*

(i) *The map φ^G is surjective on k-points.*

(ii) *For any open subset $U \subseteq X /\!/ G$, the map $\mathcal{O}_{X/\!/G}(U) \to \mathcal{O}_X((\varphi^G)^{-1}(U))$ is an isomorphism onto $\mathcal{O}_X((\varphi^G)^{-1}(U))^G$.*

(iii) *The image of every $G(k)$-invariant closed subset is closed.*

(iv) *If W_1 and W_2 are disjoint, $G(k)$-invariant and closed, then there exists $f \in A^G$ such that for any $p_1 \in W_1(k)$ and $p_2 \in W_2(k)$, we have $f(p_1) = 0$ and $f(p_2) = 1$.*

(v) *φ^G is affine.*

(vi) *φ^G is an FTSch/k-categorical quotient.*

Proof. We follow [20, p. 31] and [28, p. 28]. Let $p \in (X /\!/ G)(k)$ be a k-point with maximal ideal $\mathfrak{m}$. We claim $\mathfrak{m} \otimes_{A^G} A$ is a proper ideal in A. Indeed if not, then $1 = \sum f_i \otimes a_i$ where $f_i \in \mathfrak{m}, a_i \in A$, and so

$$
1 = E(1) = \sum E(f_i \otimes a_i) = \sum f_i E(a_i) \in \mathfrak{m},
$$

a clear contradiction. In particular, $\mathfrak{m} \otimes_{A^G} A$ is contained in a maximal ideal $\mathfrak{m}'$ in A associated to some $q \in X(k)$, and it thus follows that $\varphi^G(q) = p$ as desired. This proves (i).

(ii) follows directly from (i) in Corollary 2.2.11.

To prove (iii), let W be an invariant closed subset corresponding to an invariant ideal $I \subseteq A$, and suppose for contradiction its image is not closed. We first claim that there must be a closed point in $\overline{\varphi^G(W)} \smallsetminus \varphi^G(W)$. To this end, recall that by Chevalley's theorem on constructible sets ([15, Théorèm

1.8.4]), the image $\varphi^G(W)$ is contructible (i.e. a finite union of locally closed subsets), and hence its complement in its closure $\overline{\varphi^G(W)} \smallsetminus \varphi^G(W)$ is also constructible. Let $U \cap V$ be a nonempty locally open subset of $\overline{\varphi^G(W)} \smallsetminus \varphi^G(W)$, where U is open and V is closed (both in X), and let $p \in U \cap V$. We claim that $U \cap V$ must contain some closed point of $\overline{\{p\}}$. Indeed, V clearly does, and observe that if U does not, then all the closed points of $\overline{\{p\}}$ are contained in the complement $X \smallsetminus U$, which is closed. But this means the closure of these closed points is also contained in $X \smallsetminus U$, and since $\operatorname{Spec} A^G$ is of finite type over k, these closed points are dense in $\overline{\{p\}}$ ([17, II Ex. 3.14]), so in particular $p \notin U$, a clear contradiction. This proves the claim, and thus we may find some $q \in \overline{\varphi^G(W)}(k) \smallsetminus \varphi^G(W)(k)$. Write $W' := (\varphi^G)^{-1}(q)$, and observe that W' is closed, invariant and nonempty by (i). Denote its ideal by $I' \subseteq A$. By property (ii) of Corollary 2.2.11, it follows that

$$(I + I') \cap A^G = I \cap A^G + I' \cap A^G,$$

and translated into geometric language, this says that

$$\overline{\varphi^G(W \cap W')} = \overline{\varphi^G(W)} \cap \{q\} = \{q\}.$$

In particular, this means that $W \cap W' \neq \varnothing$, contradicting the fact that q is contained in the complement of $\varphi^G(W)$. This proves (iii).

To prove (iv), let I_1, I_2 be the ideals corresponding to W_1, W_2 respectively. The assertion that they are disjoint is equivalent to the statement $I_1 + I_2 = A$, whence

$$A^G = (A^G \cap I_1) + (A^G \cap I_2).$$

Let $f \in A^G \cap I_1$ be such that $1 - f \in A^G \cap I_2$. Then for any $p_1 \in W_1(k), p_2 \in W_2(k)$, we have $f(p_1) = 0$ and $1 - f(p_2) = 0$, as desired.

(v) follows directly from the definition.

Finally, the proof of (vi) is essentially the second and third paragraphs of the proof of Theorem 1.3.13. Let $f : X \to Z$ be another G-invariant morphism, and let $\{U_i = \operatorname{Spec} A_i\}$ be an open affine cover of Z. Then as in the aforementioned proof, we can cover $X /\!/ G$ with open subsets $\{V_i\}$ such that $(\varphi^G)^{-1}(V_i) = f^{-1}(U_i)$ and by (ii) above, the restricted map $f|_{f^{-1}(U_i)}$ factors uniquely through $\varphi^G|_{(\varphi^G)^{-1}(V_i)} : (\varphi^G)^{-1}(V_i) \to V_i$, and one can check these glue into a global morphism. $\square$

Before we look at some examples, we will need to take a closer look at the action of $G(k)$ on $X(k)$. To this end, we first make the following definition:

Definition 2.2.16. Let G be a reductive group acting on X, and let $p \in X(k)$. The *orbit* of p, denoted $G \cdot p$, is the set $\{g \cdot p \mid g \in G(k)\} \subseteq X(k)$. The *stabiliser* of p, denoted G_p is the fibred product $G \times_X \operatorname{Spec} k$, given by the following diagram:

$$
\begin{array}{ccc}
G \times_X \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k \\
\downarrow & & \downarrow{\scriptstyle p} \\
G & \xrightarrow{\ \sigma(\mathrm{id} \times p)\ } & X
\end{array}
$$

In fact, since X is a variety, we can say more: since $G \cdot p$ is just the set-theoretic image of the morphism $\sigma(-, p) : G \to X$ in $X(k)$, and by Chevalley's theorem is a constructible subset of $X(k)$ (where $X(k)$ is identified as the closed points of X and thus given the induced topology). So we can write

$$G \cdot p = \bigcup_{i=1}^{n} (U_i \cap V_i),$$

where U_i is open and V_i is closed, and we may assume without loss of generality $V_i = \overline{U_i \cap V_i}$, the closure of $U_i \cap V_i$. Since closure commutes with unions, it follows

$$\overline{G \cdot p} = \bigcup V_i.$$

Now observe that $U = (\bigcup U_i) \cap G \cdot p$ is a dense open subset of $\overline{G \cdot p}$ (because $\overline{U} \cap \overline{G \cdot p}$ necessarily contains each $\overline{U_i \cap V_i} = V_i$ and $\bigcup V_i = \overline{G \cdot p}$ itself is closed), and since

$$G \cdot p = \bigcup_{g \in G(k)} g \cdot U$$

it follows that $G \cdot p$ is itself locally closed. Since k is algebraically closed, we may thus identify $G \cdot p$ as the set of closed points of a closed subset of an open subscheme, and equipping it with the reduced closed subscheme structure of this open subscheme, we give $G \cdot p$ the natural structure of a scheme. By abuse of language, we will use the word "orbit" to denote both the set $G(k) \cdot p \subseteq X(k)$, and the scheme described above. It will either be clear from context, or unimportant which is meant.

Our next result is valid for any variety X, not just affine varieties.

Proposition 2.2.17. *Let X be a variety and given any k-point $p \in X(k)$, the morphism $\sigma(-, p) : G \to G \cdot p$ is flat. In particular, we have*

$$\dim G = \dim G_p + \dim G \cdot p.$$

Moreover, $\sigma(-, p)$ is proper if and only if $G \cdot p$ is closed and G_p is proper over k.

Proof. [20, p. 19] and [28, pp. 10-11] □

With this in mind, we are now ready to describe the points of the action of G:

Definition 2.2.18. A point $p \in X(k)$ is *polystable* if $G \cdot p$ is closed. Furthermore p is *stable* if it is polystable, and $\dim G_p = 0$. An orbit is *(poly)stable* if one (equivalently all) of its points is.

With the terminology developed, we are ready to study our first example:

Example 2.2.19. Recall Example 2.0.1. As remarked, the orbits consist of the axes without the origin, the origin and for every $u \in \mathbb{G}_m(k)$ the hyperbola $xy = u$. Clearly the "typical" orbits are the hyperbolas, which are closed, and the stabiliser for each such point is trivial, hence they are stable. The origin itself is polystable, but not stable, and the axes are neither stable nor polystable. We note two things: firstly that the stable points form an open subset and secondly that the closure of the union of the two non-closed orbits are G-invariant, and their intersection contains a single polystable orbit (the origin).

Now it is easy to see that the ring of invariants is $k[x,y]^{\mathbb{G}_m} = k[xy]$, hence the GIT quotient is

$$\mathbb{A}^2 = \operatorname{Spec} k[x,y] \to \operatorname{Spec} k[xy] = \mathbb{A}^1,$$

where the map sends each stable orbit $\{xy = u\}$ to $u \in k \cong \mathbb{A}^1(k)$ and the three orbits that are not stable, which intersect each other, to the origin. However, even though this is an FTSch/k-categorical quotient (which will henceforth just be referred to as a categorical quotient), it is not an orbit space (since there are three orbits which are merged), but note that if we restrict to the open subset of stable points $(\mathbb{A}^2)^s := \mathbb{A}^2 \smallsetminus \{xy = 0\}$, the quotient is in fact an orbit space. Indeed, the restricted quotient is just

$$(\mathbb{A}^2)^s = \operatorname{Spec} k[x,y,(xy)^{-1}] \to k[(xy)^{\pm 1}] = \mathbb{A}^1 \smallsetminus \{0\}$$

and the set-theoretic fibre of each $u \in k^*$ is just the hyperbola $\{xy = u\}$.

In fact, this example demonstrates very typical behaviour of the affine GIT quotient, which we will deduce as consequences of Theorem 2.2.15. Firstly observe that since the quotient $\varphi^G : X \to X /\!\!/ G$ is, in fact, a quotient, orbits in $X(k)$ are contracted to points in $X /\!\!/ G$, and in particular, property (iii) of Theorem 2.2.15 implies that orbit closures are contracted to a point. By property (iv), the converse is also true: two points are mapped to the same point if and only if their orbit closures intersect. By the same property, it follows that the set-theoretic fibre (i.e. preimage) of every $p \in Y(k)$ contains a unique polystable orbit. In particular, every equivalence class of orbits (the relation being intersection of closure) contains a unique polystable orbit, and thus the set $Y(k)$ is in canonical 1-1 correspondence with the polystable orbits of X.

The final observation in the above example (about restricting to the stable locus) is formalised thus:

Definition 2.2.20. Let G be a reductive algebraic group acting on a scheme X. A G-invariant morphism $X \to Y$ is a *geometric quotient* if all the properties of Theorem 2.2.15 are satisfied, and moreover $Y(k)$ is an orbit space of the $G(k)$-action on $X(k)$.

Theorem 2.2.21. *Let G be a reductive affine algebraic group acting on an affine variety X. Then the set of stable k-points are the k-points of a (possibly empty) open subscheme X^s, and the restricted map*

$$X^s \to X^s /\!\!/ G$$

is a geometric quotient.

Proof. This follows the proof in [20, p. 19, 32]. Firstly, we claim that the set of points p such that $\dim G_p > 0$ is closed. Indeed, consider the following diagram:

$$
\begin{array}{ccc}
S = (G \times X) \times_{X \times X} X & \stackrel{\varphi}{\longrightarrow} & X \\
\downarrow & & \downarrow{\scriptstyle \Delta} \\
G \times X & \xrightarrow{(\sigma \times \pi_X)} & X \times X
\end{array}
$$

where $\Delta : X \to X \times X$ is the diagonal map and S is the fibred product of the diagram. Now observe that the k-points of S are exactly the pairs (g, p) such that $g \cdot p = p$. By Chevalley's semicontinuity theorem ([15, Théorème 13.1.3]), the subset

$$V := \{(g, p) \in S(k) \mid \dim G_p = \dim S_p > 0\}$$

is closed. Now define T to be the fibred product $S \times_{G \times X} X$ given by the following diagram:

$$
\begin{array}{ccc}
T & \longrightarrow & S \\
\downarrow & & \downarrow \\
\operatorname{Spec} k \times X \cong X & \xrightarrow{e \times \mathrm{id}} & G \times X
\end{array}
$$

and observe that V pulls back to the set $\{(e, p) \mid \dim G_p > 0\}$. Since X is separated over k, the diagonal is a closed immersion, and in particular it is proper. It thus follows that $T \to X$ is a closed map, which proves the claim.

Now let p be a stable point. We will find an open neighbourhood of stable points containing p. Observe that V, which we saw was closed, and is clearly G-invariant, is disjoint from p, and hence by property (iv), p and V are mapped to disjoint sets in the GIT quotient. In particular, there is some invariant $f \in A^G$, such that $f(V) = 0$, and $f(p) \neq 0$ (for example, suppose take $1 - F$ where $F \in A^G$ vanishes exactly on $\varphi^G(p)$). We claim that the k-points of X_f are stable. Indeed, since $X_f(k) \cap V = \varnothing$, it suffices to show that the orbits of X_f are closed. So fix some $q \in X_f(k)$, suppose for contradiction its orbit is not closed. Since $G \cdot q$ is open and dense in $\overline{G \cdot q}$, its complement in $\overline{G \cdot q}$ must be of smaller dimension, and moreover contains a polystable orbit, say $G \cdot r$. It must therefore be that $\dim G_r > 0$ by, so $r \in V$, and hence $f(r) = 0$. But since f is G-invariant and does not vanish on q, it does not vanish anywhere on $\overline{G \cdot q}$, which is a contradiction. It thus follows that the k-points of X_f are all stable, and hence we conclude that X^s is open.

Finally, we prove that this is a geometric quotient. To this end, we simply need to show that the set-theoretic fibre of each k-point in $X^s /\!\!/ G$ is a single stable orbit. But this is obvious; indeed for each $p \in X^s$, the set-theoretic fibre $(\varphi^G)^{-1}(\varphi^G(p))$ contains a unique polystable orbit, which must be $G \cdot p$. If it contains any other orbit, say $G \cdot q$, then $G \cdot p \cap \overline{G \cdot q}$ is nonempty, and since $\overline{G \cdot q}$ is G-invariant, it follows $G \cdot p$ is in the complement of $G \cdot q$ in $\overline{G \cdot q}$, and hence has dimension strictly less, which contradicts the fact that p is stable. $\qquad\square$

Example 2.2.22. We are now ready to revisit the elliptic curves example from last chapter, and assimilate it into our new framework. In the previous chapter, we saw how $k^* = \mathbb{G}_m(k)$ acted on $S = \operatorname{Spec} k[a, b, \Delta^{-1}]$. This can be formalised as an algebraic action $\sigma : \mathbb{G}_m \times S \to S$ as dual to the homomorphism $\sigma^\sharp : k[a, b, \Delta^{-1}] \to k[t^{\pm 1}] \otimes k[a, b, \Delta^{-1}]$ given by $a \mapsto t^4 \otimes a$, $b \mapsto t^6 \otimes b$. To check that this is a group action, observe that

$$((\mathrm{id}_G^\sharp \otimes \sigma^\sharp) \circ \sigma^\sharp)(a) = t^4 \otimes t^4 \otimes a = ((\mu^\sharp \otimes \mathrm{id}_S^\sharp) \circ \sigma^\sharp)(a)$$

and similarly with b, and

$$((e^\sharp \otimes \mathrm{id}_S^\sharp) \circ \sigma^\sharp)(a) = 1 \otimes a.$$

Let $(p, q) \in S(k)$ be a k-point. Then the orbit of (p, q) is $\{(u^4 p, u^6 q) \mid u \in k^*\}$, which is exactly the set of k-points of the closed subscheme $\operatorname{Spec} k[a, b, \Delta^{-1}]/\langle p^3 y^2 - q^3 x^3 \rangle$. In particular, every orbit is closed and stable. Now as we calculated in chapter 1, the GIT quotient is $\operatorname{Spec} k[j]$, and as we just saw, this is an orbit space since every k-point of S is stable. In particular, this tells us the following: the fact that $S \to \operatorname{Spec} k[j]$ is a categorical quotient, combined with Lemma 1.3.7 means that $\operatorname{Spec} k[j]$ satisfies the universal property (ii) in the definition of a coarse moduli space, and the fact that this is an orbit space means property (i) is also satisfied. This demonstrates the usefulness of geometric quotients.

However, the affine GIT quotient is slightly oversimplified, since every $p \in X(k)$ has an image. In particular, no orbits are thrown away; simply merged. As we will see shortly, things are more complicated in the projective case.

2.3   The Projective GIT Quotient

Our next task is to extend the notion of a GIT quotient to a projective variety. There are, however, more issues, the first and perhaps most obvious is that a projective scheme does not have a canonical homogeneous coordinate ring; such a coordinate ring is induced by a projective embedding. Indeed, even in the case of $X = \mathbb{P}^1$, we may embed X in $\mathbb{P}^2$ in the obvious way, with resulting coordinate ring $k[x_0, x_1]$, or via the 2-uple embedding, with resulting coordinate ring $k[y_0, y_1, y_2]/\langle y_0 y_2 - y_1^2 \rangle$, but these rings are not isomorphic, since the former is a UFD but the latter is not.

But even if we have an embedding $X \subseteq \mathbb{P}^n$, such that $X = \operatorname{Proj} S$, this is not enough, because the action does not lift canonically to S; indeed:

Example 2.3.1. Let $X = \mathbb{P}^n$. Let $\mathbb{G}_m$ act on X as follows: for any $\lambda \in \mathbb{G}_m(k), p = [p_0 : ... : p_n] \in X(k)$, we define

$$\lambda \cdot p := [\lambda^{-1} p_0 : \lambda p_1 : ... : \lambda p_n]$$

which extends uniquely to an algebraic action. We embed X in itself via the identity, with resulting coordinate ring $S = k[x_0, ..., x_n]$. However, the action of G may be lifted to one on S in many ways; for example

$$\lambda \cdot x_0 := \lambda^{-1} x_0, \ \lambda \cdot x_i := \lambda x_i$$

and

$$\lambda \cdot x_0 := x_0, \ \lambda \cdot x_: = \lambda^2 x_i.$$

just to name two.

We must therefore **choose** a lift of the action, which is accomplished as follows: recall that a projective embedding is equivalent to picking a very ample line bundle $\mathcal{L}$, and the resulting coordinate ring of this embedding is

$$S = \bigoplus_{r \geq 0} H^0(X, \mathcal{L}^{\otimes r}).$$

The idea is to lift our action of X to one on $\mathcal{L}$, so that the action is linear in some sense. This is encapsulated in the following definition:

Definition 2.3.2. Let $\sigma : G \times X \to X$ be an algebraic group action on a projective scheme X. A *linearisation* of this action is a line bundle $\mathcal{L}$ and an isomorphism $\Phi : \sigma^*(\mathcal{L}) \cong \pi_X^*(\mathcal{L})$, where $\pi_X : G \times X \to X$ is the projection onto the second factor, such that the following diagram commutes:

$$
\begin{array}{ccccc}
(\sigma \circ (\mathrm{id}_G \times \sigma))^* \mathcal{L} & \xrightarrow{(\mathrm{id}_G \times \sigma)^* \Phi} & (\pi_X \circ (\mathrm{id}_G \times \sigma))^* \mathcal{L} & \xrightarrow{\ =\ } & (\sigma \circ \pi_{23})^* \mathcal{L} \\
\Big\downarrow{\scriptstyle =} & & & & \Big\downarrow{\scriptstyle \pi_{23}^* \Phi} \\
(\sigma \circ (\mu \times \mathrm{id}_X))^* \mathcal{L} & \xrightarrow{(\mu \times \mathrm{id}_X)^* \Phi} & (\pi_X \circ (\mu \times \mathrm{id}_X))^* \mathcal{L} & \xrightarrow{\ =\ } & (\pi_X \circ \pi_{23})^* \mathcal{L}
\end{array}
\tag{2.2}
$$

where $\pi_{23} : G \times G \times X \to G \times X$ is the projection onto the last two factors. A linearisation is *very ample* if $\mathcal{L}$ is. By abuse of language, we will often refer to $\mathcal{L}$ itself as the linearisation.

We unwrap this definition. Of course, for any $g \in G(k)$, we may pull back $\mathcal{L}$ along the isomorphism $\sigma_g : X \to X$. The linearisation Φ allows us to identify $\mathcal{L}$ before and after the pullback. More precisely: for any open subset $U \subseteq X$ where $\mathcal{L}$ is trivial, we identify $\mathcal{L}|_U \cong \mathcal{O}_U$. Now fix a k-point g of G. Then $\sigma_g^* \mathcal{L}(U) \cong \mathcal{O}_X(gU)$ and $\pi_X^* \mathcal{L}(U) \cong \mathcal{O}_X(U)$. In particular, we have the following isomorphism:

$$
\sigma_g^* \mathcal{L}(U) \cong \mathcal{O}_X(gU) \xrightarrow{\ \Phi\ } \pi_X^* \mathcal{L}(U) \cong \mathcal{O}_X(U),
\tag{2.3}
$$

and thus, Φ may be thought of as defining a way to "shift" $\mathcal{L}$ by g.

We now make sense of (2.2) a little. Firstly, observe that these are morphisms of sheaves on $G \times G \times X$, all of which are pullbacks of $\mathcal{L}$ by various maps. The equalities follow from the axioms, for example $\sigma \circ (\mathrm{id}_G \times \sigma) = \sigma \circ (\mu \times \mathrm{id}_X)$ is just the associativity axiom of group actions. Without explicitly stating the equalities, the commutativity says

$$
(\mu \times \mathrm{id}_X)^* \Phi = \pi_{23}^* \Phi \circ (\mathrm{id}_G \times \sigma)^* \Phi
\tag{2.4}
$$

which we make sense of as follows: Let (g, h) be a k-point in $G \times G$. Then as in (2.3), the map $(\mu \times \mathrm{id}_X)^* \Phi$ induces a map

$$
\sigma_{gh}^* \mathcal{L}(U) \cong \mathcal{O}_X(ghU) \to \pi_X^* \mathcal{L}(U) \cong \mathcal{O}_X(U).
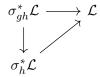$$

Now $(\sigma \circ \pi_{23})(g, h, -) = \sigma_h$, and so $\pi_{23}^* \Phi$ induces a map

$$
\sigma_h^* \mathcal{L}(U) \cong \mathcal{O}_X(hU) \longrightarrow \pi_X^* \mathcal{L}(U) \cong \mathcal{O}_X(U)
$$

and finally since $(\sigma \circ (\mathrm{id}_G \times \sigma))(g, h, -) = \sigma_g \circ \sigma_h$, the pullback $(\mathrm{id}_G \times \sigma)^* \Phi$ induces a map

$$
\sigma_g^*(\sigma_h^* \mathcal{L})(U) \cong \mathcal{O}_X(ghU) \longrightarrow \sigma_h^* \mathcal{L}(U) \cong \mathcal{O}_X(hU)
$$

Put together, this means the following diagram commutes:

$$
\begin{array}{ccc}
\sigma_{gh}^* \mathcal{L} & \longrightarrow & \mathcal{L} \\
\Big\downarrow & \nearrow & \\
\sigma_h^* \mathcal{L} & &
\end{array}
$$

so in particular $G(k)$ acts on $\mathcal{L}$ via automorphisms.

Of course, this means that $G(k)$ also acts on all tensor powers of $\mathcal{L}$, and in the case $\mathcal{L}$ is very ample, taking global sections shows that $G(k)$ acts on the graded homogeneous coordinate ring $S = \bigoplus_{r \geq 0} H^0(X, \mathcal{L}^{\otimes r})$, and moreover it is not hard to see that this action preserves the grading.

Example 2.3.3. There is a natural linearisation of the action described in Example 2.3.1 on $\mathcal{O}_X(1)$. To see this, we first note that both $\sigma^*(\mathcal{O}_X(1))$ and $\pi_X^*(\mathcal{O}_X(1))$ are abstractly isomorphic to $\mathcal{O}_{\mathbb{G}_m \times X}(1)$. We define the isomorphism $\sigma^*(\mathcal{O}_X(1)) \to \pi_X^*(\mathcal{O}_X(1))$ to be $x_0 \mapsto t^{-1} x_0$ and $t_i \mapsto t x_i$ for $i \neq 0$.

Once again, we check that this is in fact a linearisation. Clearly it is an isomorphism, so it suffices to show that (2.4) holds. To this end, we first observe that the various pullbacks of $\mathcal{O}_X(1)$ to $\mathbb{G}_m \times \mathbb{G}_m \times X$ are abstractly isomorphic to the $\mathcal{O}_{\mathbb{G}_m \times \mathbb{G}_m \times X}$-module $\mathcal{O}_{\mathbb{G}_m \times \mathbb{G}_m \times X}(1)$, and since

$$f^*(\mathcal{O}_{\mathbb{G}_m \times X}(1)) = \mathcal{O}_{\mathbb{G}_m \times \mathbb{G}_m \times X} \otimes_{f^{-1}\mathcal{O}_{\mathbb{G}_m \times X}} f^{-1}(\mathcal{O}_{\mathbb{G}_m \times X}(1))$$

where $f : \mathbb{G}_m \times \mathbb{G}_m \times X \to \mathbb{G}_m \times X$ is any map, we may write its elements as sums of $f \otimes g \otimes h x_i$, where $f, g \in \mathcal{O}_{\mathbb{G}_m}$ and $h \in \mathcal{O}_{\mathbb{G}_m \times X}$ (the X component of $\mathcal{O}_{\mathbb{G}_m \times \mathbb{G}_m \times X}$ is absorbed by h). With this in mind, we compute:

$$(\mu \times \mathrm{id}_X)^* \Phi(1 \otimes 1 \otimes x_0) = 1 \otimes 1 \otimes t^{-1} x_0 = t^{-1} \otimes t^{-1} \otimes x_0$$

and similarly for any other x_i. We also have

$$(\mathrm{id}_G \times \sigma)^* \Phi(1 \otimes 1 \otimes x_0) = 1 \otimes 1 \otimes t^{-1} x_0 = t^{-1} \otimes 1 \otimes x_0$$

and finally

$$\pi_{23}^* \Phi(t^{-1} \otimes 1 \otimes x_0) = t^{-1} \otimes t^{-1} \otimes x_0$$

as desired.

Now the homogeneous coordinate ring of this embedding is just

$$S = \bigoplus_{r \geq 0} H^0(X, \mathcal{O}_X(1)^{\otimes r}) = k[x_0, ..., x_n]$$

as expected, and there is an induced action of $\mathbb{G}_m(k)$ on S given by $\lambda \cdot x_0 = \lambda^{-1} x_0$ and $\lambda \cdot x_i = \lambda x_i$ for $i \neq 0$, and in particular observe that this action preserves the grading on S.

Now we may define our quotient. We fix the following data: let X be a projective variety, G a reductive affine algebraic group, $G \times X \to X$ an action, $\mathcal{L}$ a very ample linearisation and $S = \bigoplus_{r \geq 0} H^0(X, \mathcal{L}^{\otimes r})$ the homogeneous coordinate ring. We denote S^G the subring of invariant elements of S, and we write S_+ for the irrelevant ideal $\bigoplus_{r > 0} S_{\deg r}$, and similarly write S_+^G for the S^G-ideal $S_+ \cap S^G$.

Definition 2.3.4. A k-point p is *semistable* (with respect to $\mathcal{L}$) if there is a homogeneous invariant $\sigma \in S^G$ of positive degree such that $\sigma(p) \neq 0$, or equivalently $p \in X_\sigma(k)$ where $X_\sigma = \mathrm{Spec}\, S[\sigma^{-1}]_{\deg 0}$. If p is not semistable, then it is *unstable*. The *semistable locus*, denoted X^{ss} is the open subscheme $X \smallsetminus V(S_+^G)$, where $V(S_+^G)$ is the closed subset associated to the homogeneous ideal $\langle S_+^G \rangle$ in S. Note that the homogeneous elements of S_+^G generate this ideal, so it is in fact homogenous. We say p is

polystable if it is semistable, and its orbit is closed in the semistable locus. Furthermore, p is *stable* if it is polystable, and additionally its stabiliser has dimension zero. The *projective GIT quotient* is the map

$$X^{ss} \to X /\!\!/_{\mathcal{L}} G := \operatorname{Proj} S^G$$

induced by the inclusion $S^G \subseteq S$.

Let us compare the affine and projective GIT quotients. The main difference is that in the affine case, every k-point has an image in the quotient; in other words every point is "semistable"; this is obviously not so in the projective case. Their similarities are, however, much more abundant: if $\sigma \in S_+^G$ is homogeneous, it is not hard to check that $X_\sigma = \operatorname{Spec} S[\sigma^{-1}]_{\deg 0}$ is invariant, and that the restriction of the projective GIT quotient to X_σ is just the affine GIT quotient $X_\sigma \to \operatorname{Spec} S[\sigma^{-1}]_{\deg 0}^G$, and since X^{ss} is covered by these affine open subsets, it follows that the projective GIT quotient is just a collection of affine GIT quotients glued together. One can check that the statements of Theorem 2.2.15 hold (indeed, statements (i) - (v) are local on the target, and (vi) can be checked using the same argument), and in particular the projective GIT quotient is a categorical quotient for the restricted action $G \times X^{ss} \to X^{ss}$. By a similar argument to the affine case, it can also be shown that the stable locus is open and that the restriction to the stable locus is a geometric quotient.

Example 2.3.5. Retain the notation and hypotheses in Example 2.3.3. It can be shown ([28, p. 37]) that the ring of invariants S^G is just $k[x_0 x_1, ..., x_0 x_n]$. It follows that $p = [p_0 : ... : p_n]$ is semistable if and only if p_0 is nonzero, and some other p_i for $i > 0$ is nonzero. In particular, the semistable locus can be identified with $\mathbb{A}^n \setminus \{0\}$. Now on the semistable locus, the action is just multiplication by λ^2, so every point is polystable, the orbit just being the line passing through our point and the origin in $\mathbb{A}^n$, minus the origin itself. In fact, every point is stable, since the action is free. Of course, this makes sense because our projective GIT quotient is just

$$\mathbb{P}^{n-1} = \operatorname{Proj} k[x_0 x_1, ..., x_0 x_n]$$

and this is a geometric quotient.

Example 2.3.6. Of course, there is another linearisation on $\mathcal{O}_X(1)$ given by $x_0 \mapsto x_0$ and $x_i \mapsto t^2 x_i$ for $i > 0$. Clearly $k[x_0]$ is the ring of invariants, so the projective GIT quotient with respect to this linearisation is simply $\operatorname{Spec} k$. Indeed, the semistable locus is the open set given by $x_0 \neq 0$, which is isomorphic to $\mathbb{A}^n$. With this interpretation, the action of $\mathbb{G}_m$ is just scaling, and the closure of every orbit contains the origin in $\mathbb{A}^n$ (or equivalent the point $[p_0 : 0 : ... : 0] \in \mathbb{P}^n$), which is the unique polystable orbit of this linearisation. In particular, the stable locus is empty. This shows that projective GIT is heavily dependent on our choice of linearisation. However, we will often fix a single linearisation to work with, and the problem of choosing different linearisations will not be discussed in this thesis.

2.4 The Hilbert-Mumford Criterion

As we have just seen, stability is very important. However, with our definition, it is rather difficult to calculate. In this section, we will develop the *Hilbert-Mumford criterion*, which gives a numerical

criterion for stability in terms of 1-parameter subgroups. We begin with a closer examination of our current definition for stability, which requires the following definition:

Definition 2.4.1. Let X be a projective scheme and let $\mathcal{L}$ be a very ample line bundle. Write S for the homogeneous coordinate ring

$$S := \bigoplus_{r \geq 0} H^0(X, \mathcal{L}^{\otimes r}).$$

We define the *affine cone* of X to be the affine scheme $\widetilde{X} := \operatorname{Spec} S$.

To make sense of the affine cone, firstly recall that $\mathcal{L}$ embeds X as a closed subscheme of $\mathbb{P}^n$, where $n = h^0(X, \mathcal{L}) - 1 = \dim H^0(X, \mathcal{L}) - 1$. The k-points of $\mathbb{P}^n$ are just the 1-dimensional subspaces of k^{n+1}, and thus the k-points of X may be interpreted as a collection of lines through the origin in k^{n+1}. The k-points of $\widetilde{X}$ may, in turn, be thought of as the union of these lines. For example, the affine cone of $\mathbb{P}^n$ is just $\mathbb{A}^{n+1}$.

There is a well-defined notion of an origin, which corresponds to the irrelevant ideal S_+, which is clearly maximal, and there is a natural map $\operatorname{Spec} S \smallsetminus \{0\} \to \operatorname{Proj} S$, which we define as follows: let $f \in S_{\deg 1}$. Then there is an inclusion $S[f^{-1}]_{\deg 0} \subseteq S[f^{-1}]$, which induces a morphism $\operatorname{Spec} S[f^{-1}] \to \operatorname{Spec} S[f^{-1}]_{\deg 0}$. Since $\mathcal{L}$ is very ample, it follows S is generated by a finite set of these $f \in S_{\deg 1}$ as a k-algebra, and so $X_f = \operatorname{Spec} S[f^{-1}]_{\deg 0}$ cover $\operatorname{Spec} S \smallsetminus \{0\}$. On the level of k-points, this is just $(p_0, ..., p_n) \mapsto [p_0 : ... : p_n]$.

Now suppose X is a projective variety, and let G be a reductive affine algebraic group acting on X. Further, let $\mathcal{L}$ be a very ample linearisation, and let S be the homogeneous coordinate ring. We claim the the linearisation naturally induces an action on the affine cone. Indeed, by the adjunction property of pullbacks and pushforwards, there is a natural map (the unit map of the adjunction) $\mathcal{L} \to \sigma_* \sigma^*(\mathcal{L})$. Taking global sections, we have

$$H^0(X, \mathcal{L}) \to H^0(G \times X, \sigma^*(\mathcal{L})) \cong H^0(G \times X, \pi_X^*(\mathcal{L})) \cong H^0(G, \mathcal{O}_G) \otimes H^0(X, \mathcal{L})$$

where the final isomorphism comes from the Künneth formula ([44, Lemma 33.29.1]). We can check that this induces a map $\tilde{\sigma}^* : S \to \mathcal{O}_G(G) \otimes S$ which satisfies the co-action axioms; and in particular this induces a group action $G \times \widetilde{X} \to \widetilde{X}$. Moreover, since the co-action homomorphism is linear on $H^0(X, \mathcal{L})$, by linear algebra this means that G acts linearly (i.e. via a representation $G \to \operatorname{GL}_{n+1}$) on $\widetilde{X}$.

Example 2.4.2. Recall Example 2.3.3. The induced action on $\widetilde{X} = \mathbb{A}^{n+1}$ is just

$$\lambda \cdot (p_0, ..., p_n) = (\lambda^{-1} p_0, \lambda p_1, ..., \lambda p_n)$$

on the level of k-points. More rigorously, the co-ordinate rings are $k[t^{\pm 1}]$ and $k[x_0, ..., x_n]$, and co-action homomorphism is given by $x_0 \mapsto t^{-1} \otimes x_0$ and $x_i \mapsto t \otimes x_i$ for $i > 0$.

We now present our first criterion for stability:

Theorem 2.4.3 (Topological criterion for stability). *Let X be a projective variety, let G be a reductive affine algebraic group with a linearisation on the very ample line bundle $\mathcal{L}$, and let S denote the resulting coordinate ring.*

(i) *A k-point p is semistable if and only if for any lift $\tilde{p} \in \widetilde{X}(k)$, the closure of the $\tilde{p}$ orbit in $\widetilde{X}$, $\overline{G \cdot \tilde{p}}$, does not contain the origin.*

(ii) *A k-point p is polystable if and only if the orbit of any of its lifts is closed in $\widetilde{X}$.*

(iii) *A k-point p is stable if and only if for any lift $\tilde{p}$ its orbit is closed in $\widetilde{X}$ and $\dim G_{\tilde{p}} = 0$.*

Proof. Fix a k-point p and a lift $\tilde{p}$. If p is semistable, then there is some $r > 0$ and $\sigma \in S^G_{\deg r}$ such that $\sigma(p) \neq 0$. Let $f = \sigma - \sigma(\tilde{p})$. Then f is invariant, and hence constant on $G \cdot \tilde{p}$. Observe that $f(0) = \sigma(0) - \sigma(\tilde{p}) = -\sigma(\tilde{p})$ (since σ is homogeneous of positive degree, it follows $\sigma(0) = 0$), which means that there is some function which vanishes on $G \cdot \tilde{p}$ but not 0, and hence 0 is not in the orbit closure of $\tilde{p}$. Conversely, suppose 0 is not in the orbit closure of $\tilde{p}$. Then $\overline{G \cdot \tilde{p}}$ and the origin are both G-invariant closed subsets of $\widetilde{X}$, and it can be shown ([28, Corollary 1.2]) there is some invariant $f \in S$ such that $f(0) = 0$ for all $g \in G(k)$ but $f(g \cdot p) \neq 0$. Clearly then f has no degree zero component. Now let $f = \sum_{i>0} f_i$ be the homogeneous decomposition of f, with f_i of degree i. In particular, some f_r must not vanish on $\tilde{p}$, and since G preserves each homogeneous component of S, it follows that f_r is invariant, and hence $f_r(p) \neq 0$, so p is semistable. This proves (i).

To prove (ii), firstly suppose p is semistable (if it is not, then it cannot be polystable, and the closure of its orbit contains the origin). Then $p \in X_\sigma(k)$ for some invariant homogeneous σ of positive degree. Observe then that X_σ is G-invariant, and so $G \cdot p \subseteq X_\sigma(k)$. Now pick a lift $\tilde{p}$ of p, and consider the closed subscheme $V = \operatorname{Spec} S/\langle \sigma - \sigma(\tilde{p}) \rangle$ of $\widetilde{X}$, which clearly contains the orbit $G \cdot \tilde{p}$. Now the map $\widetilde{X} \smallsetminus \{0\} \to X$ restricts to a map $\varphi : V \mapsto X_\sigma$, which is a morphism of affine schemes, induced by the canonical ring homomorphism

$$S[\sigma^{-1}]_{\deg 0} \to S/\langle \sigma - \sigma(\tilde{p}) \rangle,$$

$$\frac{f}{\sigma} \mapsto \frac{f}{\sigma(\tilde{p})},$$

and since the homomorphism is surjective, the morphism φ is finite, and hence closed. In particular, if $G \cdot \tilde{p}$ is closed in $\widetilde{X}$, then it is closed in V, and hence $G \cdot p$ is closed in X_σ. Conversely, suppose $G \cdot p$ is closed in X_σ, and let V be as above. We claim the preimage of $G \cdot p$ in V is equal to $G \cdot \tilde{p}$. To this end, firstly observe that clearly $G \cdot \tilde{p}$ is contained in $\varphi^{-1}(G \cdot p) \cap V(k)$, so suppose $\tilde{q} \in \varphi^{-1}(G \cdot p) \cap V(k)$, and we may suppose without loss of generality $\varphi(\tilde{q}) = p$. Then $\tilde{q}$ and $\tilde{p}$ lie in the same fibre over p and thus, considered as points in an ambient $\mathbb{A}^n(k) = k^n$, differ by a scalar multiple, say $\tilde{q} = u\tilde{p}, u \in \mathbb{G}_m(k)$, and so

$$\sigma(\tilde{p}) = \sigma(\tilde{q}) = \sigma(u\tilde{p}) = u^{\deg \sigma}\sigma(\tilde{p}),$$

whence $u = 1$, as claimed. In particular, this means $G \cdot \tilde{p}$ is closed in V, and hence in $X(k)$ as desired. This proves (ii).

Finally, to prove (iii), it suffices to show that if p is polystable then $\dim G_p = 0$ if and only if $\dim G_{\tilde{p}} = 0$. Clearly $\dim G_{\tilde{p}} \leq \dim G_p$, and so it suffices to show that if $\dim G_{\tilde{p}} = 0$ then $\dim G_p = 0$. To this end, observe that finite morphisms are stable under base change ([14, Proposiion 6.1.5]), and

hence the leftmost down arrow below is also finite:

$$
\begin{array}{ccccc}
G_{\tilde{p}} & \longrightarrow & G \times V & \longrightarrow & V \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\varphi} \\
G_p & \longrightarrow & G \times X_\sigma & \longrightarrow & X_\sigma.
\end{array}
$$

This completes the proof. $\square$

The main issue with the above is that it is oftentimes very difficult to compute the closure of an orbit, and in fact it may even be unknown what the homogeneous coordinate ring of our linearisation is in the first place! Hence we will require a more computation-friendly criterion. This is the *Hilbert-Mumford criterion*, which relates stability to 1-parameter subgroups, which we will now define:

Definition 2.4.4. Let G be an algebraic group. A *1-parameter subgroup*, or just *1-PS*, is an algebraic group homomorphism $\lambda : \mathbb{G}_m \to G$.

Now let X be a scheme, separated over k and let $f : \mathbb{G}_m \to X$ be any morphism. Then by the valuative criterion for separation, f has at most one extension to a morphism $f^\sharp : \mathbb{A}^1 \to X$. If this extension does exist, we define the *limit* of f at 0, denoted $\lim_{t \to 0} f(t)$, to be

$$
\lim_{t \to 0} f(t) := f^\sharp(0).
$$

If this extension does not exist, we say the *limit does not exist*.

The idea is to reinterpret stability in terms of whether or not certain limits exist (which, in practice, usually amounts to checking if negative powers of t turn up in an expression). However, since we will only ever take $G = \mathrm{SL}_m$ for some $m \in \mathbb{N}$ in practical applications in this thesis, and since this case is easier to prove, we will henceforth assume $G = \mathrm{SL}_m$. It is worth noting that this method works for GL_m as well.

Theorem 2.4.5. *Let* SL_m *act on a projective variety* X *with a very ample linearisation* $\mathcal{L}$ *embedding* X *in* $\mathbb{P}^n$. *Let* $p \in X(k)$.

(i) *p is stable if and only if for any lift $\tilde{p}$ and nontrivial 1-PS $\lambda : \mathbb{G}_m \to \mathrm{SL}_m$, the limit*

$$
\lim_{t \to 0} \lambda(t) \cdot \tilde{p}
$$

   does not exist.

(ii) *p is semistable if and only if for any $\tilde{p}$ and 1-PS $\lambda : \mathbb{G}_m \to \mathrm{SL}_m$, we have*

$$
\lim_{t \to 0} \lambda(t) \cdot \tilde{p} \neq 0.
$$

Before we give the proof, we require the following result:

Lemma 2.4.6 (Iwahori's Decomposition Theorem)**.** *Let R be a DVR with valuation v, fraction field K and uniformiser ϖ. Then given any $M \in \mathrm{SL}_m(K)$, there exists A, B and $\Lambda = \mathrm{diag}(\varpi^{r_1}, ..., \varpi^{r_m})$ such that*

$$M = A\Lambda B.$$

Proof. Following [27, p. 215], we induct on m, with $m = 1$ being trivial. Firstly, denote the entries of M by m_{ij}. Multiplying left and right by permutation matrices we may assume

$$v(m_{1,1}) \leq v(m_{ij})$$

for all i, j, and multiplying by diagonal elements of $\mathrm{SL}_m(R^*)$, where R^* is the group of units in R, we may assume $m_{1,1} = \varpi^{v(m_{1,1})}$. Now considering only the top and left entries, we have

$$\begin{pmatrix} 1 & 0 & ... & 0 \\ -\frac{m_{2,1}}{m_{1,1}} & 1 & ... & 0 \\ ... & & ... & \\ -\frac{m_{m,1}}{m_{1,1}} & 0 & ... & 1 \end{pmatrix} M \begin{pmatrix} 1 & -\frac{m_{1,2}}{m_{1,1}} & ... & -\frac{m_{1,m}}{m_{1,1}} \\ 0 & 1 & ... & 0 \\ ... & & & ... \\ 0 & 0 & ... & 1 \end{pmatrix} = \begin{pmatrix} m_{1,1} & 0 & ... & 0 \\ 0 & & & \\ ... & & M^\flat & \\ 0 & & & \end{pmatrix},$$

where $M^\flat$ is some $(m - 1) \times (m - 1)$ matrix, and the result follows from the inductiive hypothesis applied to $M^\flat$. $\square$

Proof of Theorem 2.4.5. Inspired by [27, pp. 216-218], we begin with some generalities. Since the action of SL_m on $\widetilde{X} \subseteq \mathbb{A}^{n+1}$ is linear, there is a representation $\rho : \mathrm{SL}_m \to \mathrm{GL}_{n+1}$. The diagonal subgroup $T \cong (\mathbb{G}_m)^m$ of SL_m, induces a weight-space decomposition

$$V = k^{n+1} = \bigoplus_{\gamma \in \mathbb{Z}^m} V_\gamma,$$

where if $\gamma = (r_1, ..., r_m)$, then

$$\mathrm{diag}(t_1, ..., t_m) \cdot v = \prod_i t_i^{r_i} \cdot v$$

for all $v \in V_\gamma$. To see this, observe we can apply the usual weight space decomposition of Theorem 2.1.13 to each $\mathbb{G}_m$ component of T (which will look like $\mathrm{diag}(1, ..., t_i, ..., 1)$), and take refinements as i moves up, noting that if $\mathrm{diag}(1, ..., t_i, ..., 1) \cdot v = t_i^{r_i} v$ and $\mathrm{diag}(1, ..., t_j, ..., 1) \cdot v = t_j^{r_j} v$ then $\mathrm{diag}(1, ..., t_i, ..., t_j, ..., 1) \cdot v = t_i^{r_i} t_j^{r_j} v$. In particular, we have a basis $\{e_{\gamma,i}\}$ of V where $e_{\gamma,i} \in V_\gamma$.

Now to prove (i), we will first show that if a 1-PS attains a limit then p is not stable. To this end, suppose we have a 1-PS $\lambda : \mathbb{G}_m \to \mathrm{SL}_m$ and suppose $\lim_{t \to 0} \lambda(t) \cdot \tilde{p} = \tilde{q}$ for some $\tilde{q} \in X(k)$. If $\tilde{q} \notin (\mathrm{SL}_m) \cdot \tilde{p}$ we are done, otherwise we observe that the action of λ fixes $\tilde{q}$, hence $\mathrm{im}\,\lambda \subseteq (\mathrm{SL}_m)_{\tilde{q}}$. But since λ is nontrivial and $\mathbb{G}_m$ is connected, it follows that $(\mathrm{SL}_m)_{\tilde{q}}$, and hence $(\mathrm{SL}_m)_{\tilde{p}}$ cannot be finite.

Conversely, suppose that p is not stable. Then either $\dim(\mathrm{SL}_m)_{\tilde{p}} > 0$ or $(\mathrm{SL}_m) \cdot \tilde{p}$ is not closed. In the first case, note that since $(\mathrm{SL}_m)_{\tilde{p}}$ is affine (it is the fibred product of two affine schemes over an affine scheme) of positive dimension, it is not proper over $\mathrm{Spec}\,k$, and similarly, if $(\mathrm{SL}_m) \cdot \tilde{p}$ is not closed, then the map $\mathrm{SL}_m \to \widetilde{X}$ given by multiplication by $\tilde{p}$ is not closed. In either case, by the noetherian version of the valuative criterion for properness ([44, Lemma 32.15.1]), there exists a

DVR, say R with residue field k, uniformiser ϖ, fraction field $K \supseteq k$ with the property $R = k \oplus \varpi R$ (in particular there is a subring $k[\varpi^{\pm 1}] \subseteq K$ isomorphic to $k[t^{\pm 1}]$), and an element $M \in \mathrm{SL}_m(K) \smallsetminus \mathrm{SL}_m(R)$ such that $M \cdot \tilde{p}$ specialises to some $\tilde{q}$ (that is, the image of $M \cdot \tilde{p}$ in $V := k^{n+1}$ is $\tilde{q}$), for which we will adopt the notation $M \cdot \tilde{p} \to \tilde{q}$. In the former case, this $\tilde{q}$ is just $\tilde{p}$; in the latter it is some point in the boundary. Applying Iwahori's theorem, this means we can write $M = A\Lambda B$ where $\Lambda = \mathrm{diag}(\varpi^{r_1}, ..., \varpi^{r_m})$ and moreover Λ is nontrivial, since M is not contained in $\mathrm{SL}_m(R)$. In particular, we have

$$(A\Lambda B) \cdot \tilde{p} \to \tilde{q}.$$

We note that by definition A specialises to a matrix of determinant 1, and since matrix multiplication commutes with ring homomorphisms (and group actions are associative), it follows $\Lambda B \cdot \tilde{p} \to \tilde{q}$. Now using the basis $\{e_{\gamma,i}\}$ constructed in the beginning of the proof, we may write

$$B \cdot \tilde{p} = \sum b_{\gamma,i} e_{\gamma,i} \in R^n, \tag{2.5}$$

and the statement $\Lambda B \cdot \tilde{p} = \tilde{q}$ implies that for each $\gamma = (a_1, ..., a_m)$, we have

$$v_R(\varpi^{\sum a_i r_i} b_{\gamma,i}) \geq 0, \tag{2.6}$$

where v_R is the valuation. In particular, this is saying that if $v_R(b_{\gamma,i}) = 0$ then $\sum a_i r_i \geq 0$. But that means that for any $\bar{b}_{\gamma,i} \neq 0$, where $\bar{b}_{\gamma,i}$ is the image of $b_{\gamma,i}$ in the residue field k, we have

$$v_R(\varpi^{\sum a_i r_i} \bar{b}_{\gamma,i}) \geq 0 \tag{2.7}$$

and in particular, it follows that $\Lambda \overline{B} \cdot \tilde{p}$, where $\overline{B}$ is the reduction of B mod ϖ, has a specialisation, or more precisely, it is of the form

$$\Lambda \overline{B} \cdot \tilde{p} = u + \varpi v \in V \otimes R, \tag{2.8}$$

where $u \in V$, $v \in V \otimes R$. Furthermore, we note that since $\Lambda \in \mathrm{SL}_m(k[\varpi^{\pm 1}])$, it follows that $\Lambda \overline{B} \cdot \tilde{p}$ is actually contained in $V \otimes k[\varpi] \cong V \otimes k[t]$. Now using the isomorphism $k[t^{\pm 1}] \cong k[\varpi^{\pm 1}]$, the image of $\Lambda \in \mathrm{SL}_m(k[\varpi^{\pm 1}])$ in $\mathrm{SL}_m(k[t^{\pm 1}])$ is the matrix

$$\lambda = \mathrm{diag}(t^{r_1}, ..., t^{r_m}) \in \mathrm{SL}_m(k[t^{\pm 1}]),$$

and (2.8) may be reinterpreted as saying

$$\lambda \overline{B} \cdot \tilde{p} = u + tv \in V \otimes k[t]$$

for some $u \in V$, $v \in V \otimes k[t]$. But the morphism $\mathbb{G}_m = \mathrm{Spec}\, k[t^{\pm 1}] \to \mathrm{SL}_m$ induced by λ is a 1-PS, and it thus follows that $\overline{B}^{-1} \lambda \overline{B}$ is a 1-PS with limit

$$\lim_{t \to 0} \overline{B}^{-1} \lambda(t) \overline{B} \cdot \tilde{p} = \overline{B}^{-1} u$$

as desired.

Finally, to prove (ii), observe that if $\lim_{t \to 0} \lambda(t) \cdot \tilde{p} = 0$ for some 1-PS λ then 0 is in the closure of $(\mathrm{SL}_m) \cdot \tilde{p}$, whence p is unstable. Conversely, if 0 is in the closure of $(\mathrm{SL}_m) \cdot \tilde{p}$, then similar to above,

by the valuative criterion of properness there is some $M \in \mathrm{SL}_m(K) \smallsetminus \mathrm{SL}_m(R)$ such that $M \cdot \tilde{p} \to 0$, and reasoning exactly as before, we deduce the analogue of (2.5), but since $\Lambda B \cdot \tilde{p} \to 0$ this time, strict inequality holds in (2.6), and hence (2.7) too. It thus follows that the $\Lambda \overline{B} \cdot \tilde{p} \to 0$, and thus $\overline{B}^{-1} \lambda \overline{B} \cdot \tilde{p} \to 0$ too, as desired. $\qquad \square$

In summary, we have shown that stability and semistability (although not polystability) are encoded in whether the induced limits of 1-PS's exist. Our final task is to find a numerical criterion that tells us whether these limits do indeed exists. To this end, we have the following definition:

Definition 2.4.7. Let G be a reductive affine algebraic group acting on $X \subseteq \mathbb{P}^n$, with a linearisation on $\mathcal{O}_X(1)$. Then we have a linear action of G on $\widetilde{X} \subseteq \mathbb{A}^{n+1}$. Now given a 1-PS $\lambda : \mathbb{G}_m \to G$, we have a weight space decomposition

$$k^{n+1} =: V = \bigoplus_{r \in \mathbb{Z}} V_r.$$

Choose a basis $\{e_i\}$ for V such that $\lambda \cdot e_i = \lambda^{r_i} e_i$ for all $\lambda \in \mathbb{G}_m(k)$. Now let $p \in X(k)$, and let $\tilde{p} \in \widetilde{X}(k)$ be a lift. We may write $\tilde{p} = \sum p_i e_i$. The *Hilbert-Mumford weight of λ at p with respect to* $\mathcal{O}_X(1)$, denoted $\mu^{\mathcal{O}_X(1)}(p, \lambda)$, is the integer

$$\mu^{\mathcal{O}_X(1)}(p, \lambda) := \max\{-r_i \mid p_i \neq 0\}.$$

Note that this does not depend on the choice of $\tilde{p}$ nor the basis.

We prove a very useful property of the Hilbert-Mumford weight:

Lemma 2.4.8. *For any $g \in G(k)$, we have $\mu^{\mathcal{O}_X(1)}(p, \lambda) = \mu^{\mathcal{O}_X(1)}(gp, g\lambda g^{-1})$.*

Proof. Observe that if $v \in V_i$, then for any $u \in \mathbb{G}_m(k)$ we have

$$g\lambda(u)g^{-1}(gv) = g\lambda(u)v = gu^i v = u^i gv.$$

Hence writing $p = \sum p_i e_i$ and applying this to each $p_i e_i$ we deduce the result. $\qquad \square$

Of course, it is a simple observation that if $\mu^{\mathcal{O}_X(1)}(p, \lambda) < 0$, then all the r_i are positive, and we have $\lim_{t \to 0} \lambda(t) \cdot \tilde{p} = 0$. If $\mu^{\mathcal{O}_X(1)}(p, \lambda) = 0$, then all the r_i are nonnegative, and hence the limit $\lim_{t \to 0} \lambda(t) \cdot \tilde{p}$ exists, but may not be zero. In summary, we have:

Theorem 2.4.9 (The Hilbert-Mumford Criterion for SL_m). *Let SL_m act on a projective variety X and suppose we have a very ample linearisation on $\mathcal{O}_X(1)$. Let p be a k-point. Then p is semistable if and only if $\mu^{\mathcal{O}_X(1)}(p, \lambda) \geq 0$ for all nontrivial 1-PS λ, with stability holding if and only if the condition holds with strict inequality.*

Proof. This follows directly from Theorem 2.4.5 and the above observation. $\qquad \square$

Remark 2.4.10. The Hilbert-Mumford criterion is true for general reductive algebraic groups, not just for SL_m. In fact, our argument could easily (in fact, almost word-for-word) be adapted for GL_m too. However, the general statement requires a stronger version of Iwahori's decomposition theorem, such as [1, Theorem 1.1], which is valid for reductive algebraic groups in general, but requires our DVR

be complete, and replaces the diagonal Λ with an element of $G(K)$ of the form $\operatorname{Spec} K \to \mathbb{G}_m \to G$, where the second arrow is some 1-PS, and the first is dual to $t \mapsto \varpi$ as before. The argument is similar to the one given here, although differences include taking R to be a complete DVR in the valuative criterion of properness (which is always possible, since we can just replace R by its completion) in order to invoke the general Iwahori decomposition and replacing the matrix arguments with more abstract arguments (although they achieve the same effect). The complete proof can be found in [28, pp. 53-54].

Now that the tools have been developed, we conclude this chapter with a revisit to a previous example, to see if our new tools can shed more light.

2.5 Conics Revisited

Recall that in the conics example in Chapter 1 (§1.3.1), we remarked (Remark 1.3.5) that it is crucial we are not defining conics up to projective transformations, since we get jump phenomena. Now that we have developed the techniques of GIT, we will try to fit this within our framework. We begin with a formal definition of our problem:

Definition 2.5.1. Let k be an algebraically closed field of characteristic zero. The *moduli problem of conics in $\mathbb{P}^2$ up to projective transformations* is the functor

$$\mathcal{M}^\flat : \mathsf{FTSch}/k \to \mathsf{Sets}, \ S \mapsto \{\text{families over } S\}/\operatorname{Aut}_S(\mathbb{P}^2 \times S).$$

Note that $\mathcal{M}^\flat$ is essentially the same problem, the only difference being that the equivalence relation on our families have changed. In particular, Lemmas 1.3.3 and 1.3.4 still hold, and thus while the family $\mathfrak{X}$ in the statement of Theorem 1.3.2 is no longer universal, it is still a locally versal family.

Now observe that SL_3 acts on $\mathbb{P}^5$ (which parameterises $\mathfrak{X}$) by acting inversely in the usual way on the variables x, y, z. More precisely, the usual action **composed with inversion** $\operatorname{SL}_3 \times \mathbb{P}^2 \to \mathbb{P}^2$ induces the following diagram:

$$
\begin{array}{ccc}
(\operatorname{SL}_3 \times \mathbb{P}^2 \times \mathbb{P}^5) \times_{\mathbb{P}^2 \times \mathbb{P}^5} \mathfrak{X} & \longrightarrow & \mathfrak{X} \\
\downarrow & & \downarrow \\
\operatorname{SL}_3 \times \mathbb{P}^2 \times \mathbb{P}^5 & \longrightarrow & \mathbb{P}^2 \times \mathbb{P}^5
\end{array}
$$

and it is clear that $(\operatorname{SL}_3 \times \mathbb{P}^2 \times \mathbb{P}^5) \times_{\mathbb{P}^2 \times \mathbb{P}^5} \mathfrak{X}$ is a family of conics over $\operatorname{SL}_3 \times \mathbb{P}^5$, and since $\mathbb{P}^5$ is a fine moduli space for the moduli problem of conics in $\mathbb{P}^2$, this is equivalent to a morphism $\operatorname{SL}_3 \times \mathbb{P}^5 \to \mathbb{P}^5$. One can check that this is a group action, and acts on k-points as described, and clearly two fibres in $\mathfrak{X}$ are equivalent if and only if they lie over points in the same orbit. This action has an obvious linearisation on $\mathcal{O}_{\mathbb{P}^5}(1)$, and thus induces an action $\operatorname{SL}_3 \times \mathbb{A}^6 \to \mathbb{A}^6$. So now we ask: what do stability semi-stability look like in this context? The first is very easy to answer: since $\dim \operatorname{SL}_3 = 8$ and $\dim \mathbb{P}^5 = 5$, for purely dimensional reasons no point is stable. We will now look at semistability.

To begin, observe that we may scale the x, y, z uniformly (i.e. multiply them by the same scalar) as we please, hence we may scale any $\mathrm{GL}_3(k)$ operation on the x, y, z so that it ends up in $\mathrm{SL}_3(k)$. For example, in order to interchange x, y, even though the usual permutation matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

has determinant -1, we may multiply A by -1 so that the resulting determiant is 1, and we end up with the desired operation. In particular, we may use any $\mathrm{GL}_3(k)$ operation, and we will do so without further comment.

Next observe that we may regard the equation $f = s_0 x^2 + \ldots + s_5 zx$ of any conic as a quadratic form Q on $V = k^3$, and hence we have an associated symmetric bilinear form β with the matrix

$$J = \begin{pmatrix} 2s_0 & s_1 & s_5 \\ s_1 & 2s_2 & s_3 \\ s_5 & s_3 & 2s_4 \end{pmatrix}.$$

Note that $J(x, y, z)^t$ is just the Jacobian of f. We can interpret β as a map $\beta : V \to V^*$. Observe that $X = \operatorname{Proj} k[x, y, z]/\langle f \rangle$ is nondegenerate if and only if β is injective.

Theorem 2.5.2. *A conic X over k is semistable with respect to the linearisation of the SL_3 action if and only if X is nondegenerate.*

Proof. Let X be a conic defined by some $f \in k[x, y, z]_{\deg 2}$ and let Q and $\beta : V \to V^*$ be as above. Suppose firstly X is denerate, and let $v_0 \in \ker \beta \smallsetminus \{0\}$. Let v_1, v_2 be orthogonal to each other (that is, $\beta(v_1, v_2) = 0$) so that $\{v_0, v_1, v_2\}$ is an orthogonal basis for V, and define a 1-PS as follows: let $g \in \mathrm{SL}_3(k)$ be (a scale of) the element sending (v_0, v_1, v_2) to (e_1, e_2, e_3), and define the 1-PS $\lambda : \mathbb{G}_m \to \mathrm{SL}_3$ represented as

$$g^{-1} \operatorname{diag}(t^2, t^{-1}, t^{-1}) g \in \mathrm{SL}_3(k[t^{\pm 1}]).$$

Then one can check that $g \cdot f = ay^2 + bz^2$, for some $a, b \in k$, and

$$(g\lambda(u)g^{-1}) \cdot (g \cdot X) = \operatorname{diag}(u^2, u^{-1}, u^{-1}) \cdot \operatorname{Proj} k[x, y, z]/\langle ay^2 + bz^2 \rangle = \operatorname{Proj} k[x, y, z]/\langle u^2 ay^2 + u^2 bz^2 \rangle$$

for any $u \in \mathbb{G}_m(k)$, and thus

$$\mu(X, \lambda) = \lambda(g \cdot X, g\lambda g^{-1}) = -2 < 0$$

whence X is unstable, as desired.

Conversely, suppose $X = \operatorname{Proj} k[x, y, z]/\langle f \rangle$ is unstable. Then there is some 1-PS $\lambda : \mathbb{G}_m \to \mathrm{SL}_3$ such that $\mu(X, \lambda) < 0$. The observation that stability is only dependent on the orbit combined with Lemma 2.4.8 allows us to suppose without loss of generality that λ diagonal, and hence represented by some

$$\operatorname{diag}(t^{r_1}, t^{r_2}, t^{r_3}) \in \mathrm{SL}_3(k[t^{\pm 1}]),$$

where $r_1 + r_2 + r_3 = 0$, and we may furthermore assume without loss of generality $r_1 \leq r_2 \leq r_3$. Write $f = s_0 x^2 + s_1 xy + s_2 y^2 + s_3 yz + s_4 z^2 + s_5 zx$ and observe that

$$\lambda(u) \cdot f = s_0 u^{-2r_1} x^2 + s_1 u^{-r_1-r_2} xy + s_2 u^{-2r_2} y^2 + s_3 u^{-r_2-r_3} yz + s_4 u^{-2r_3} z^2 + s_5 u^{-r_3-r_1} zx.$$

In order for μ to be negative, it follows that all the powers of u in the above expression with nonvanishing coefficient must be strictly positive. With this in mind, observe that since $r_1 < 0$ and $r_3 > 0$, it must be that $s_4 = 0$. Here we have a trichotomy about the sign of r_2: if $r_2 = 0$, then it follows $r_1 = -r_3$ and s_3 and s_5 are both 0. We then see

$$\det J = 2 s_2 s_5^2 = 0.$$

Now supposing that $r_2 > 0$, it once again follows $s_3 = 0$ and also $s_2 = 0$, and so $\det J = 0$. Finally, if $r_2 < 0$, then since $r_2 < r_3$, it follows $-r_2 - r_3 < -2r_2 < 0$ and hence we must have $s_3 = 0$ one final time. Similarly, it follows $-r_3 - r_1 = r_2 < 0$ and so $s_5 = 0$, whence $\det J = 0$ too. This completes the proof. $\square$

Since any two nondegenerate conics are equivalent, it follows that $\mathbb{P}^5 /\!/_{\mathcal{O}_{\mathbb{P}^5}(1)} \mathrm{SL}_3 = \operatorname{Spec} k$.

Remark 2.5.3. We can make a little tweak to this situation as follows: firstly note that every quadratic form is diagonalisable ([39, IV, Theorem 1]), and so we may assume every conic is of the form $\operatorname{Proj} k[x, y, z]/\langle px^2 + qy^2 + rz^2 \rangle$. Then analogously, the space of diagonal forms has a natural locally versal family parameterised by $\mathbb{P}^2 = \operatorname{Proj} k[a, b, c]$ with the family cut out by $ax^2 + by^2 + cz^2$. Then we need only consider the diagonal $\mathbb{G}_m^2 \subseteq \mathrm{SL}_3$ (we only need two copies of $\mathbb{G}_m$ because the third diagonal entry is given by the reciprocal of the product of the first two, since we are working with SL_3) acion on $\mathbb{P}^2$, with the obvious linearisation on $\mathcal{O}_{\mathbb{P}^2}(1)$, and with respect to this linearisation, doing the exact same calculation we find that nondegenerate conics are in fact **stable**, not just semistable (and in fact, stability and semistability coincide). Of course, the GIT quotient will be the same in both cases.

This innocuous calculation and result is an illustration of how geometric invariant theory is commonly used. Let $\mathcal{M}$ be a moduli problem, and suppose $X \to S$ is a locally versal family. Then if an algebraic group G acts on S parameterising equivalent families, restricting to the stable locus (possibly with respect to a very ample linearisation $\mathcal{L}$) can filter out "bad" points; this is particularly useful if $\mathcal{M}$ has a jump phenomenon, as we have just seen. Moreover, it is not uncommon for stability to coincide with a "natural" condition on the underlying naïve moduli problem of $\mathcal{M}$ (for example, nonsingularity in the above case). And furthermore, the GIT quotient $S^s \to S^s /\!/_{\mathcal{L}} G$ is a geometric quotient, so first and foremost it is a categorical quotient, and one can use the fact $X \to S$ is locally versal to build a natural transformation $\eta : \mathcal{M} \to \operatorname{Hom}(-, S^s /\!/_{\mathcal{L}} G)$ which will satisfy property (ii) in the definition of a coarse moduli space. Moreover, since it is an orbit space, property (i) will also be satisfied, and in particular, we can salvage a coarse moduli space of "stable" objects even if we have a jump phenomenon. In the next chapter, we will apply this idea to construct the moduli space of stable vector bundles.

Chapter 3

Stable Vector Bundles and their Moduli

In this chapter, we will combine everything we have learnt so far with a study of vector bundles to construct the moduli space of stable vector bundles. Let X be a fixed complete nonsingular curve of genus g over an algebraically closed field k of characteristic zero, equipped with a very ample line bundle $\mathcal{O}_X(1)$.

3.1   Stable and Semistable Bundles

To motivate the definitions that follow, we begin by stating our moduli problem.

Definition 3.1.1. The *moduli problem of vector bundles of signature* (n, d) *on* X (see Definition A.1.7) is the functor $\mathcal{V}_{n,d} : \mathsf{Var}/k \to \mathsf{Sets}$ defined as follows: for any variety S/k, we define a family of vector bundles over S to be a coherent sheaf $\mathcal{E}$ on $X \times S$, flat over S such that for any $s \in S(k)$, the fibre $\mathcal{E}_s$ defined to be the pullback of $\mathcal{E}$ along the map $s \times \mathrm{id} : \operatorname{Spec} k \times X \to S \times X$ is a locally free sheaf of signature (n, d) on X. Two families $\mathcal{E}, \mathcal{F}$ are *equivalent* if and only if there exists some line bundle $\mathcal{L}$ on S such that

$$\mathcal{E} \cong \mathcal{F} \otimes \pi_S^* \mathcal{L}.$$

We may also pull back families in the obvious way, and clearly equivalent families are pulled back to equivalent families. We then define $\mathcal{V}_{n,d}$ to be

$$\mathcal{V}_{n,d}(S) := \{\text{families over } S\}/\sim .$$

Of course, there are some interesting choices we have made, which we will briefly justify now, but will become clearer as more details are presented. Firstly, the reason we work with varieties is that Grauert's theorem ([17, III Corollary 12.9]) is needed in the argument, and Grauert's theorem requires an integrality condition. Of course, we could just work with integral schemes of finite type over k (note that we are constrainted to be working with schemes of finite type, since this is as far as our work on GIT takes us), but working with varieties seems more natural. Also, the equivalence relation may seem unusual, the reason for this is that we will have a GL_N (where N is some constant which will be defined later) action on some base space, and this extra condition allows us to work with SL_N instead, so we have an extra hypothesis to use.

However, even in the simple case of signature (2,0)-bundles over $\mathbb{P}^1 = \operatorname{Proj} k[x, y]$, there is a jump phenomenon:

Example 3.1.2. We consider the following family $\mathcal{E}$ over the affine line $\mathbb{A}^1 = \operatorname{Spec} k[t]$: let U_0 (resp. U_1) be the open subset where x (resp. y) does not vanish. Then we glue $\mathcal{O}^2_{U_0 \times \mathbb{A}^1}$ and $\mathcal{O}^2_{U_1 \times \mathbb{A}^1}$ on the overlap via the transition function

$$g_{0,1} := \begin{pmatrix} \frac{x}{y} & t \\ 0 & \frac{y}{x} \end{pmatrix} \in \operatorname{GL}_2(k[t, \frac{y}{x}, \frac{x}{y}]) = \operatorname{GL}_2(\Gamma(U_0 \cap U_1, \mathcal{O}_{\mathbb{A}^1 \times \mathbb{P}^1})).$$

More precisely, on U_0 we have the standard basis $r_1, r_2 \in \Gamma(U_0, \mathcal{O}^2_{U_0 \times \mathbb{A}^1})$, and on U_1 the standard basis $s_1, s_2 \in \Gamma(U_0, \mathcal{O}^2_{U_1 \times \mathbb{A}^1})$. We then define an isomorphism sending $s_1 \mapsto (x/y)r_1$ and $s_2 \mapsto tr_1 + (y/x)r_2$, and define $\mathcal{E}$ to be the gluing of the two sheaves on the overlap via this isomorphism. Since $\mathcal{E}$ is locally free, it is flat over $\mathbb{A}^1 \times \mathbb{P}^1$ and by the transitivity of flatness is also flat over $\mathbb{A}^1$.

Now we claim that the fibre $\mathcal{E}_0$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, but every other fibre is isomorphic to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ (and of course, these are not isomorphic, the latter has a nowhere-vanishing global section, but the former does not). To see this, first observe that clearly when $t = 0$, the transition map is just $\operatorname{diag}(y/x, x/y)$, which is the transition map of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. However, for a nonzero λ, we will define an isomorphism $\varphi : \mathcal{E}_\lambda \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ as follows: let e_1, e_2 be the standard basis of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. We define φ to be the map

$$r_1 \mapsto \frac{y}{x}e_1 - e_2, \quad r_2 \mapsto -\lambda e_1$$

on U_0 and

$$s_1 \mapsto e_1 - \frac{x}{y}e_2, \quad s_2 \mapsto -\lambda e_2$$

on U_1. To see that these glue, observe that $s_1 = (x/y)r_1$ and $s_2 = \lambda r_1 + (y/x)r_2$ on the overlap, and so

$$\varphi(s_1) = \varphi(\frac{x}{y}r_1) = e_1 - \frac{x}{y}e_2, \quad \varphi(s_2) = \varphi(\lambda r_1 + \frac{y}{x}r_2) = \lambda(\frac{y}{x}e_1 - e_2) - \lambda\frac{y}{x}e_1 = -\lambda e_2,$$

as expected. Finally, observe that φ maps local free generators to local free generators, and so is an isomorphism.

In fact, we can generalise this example: observe that for any $\lambda \in k$ above there is a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{E}_\lambda \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0,$$

with the only split fibre being at $\lambda = 0$. In particular, the $\mathcal{E}_\lambda$ are canonically identified with elements of the group $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1))$, and this idea is what we use to generalise the example. Firstly, let us look explicitly at how the Ext^1 group parameterises extensions, using Čech cohomology. Suppose we have a short exact sequence of vector bundles

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0. \tag{3.1}$$

Since these sheaves are locally free, there is an open affine cover $\{U_\alpha\}$ on which the restricted sequence does split, so let us choose a collection of splittings $\{f_\alpha : \mathcal{G}|_{U_\alpha} \to \mathcal{F}|_{U_\alpha}\}$. On overlaps, we

have two splittings and we see that their differences $f_{\alpha\beta} := f_\alpha - f_\beta$ have image in $\mathcal{E}|_{U_\alpha \cap U_\beta}$, and satisfy a cocycle condition, and in particular the $\{f_{\alpha\beta}\}$ form a Čech 1-cocycle for the sheaf $\mathcal{H}om(\mathcal{G}, \mathcal{E})$ with respect to the cover $\{U_\alpha\}$, and hence define an element of $H^1(X, \mathcal{H}om(\mathcal{G}, \mathcal{E})) \cong \mathrm{Ext}^1(\mathcal{G}, \mathcal{E})$. Moreover, it is not hard to check that a different choice of splittings gives a cohomologous cocycle, and thus the class $\delta_{\mathcal{F}} := [\{f_{\alpha\beta}\}] \in H^1(X, \mathcal{H}om(\mathcal{G}, \mathcal{E}))$ depends only on $\mathcal{F}$. Moreover, if (3.1) does split then $\delta_{\mathcal{F}} = 0$, and conversely as can easily be seen.

Even more explicitly, let us compute some transition functions of $\mathcal{F}$. Let $\{e_1^\alpha, ..., e_{\mathrm{rk}\,\mathcal{E}}^\alpha\}$ denote a frame for $\mathcal{E}$ on U_α (or equivalently the image of the standard basis of $\Gamma(U_\alpha, \mathcal{O}_X)^{\mathrm{rk}\,\mathcal{E}}$ in $\Gamma(U_\alpha, \mathcal{E})$) via a choice of local trivialisation, let $\{e_{\alpha\beta} \in \mathrm{GL}_{\mathrm{rk}\,\mathcal{E}}(\Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X))\}$ denote the transition functions satisfying

$$e_j^\beta = \sum (e_{\alpha\beta})_{ij} e_i^\alpha$$

on overlaps, and we similarly define $\{g_1^\alpha, ..., g_{\mathrm{rk}\,\mathcal{G}}^\alpha\}$, and $\{g_{\alpha\beta}\}$. Now letting $\{f_\alpha : \mathcal{G}|_{U_\alpha} \to \mathcal{F}|_{U_\alpha}\}$ denote a choice of local splittings as before, the set

$$\{e_1^\alpha, ..., e_{\mathrm{rk}\,\mathcal{E}}^\alpha, f_\alpha(g_1^\alpha), ..., f_\alpha(g_{\mathrm{rk}\,\mathcal{G}}^\alpha)\}$$

forms a frame for $\mathcal{F}$ on U_α. The e_i are related as before, but observe that on $U_{\alpha\beta}$ we have

$$f_\beta(g_j^\beta) = f_\alpha(g_j^\beta) - f_{\alpha\beta}(g_j^\beta) = (\sum_i (g_{\alpha\beta})_{ij}(f_\alpha(g_i^\alpha))) - f_{\alpha\beta}(g_j^\beta),$$

and so the transition matrix is of the form

$$\begin{pmatrix} e_{\alpha\beta} & -[f_{\alpha\beta}]_{\alpha\beta} \\ 0 & g_{\alpha\beta}, \end{pmatrix}$$

where $[f_{\alpha\beta}]_{\alpha\beta}$ is the matrix of $f_{\alpha\beta} \in \mathrm{Hom}(\mathcal{G}|_{U_{\alpha\beta}}, \mathcal{E}|_{U_{\alpha\beta}})$ with respect to the bases $\{e_i^\beta\}$, $\{f_\beta(g_i)^\beta\}$. With this in mind, we have the following result:

Theorem 3.1.3. *Suppose there is a vector bundle $\mathcal{F}$ of signature (n, d) which fits in a short exact sequence*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

that does not split. Then $\mathcal{V}_{n,d}$ does not have a coarse moduli space.

Proof. Fix a sufficiently fine open cover $\{U_\alpha\}$ and frames and transition maps retaining the notations as above. We now build a family $\{\mathcal{F}_t\}$ parameterised by $\mathbb{A}^1 = \mathrm{Spec}\,k[t]$ by gluing the frames via the transition function

$$\begin{pmatrix} e_{\alpha\beta} & -t[f_{\alpha\beta}]_{\alpha\beta} \\ 0 & g_{\alpha\beta} \end{pmatrix}.$$

Observe that the fibre $\mathcal{F}_0$ clearly splits, but we claim that all the other fibres are isomorphic to $\mathcal{F} = \mathcal{F}_1$. Indeed, fixing $\lambda \in k^*$, we define a map $\varphi : \mathcal{F} \mapsto \mathcal{F}_\lambda$ sending $f_\alpha(g_i^\alpha)$ to itself, but e_i^α to λe_i^α. To see that these glue, note that the e_i^α will clearly not cause problems, and observe

$$\varphi(f_\beta(g_j^\beta)) = \varphi((\sum_i (g_{\alpha\beta})_{ij}(f_\alpha(g_i^\alpha))) - f_{\alpha\beta}(g_j^\beta)) = (\sum_i (g_{\alpha\beta})_{ij}(f_\alpha(g_i^\alpha))) - \lambda f_{\alpha\beta}(g_j^\beta) = f_\beta(g_j^\beta).$$

It is clear this is an isomorphism, and that both bundles do not split, and hence there is a jump phenomenon, and so $\mathcal{V}_{n,d}$ has no coarse moduli space. $\qquad\square$

Remark 3.1.4. To tease our eventual defection into the analytic theory, we will give an alternative description of the Ext^1 group in the special case $k = \mathbb{C}$ using Dolbeault cohomology. In this case, $X(\mathbb{C})$ is just a compact Riemann surface, and there is a natural equivalence of categories between holomorphic vector bundles on $X(\mathbb{C})$ and algebraic vector bundles on X (in fact, this equivalence extends to coherent sheaves), so we identify them. Now a holomorphic vector bundle may be thought of as a smooth vector bundle E equipped with a Dolbeault operator $\bar{\partial}_{\mathcal{E}} : \Omega^0(E) \to \Omega^{0,1}(E)$ (this will be elaborated upon in the next chapter), where $\Omega^0(E)$ is the space of global E-valued 0-forms and similarly with $\Omega^{0,1}(E)$, and using $\bar{\partial}_{\mathcal{E}}$ we may build a complex of abelian group $\Omega^{0,q}(E) \to \Omega^{0,q+1}(E)$ which are the global sections of an acyclic resolution of E, so in particular its cohomology $H^{0,q}_{\bar{\partial}_{\mathcal{E}}}(E)$ is canonically isomorphic to the sheaf cohomology $H^q(X, \mathcal{E})$ (see [23, Corollary 2.6.25] for details). In particular, in the above situation, there is an isomorphism $\mathrm{Ext}^1(\mathcal{G}, \mathcal{E}) \cong H^1(X, \mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{E})) \cong H^{0,1}_{\bar{\partial}_{\mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{E})}}(\mathcal{H}\mathrm{om}(G, E))$, where E, G are the underlying smooth bundles of $\mathcal{E}, \mathcal{G}$ and $\mathcal{H}\mathrm{om}(G, E)$ is the bundle of smooth homomorphisms from G to E. To give an explicit isomorphism, firstly observe that the sequence

$$0 \to E \to F \to G \to 0$$

splits smoothly, so we have a splitting map $f : G \to F$. Composing with $\bar{\partial}_{\mathcal{F}}$ and projecting onto E, we have a map from G to $\Omega^{0,1}(E)$; i.e. an element β of $H^{0,1}_{\bar{\partial}_{\mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{E})}}(\mathcal{H}\mathrm{om}(G, E))$, and one can check this process defines the isomorphism. β is called the *second fundamental form* of the extension, and will be important in the next chapter.

And this unfortunately kills any hope of finding a moduli space. However, as with the conics revisited example in the previous chapter, one could hope that a stability condition, similar to nondegeneracy of conics in that example, will fix this problem. We will dedicate the rest of this chapter to showing that this is indeed the case. To define this stability condition, which we will just call stability, we first make the following definition

Definition 3.1.5. Let $\mathcal{E}$ be a vector bundle. The *slope* of $\mathcal{E}$, denoted $\mu(\mathcal{E})$, is defined to be

$$\mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\mathrm{rk}\, \mathcal{E}}$$

Lemma 3.1.6. *Let*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be a short exact sequence of vector bundles. Then $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$ if and only if $\mu(\mathcal{G}) \geq \mu(\mathcal{F})$, with equality holding in one if and only if in the other.

And finally, the definition:

Definition 3.1.7. Let $\mathcal{E}$ be a vector bundle. Then $\mathcal{E}$ is *stable* (resp. *semistable*) if for every proper subbundle $\mathcal{F}$, we have

$$\mu(\mathcal{F}) < \mu(\mathcal{E})$$
$$(\leq)$$

$\mathcal{E}$ is *polystable* if it is a direct sum of stable bundles of the same slope.

A *family* of (semi)-stable vector bundles over a variety S over k is a family of vector bundles whose fibres are all (semi)-stable. The *moduli problem of stable vector bundles of signature* (n, d) is the functor $\mathcal{V}^s_{n,d}$ taking a variety S to the set of families of stable vector bundles of signature (n, d) parameterised by S. Note that we do not define the moduli problem of semistable vector bundles.

Observe that we can alternatively define stability in terms of quotient bundles: by Lemma 3.1.6, $\mathcal{E}$ is (semi)-stable if and only if for every quotient bundle $\mathcal{G}$ we have:

$$\mu(\mathcal{E}) < \mu(\mathcal{G})$$
$$(\leq)$$

Let us mention some basic properties of stability:

Lemma 3.1.8. *Let $\mathcal{E}$ be a vector bundle.*

(i) *$\mathcal{E}$ is (semi)-stable if and only if $\mathcal{E} \otimes \mathcal{L}$ is (semi)-stable for every line bundle $\mathcal{L}$.*

(ii) *If $\operatorname{rk} \mathcal{E} = 1$ (i.e. $\mathcal{E}$ is a line bundle) then $\mathcal{E}$ is always stable.*

(iii) *If $\mathcal{E}'$ is another vector bundle then $\mathcal{E} \oplus \mathcal{E}'$ is not stable. It is semistable only if $\mu(\mathcal{E}) = \mu(\mathcal{E}')$ and both are semistable.*

(iv) *If $\varphi : \mathcal{E} \to \mathcal{F}$ is a nonzero morphism of vector bundles, and both bundles are semistable, then $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$. If both are stable, then equality occurs if and only if φ is an isomorphism.*

(v) *If $\mathcal{E}$ is stable, then it is simple (that is, $\operatorname{End}(\mathcal{E}) = k$).*

(vi) *$\mathcal{E}$ is semistable if and only if for any **subsheaf** $\mathcal{F}$ we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. Stability holds if and only if strict inequality holds.*

Proof. Note that every subbundle of $\mathcal{E} \otimes \mathcal{L}$ has the form $\mathcal{F} \otimes \mathcal{L}$ (indeed, if $\mathcal{G}$ is any subbundle of $\mathcal{E} \otimes \mathcal{L}$ we take $\mathcal{F} = \mathcal{G} \otimes \mathcal{L}^\vee$). Now observe

$$\mu(\mathcal{F} \otimes \mathcal{L}) = \frac{\deg \mathcal{F} + \operatorname{rk} \mathcal{F} \deg \mathcal{L}}{\operatorname{rk} \mathcal{F}} = \mu(\mathcal{F}) + \deg \mathcal{L}$$

and similarly for $\mu(\mathcal{E} \otimes \mathcal{L})$ and hence

$$\mu(\mathcal{E} \otimes \mathcal{L}) - \mu(\mathcal{F} \otimes \mathcal{L}) = \mu(\mathcal{E}) - \mu(\mathcal{F})$$

which proves (i). (ii) is trivial. To prove (iii), suppose without loss of generality $\mu(\mathcal{E}) \leq \mu(\mathcal{E}')$. Now

$$\mu(\mathcal{E} \oplus \mathcal{E}') = \frac{\deg(\mathcal{E}) + \deg(\mathcal{E}')}{\operatorname{rk}(\mathcal{E}) + \operatorname{rk}(\mathcal{E}')} \leq \mu(\mathcal{E}')$$

hence $\mathcal{E} \oplus \mathcal{E}'$ is not stable. If it is semistable, then equality must hold above, and clearly both must both be semistable, which proves (iii).

To prove (iv), observe that by Proposition A.1.17, the map φ factors as follows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0 \\
& & & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\
0 & \longleftarrow & \mathcal{F}'' & \longleftarrow & \mathcal{F} & \longleftarrow & \mathcal{F}' & \longleftarrow & 0
\end{array}
\tag{3.2}
$$

where $\mathcal{E}' \cong \ker \varphi$, $\mathcal{E}'' \cong \operatorname{im} \varphi$ and $\operatorname{rk} \mathcal{E}'' = \operatorname{rk} \mathcal{F}'$, $\deg \mathcal{E}'' \leq \deg \mathcal{F}'$. Since $\mathcal{E}$ and $\mathcal{F}$ are both semistable, it follows

$$
\mu(\mathcal{E}) \leq \mu(\mathcal{E}'') \leq \mu(\mathcal{F}') \leq \mu(\mathcal{F}).
\tag{3.3}
$$

Moreover, if are both are stable and φ is not an isomorphism, then either the inclusion $\mathcal{F}' \subseteq \mathcal{F}$ is proper, whence $\mu(\mathcal{F}') < \mu(\mathcal{F})$, or the inclusion $\mathcal{E}' \subseteq \mathcal{E}$ is proper, whence $\mu(\mathcal{E}) < \mu(\mathcal{E}'')$.

To prove (v), suppose $\varphi : \mathcal{E} \to \mathcal{E}$ is an endomorphism. If φ is zero, we are done. If not, we must have equality in (3.3), and by the stability of $\mathcal{E}$ we have $\mathcal{E} = \mathcal{E}'' = \operatorname{im} \varphi$, so in particular φ is an isomorphism. We have proven that any nonzero endomorphism of $\mathcal{E}$ is an isomorphism. Now fix some $p \in X(k)$; we have an induced map of fibres $\mathcal{E}_p / \mathfrak{m}_p \mathcal{E}_p \to \mathcal{E}_p / \mathfrak{m}_p \mathcal{E}_p$, and since k is algebraically closed, this map has an eigenvalue, say λ. But $\varphi - \lambda : \mathcal{E} \to \mathcal{E}$ is no longer an isomorphism, hence must be zero, and thus $\varphi = \lambda$ as desired.

Finally, to prove (vi), suppose $\mathcal{E}$ is semistable and let $\mathcal{F}$ be a subsheaf of $\mathcal{E}$. Then applying Proposition A.1.17 to the inclusion $\mathcal{F} \subseteq \mathcal{E}$, we deduce there is a subbundle $\mathcal{F}'$ of $\mathcal{E}$ with $\deg(\mathcal{F}') \geq \deg(\mathcal{F})$ and $\operatorname{rk}(\mathcal{F}') = \operatorname{rk}(\mathcal{F})$, and hence $\mu(\mathcal{F}) \leq \mu(\mathcal{F}') \leq \mu(\mathcal{E})$. If $\mathcal{E}$ is stable, then the inequality is strict. The converse is trivial. $\qquad\square$

As a first application, we will show that the jump phenomenon cannot happen when we restrict to stable bundles.

Proposition 3.1.9. *Let S be a variety over k, let $p \in S(k)$ be a k-valued point, and let $\mathcal{E}, \mathcal{F}$ be two families of vector bundles of signature (n, d) parameterised by S. Suppose $\mathcal{E}_s \cong \mathcal{F}_s$ for all points $p \neq s \in S$ and suppose further both $\mathcal{E}_p$ and $\mathcal{F}_p$ are stable. Then $\mathcal{E}_p \cong \mathcal{F}_p$.*

Proof. Following [18, p. 193], recall that the function

$$
s \mapsto \dim_{k(s)} H^0(\mathcal{H}om(\mathcal{E}_s, \mathcal{F}_s)),
$$

where $k(s)$ is the residue field of $s \in S$, is upper semicontinuous, by the semicontinuity theorem ([17, III Theorem 12.8]). Since $\mathcal{E}_s \cong \mathcal{F}_s$, it follows that $\dim_{k(s)} H^0(\mathcal{H}om(\mathcal{E}_s, \mathcal{F}_s)) > 0$ for all $s \neq p$, and hence $\dim_k H^0(\mathcal{H}om(\mathcal{E}_p, \mathcal{F}_p)) \neq 0$, so in particular there is a nonzero homomorphism $\mathcal{E}_p \to \mathcal{F}_p$. Since both are stable of the same signature, this homomorphism must be an isomorphism, by (iv) of Lemma 3.1.8. $\qquad\square$

We now give an explicit description of stability for vector bundles on $\mathbb{P}^1$.

Example 3.1.10. It is well-known that every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles ([36, Lemma 4.4.1]), and moreover every line bundle is of the form $\mathcal{O}_{\mathbb{P}^1}(n)$. Thus we may write any vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ as

$$
\mathcal{E} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(n)^{r_n},
$$

where $\mathcal{O}_{\mathbb{P}^1}(n)^{r_n}$ means r_n direct copies of $\mathcal{O}_{\mathbb{P}^1}(n)$, and all but finitely many of the r_n are zero. In this situation, stability has a very easy description: the only stable bundles are line bundles by (ii) in the above lemma, semistable and polystable bundles coincide and they all look like $\mathcal{O}_{\mathbb{P}^1}(n)^r$, and as soon as two distinct summands $\mathcal{O}_{\mathbb{P}^1}(n)$ and $\mathcal{O}_{\mathbb{P}^1}(m)$ turn up the bundle is unstable.

We consider the above example in greater detail. Given a bundle $\mathcal{E} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(n)^{r_n}$, where $n_1 < ... < n_\ell$ are the indices where $r_{n_i} \neq 0$, it is very tempting to build a filtration out of it, and indeed there is a very natural and reasonable way to do so, namely:

$$0 \subseteq \mathcal{E}_1 = \mathcal{O}_{\mathbb{P}^1}(n_\ell)^{r_{n_\ell}} \subseteq \mathcal{E}_2 = \bigoplus_{i=\ell-1}^{\ell} \mathcal{O}_{\mathbb{P}^1}(n_i)^{r_{n_i}} \subseteq ... \subseteq \mathcal{E}_\ell = \bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}^1}(n_i)^{r_{n_i}} = \mathcal{E}.$$

Note that this has the nice property that every quotient (i.e. each $\mathcal{E}_i / \mathcal{E}_{i-1}$) of the filtration is semistable, and it is ordered such that $\mu(\mathcal{E}_{i+1}/\mathcal{E}_i) > \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$. In fact, it is also easy to see that this is the *unique* filtration such that these hold. To generalise:

Proposition 3.1.11. *Let $\mathcal{E}$ be a vector bundle on X. Then there exists a unique filtration*

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq .... \subseteq \mathcal{E}_r = \mathcal{E}$$

with the property that each $\mathcal{E}_i / \mathcal{E}_{i-1}$ is semistable and $\mu(\mathcal{E}_{i+1}/\mathcal{E}_i) > \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ for each relevant i.

Before we prove this, we extract the following lemma:

Lemma 3.1.12 ([36], Lemma 5.4.1)**.** *Let $\mathcal{E}$ be a vector bundle on X. Then the set of degrees of subsheaves of $\mathcal{E}$ is bounded above.*

Proof of Proposition 3.1.11. If $\mathcal{E}$ is semistable we are done. If $\mathcal{E}$ is a line bundle then it is stable and we are done. We now induct on the rank. Suppose $\mathcal{E}$ is unstable. By the above lemma, the slopes of subbundles of $\mathcal{E}$ is bounded above, so we choose a maximal slope μ and among these a maximal rank subbundle $\mathcal{E}_1$ with slope $\mu_{\max}$. It is clear that $\mathcal{E}_1$ is semistable, and moreover we claim $\mathcal{E}/\mathcal{E}_1$ has no nontrivial subbundle of slope $\mu_{\max}$. Indeed, if it has such a subbundle, say $\mathcal{F}$, then by the correspondence theorem applied locally (it is not hard to check that we can do this and patch everything together), $\mathcal{F}$ corresponds to a subbundle $\mathcal{F}_1$ sitting strictly between $\mathcal{E}_1$ and $\mathcal{E}$, which contradicts the maximality of $\mathcal{E}_1$.

Now by the inductive hypothesis, $\mathcal{E}/\mathcal{E}_1$ has a filtration

$$0 \subseteq (\mathcal{E}/\mathcal{E}_1)_2 \subseteq ... \subseteq (\mathcal{E}/\mathcal{E}_1)_n = \mathcal{E}/\mathcal{E}_1,$$

and it is not hard to see that this lifts. Indeed, $(\mathcal{E}/\mathcal{E}_1)_i$ is a subbundle of $\mathcal{E}/\mathcal{E}_1$ and applying the correspondence theorem locally and patching together, we find an $\mathcal{E}_i$ such that $\mathcal{E}_1 \subseteq \mathcal{E}_i \subseteq \mathcal{E}$ and $\mathcal{E}_i/\mathcal{E}_1 = (\mathcal{E}/\mathcal{E}_1)_i$. Since $(\mathcal{E}_i/\mathcal{E}_{i+1}) = (\mathcal{E}/\mathcal{E}_1)_i/(\mathcal{E}/\mathcal{E}_1)_{i+1}$ it follows that this filtration satisfies the desired properties.

To prove uniqueness, suppose $(\mathcal{E}_i)_{i=1}^n$ and $(\mathcal{E}_i')_{i=1}^m$ are two filtrations satisfying the property. Let $\mu = \mu(\mathcal{E}_1)$ and let $\mu' = \mu(\mathcal{E}_1')$. Supposing without loss of generality $\mu \geq \mu'$, we consider the map $\mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}/\mathcal{E}_{m-1}'$. By assumption, $\mu \geq \mu' > \mu(\mathcal{E}/\mathcal{E}_{m-1}')$ and since both bundles are semistable, by

(iv) of Lemma 3.1.8 it follows the map is zero; in particular, $\mathcal{E}_1$ is contained in $\mathcal{E}'_{m-1}$. But applying the same argument inductively with $\mathcal{E}'_{m-i}$ in place of $\mathcal{E}$ and $\mathcal{E}'_{m-i-1}$ in place of $\mathcal{E}'_{m-1}$, we deduce $\mathcal{E}_1$ is contained in $\mathcal{E}'_1$. Now if $\mu > \mu'$, then by the same argument $\mathcal{E}_1 = 0$ which obviously cannot happen, and thus $\mu = \mu'$ and $\mathcal{E}_1$ is contained in $\mathcal{E}'_1$. But reversing the roles of $\mathcal{E}_1$ and $\mathcal{E}'_1$ we deduce $\mathcal{E}'_1$ is contained in $\mathcal{E}_1$ and thus they are equal. The result then follows by inductively applying the argument to $(\mathcal{E}_i / \mathcal{E}_1)$ and $(\mathcal{E}'_i / \mathcal{E}'_1)$. □

Definition 3.1.13. The filtration defined above is known as the *Harder-Narasimhan filtration* of $\mathcal{E}$.

We conclude this section by zooming in on a particular $\mu \in \mathbb{Q}$, and studying the category $\mathsf{V}^{ss}(\mu)$ of semistable vector bundles of slope μ. In particular, we have the following result:

Theorem 3.1.14 (Seshadri). *The category $\mathsf{V}^{ss}(\mu)$ is abelian.*

Proof. Being a subcategory of the abelian category of coherent sheaves on X, the Hom-sets naturally inherit an abelian group structure and the composition law clearly distributes over addition; moreover it is clear finite direct sums exist. Now observe that if $\varphi : \mathcal{E} \to \mathcal{F}$ is a morphism in $\mathsf{V}^{ss}(\mu)$, then Proposition A.1.17 combined with Lemma A.1.16 tell us that $\operatorname{im} \varphi$ is a subbundle and has slope μ as well, whence $\ker \varphi$ has slope μ. Clearly both are semistable, and hence $\mathsf{V}^{ss}(\mu)$ has kernels and cokernels. That every monomorphism (which is just an injective morphism, which can be seen in the supercategory of coherent sheaves) is the kernel of its cokernel and that every epimorphism is the cokernel of its kernel are immediate, and that every morphism can be factored into an epimorphism followed by a monomorphism is just the Proposition A.1.17-Lemma A.1.16 combination again.   □

And finally, we observe that for any object $\mathcal{E}$ in $\mathsf{V}^{ss}(\mu)$, every increasing and decreasing filtration terminates (we say $\mathsf{V}^{ss}(\mu)$ is *noetherian* and *artinian*), and in particular one can show that a Jordan-Hölder filtration (that is, a filtration $0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq ... \subseteq \mathcal{E}_r = \mathcal{E}$ such that every $\mathcal{E}_i / \mathcal{E}_{i-1}$ has no nontrivial subobject of $\mathsf{V}^{ss}(\mu)$ i.e. is stable) exists, and the Jordan-Hölder theorem, which states that the quotients are unique up to permutation and isomorphisms ([40, Theorem 2.1]).

Example 3.1.15. Of course, on $\mathbb{P}^1$ the only semistable bundles are of the form $\mathcal{O}_X(d)^n$, and so its Jordan-Hölder filtration is of the form

$$0 \subseteq \mathcal{O}_X(d) \subseteq \mathcal{O}_X(d)^2 \subseteq ... \subseteq \mathcal{O}_X(d)^n.$$

Example 3.1.16. In fact, one can define an equivalence relation on semistable bundles by asserting $\mathcal{E} \sim \mathcal{F}$ if and only if they have equivalent Jordan-Hölder filtrations (that is, the quotients are isomorphic up to permutation). Of course, within each class one finds a unique representative, which is the polystable bundle built by taking the direct sum of each stable quotient. Note the similarity between this and the equivalence of semistable orbits in the context of the previous chapter.

3.2   Constructing the Moduli Space

We will now construct the moduli space of stable bundles of signature (n, d). To simplify matters, we will assume $g \geq 1$. Indeed, if $g = 0$, then the result is trivial: since every vector bundle over $\mathbb{P}^1$ is

a direct sum of line bundles, the only stable bundles on $\mathbb{P}^1$ are line bundles, thus $V_{1,d}^s = \operatorname{Spec} k$ and $V_{n,d}^s = \varnothing$ if $n > 1$.

We will now suppose $g \geq 1$. Our approach is similar to the elliptic curves example in Chapter 1: we find a locally versal family equipped with a group action, and we show that the GIT quotient is the moduli space. In contrast to the aforementioned example, we do not find a locally versal first and then attach a group action; instead we find a scheme parameterising a family that contains every semistable bundle (known as a *bounded family*), but also parameterising things we do not want. We define the action first and then construct the locally versal family as the stable locus of this action. In order to construct the bounded family, we will first require the Riemann-Roch theorem:

Theorem 3.2.1 (Riemann-Roch for Vector Bundles). *Let $\mathcal{F}$ be a vector bundle on X of signature (n, d). Then*

$$\chi(\mathcal{F}) = d + n(1 - g),$$

where $\chi(\mathcal{F}) := \sum_{i=0}^{\infty} (-1)^i h^i(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F})$ is the Euler characteristic of $\mathcal{F}$.

Proof. We induct on n. For $n = 1$, the result is classical, and is proven in, for example, [17, pp. 295-296]. Now supposing true up to some $n - 1$, suppose $\mathcal{F}$ has rank n. Let $\mathcal{L}$ be a line subbundle of maximal degree (which exists because of [36, Lemma 5.4.1]). Then we claim $\mathcal{F}/\mathcal{L}$ must be locally free. Indeed, we may apply Proposition A.1.17 taking φ in the proposition statement to be the inclusion $\mathcal{L} \subseteq \mathcal{F}$, so that there is a nonzero map $\mathcal{L} \to \mathcal{F}'$ (where $\mathcal{F}'$ is as in the proposition statement). But both these bundles are line bundles, and hence are stable, and so $\deg \mathcal{L} \leq \deg \mathcal{F}'$, and since $\mathcal{L}$ is of maximal degree, equality must hold, and hence $\mathcal{F}/\mathcal{L} = \mathcal{F}''$ is locally free as claimed. Now applying the inductive hypothesis we have

$$\chi(\mathcal{F}) = \chi(\mathcal{L}) + \chi(\mathcal{F}/\mathcal{L}) = \deg \mathcal{L} + 1 - g + (d - \deg \mathcal{L}) + (n - 1)(1 - g) = d + n(1 - g)$$

as desired. $\qquad\square$

Corollary 3.2.2 (Classical Riemann-Roch Theorem). *For any line bundle $\mathcal{L}$ of degree d, we have*

$$h^0(\mathcal{L}) - h^0(\mathcal{L}^\vee \otimes \omega_X) = d + 1 - g,$$

where ω_X is the canonical bundle (which is just the cotangent bundle in this case). In particular, the degree of the canonical bundle is $2g - 2$.

Proof. Combine the above theorem and the Serre duality theorem ([17, III, Corollary 7.7]). $\qquad\square$

The importance of the Riemann-Roch theorem is that it allows us to relate the quantities we are interested in. More precisely, by the Serre vanishing theorem, for any coherent sheaf $\mathcal{F}$, and any ample line bundle $\mathcal{L}$, we have $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for any sufficiently large m and any $i > 0$. Moreover, by the definition of ampleness, we know that $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is generated by global sections, for any sufficiently large m. In particular, if $\mathcal{F}$ is locally free then the Riemann-Roch theorem tells us exactly how many global sections generate $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$. The main issue is that the "sufficiently large" criterion depends on $\mathcal{F}$ and $\mathcal{L}$. This is remedied by the following result:

Lemma 3.2.3. *Let $\mathcal{E}$ be a semistable vector bundle of signature (n, d).*

(i) *If $d > n(2g-2)$ then $H^1(X, \mathcal{E}) = 0$.*

(ii) *If $d > n(2g-1)$ then $\mathcal{E}$ is generated by global sections.*

Proof. This follows the proof given in [20, p. 68]. Suppose for contradiction $H^1(X, \mathcal{E}) \neq 0$. By the Serre duality theorem, we have

$$H^1(X, \mathcal{E}) \cong H^0(X, \mathcal{E}^\vee \otimes \omega_X) = \mathrm{Hom}(\mathcal{E}, \omega_X),$$

where ω_X is the canonical bundle (which is just the cotangent bundle here). This means that there is a nonzero homomorphism $\varphi : \mathcal{E} \to \omega_X$. Now by Lemma 3.1.8 (iv), we have

$$2g - 2 = \frac{n(2g-2)}{n} < \frac{d}{n} = \mu(\mathcal{E}) \leq \mu(\omega_X) = 2g - 2$$

which is a contradiction. This proves (i).

To prove (ii), we note that since the local ring of the generic point is a field, we need only check the stalk is generated by global sections at closed points, or equivalently k-points, since k is algebraically closed. So let p be a k-point, with local ring $\mathcal{O}_{X,p}$ and maximal ideal $\mathfrak{m}_p$. The composition $\mathcal{O}_X(U) \to \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/\mathfrak{m}_p$ induces the following short exact sequence of sheaves:

$$0 \to \mathcal{I}_p \to \mathcal{O}_X \to k_p \to 0,$$

where k_p is the skyscraper sheaf k sitting over p, and $\mathcal{I}_p$ is the kernel. Since $\mathcal{E}$ is locally free, tensoring is exact, and moreover since it is of rank n, and k_p is a skyscraper sheaf, it follows $k_p \otimes \mathcal{E} \cong k_p \otimes \mathcal{O}_X^n \cong k_p^n$. Thus tensoring the above with $\mathcal{E}$ we have

$$0 \to \mathcal{I}_p \otimes \mathcal{E} \to \mathcal{E} \to k_p^n \to 0. \tag{3.4}$$

Now we claim that $\mathcal{I}_p \cong \mathcal{L}(-p)$, where $\mathcal{L}(-p)$ is the line bundle associated to the divisor $-p$ (see Appendix B). To see this, let $U = \mathrm{Spec}\, A$ be an open affine subset. If $p \notin U$, then $\mathcal{I}_p|_U = \mathcal{O}_X|_U$. Otherwise, p is cut out by some $f \in A$, and hence $\mathcal{I}_p|_U = f\mathcal{O}_X|_U$. But this is exactly the definition of $\mathcal{L}(-p)$, as claimed. Now observe that $\mathcal{I}_p \otimes \mathcal{E} = \mathcal{E} \otimes \mathcal{L}(-p)$ is semistable and has degree $n(2g-2)$ and thus by part (i) we have $H^1(X, \mathcal{E} \otimes \mathcal{L}(-p)) = 0$. Now taking cohomology of (3.4), it follows we have a surjection

$$H^0(X, \mathcal{E}) \to H^0(X, k_p^n) = k^n.$$

Finally, we apply Nakayama's lemma on the local ring $\mathcal{O}_{X,p}$ to deduce that the map $H^0(X, \mathcal{E}) \to \mathcal{E}_p$ is surjective too. $\qquad\square$

The consequence of the above lemma combined with the Riemann-Roch theorem is that every semistable vector bundle of signature (n, d) for d sufficiently large is a quotient of $\mathcal{O}_X^{d+n(1-g)}$. Now it just so happens that there is a scheme, known as the *Quot scheme* that is a fine moduli space parameterising quotients of a given coherent sheaf, with certain constraints.

3.2.1 The Quot Functor and its Scheme

To define the moduli problem the Quot scheme represents, we first require the following result.

Proposition 3.2.4. *Let Y be a projective variety, let $\mathcal{F}$ be a coherent sheaf on Y and let $\mathcal{O}(1)$ be a very ample line bundle. For any $m \in \mathbb{Z}$, write $\mathcal{F}(m) := \mathcal{F} \otimes \mathcal{O}(m) := \mathcal{F} \otimes \mathcal{O}(1)^{\otimes m}$. Then there is a polynomial $P \in \mathbb{Q}[z]$ such that*

$$P(m) = \chi(\mathcal{F}(m))$$

for all $m \in \mathbb{Z}$.

Proof Sketch. Embed Y into some projective space via $\mathcal{O}(1)$. We use without proof the following two facts:

(i) If $F : \mathbb{Z} \to \mathbb{Z}$ is a function and there is a polynomial $Q \in \mathbb{Q}[z]$ such that $F(m) - F(m-1) = Q(m)$ for all $m \in \mathbb{Z}$, then there is some polynomial $P \in \mathbb{Q}[z]$ such that $P(m) = F(m)$ for all $m \in \mathbb{Z}$ ([17, I Proposition 7.3 (b)]).

(ii) There is some $f \in H^0(Y, \mathcal{O}(1))$ such that multiplication by f induces an injective homomorphism of sheaves $\mathcal{F}(-1) \to \mathcal{F}$ ([49, p. 489]).

With these in mind, we induct on $r = \dim \operatorname{Supp} \mathcal{F}$. If $r = 0$, then $\mathcal{F}$ is supported on a discrete subset, and hence tensoring with $\mathcal{O}(1)$ does nothing, and so P is constant, equal to $\chi(\mathcal{F})$. Now supposing true for some $r \geq 0$, we suppose $\dim \operatorname{Supp} \mathcal{F} = r + 1$, and write M for the graded module $M := \bigoplus_{m \in \mathbb{Z}} H^0(X, \mathcal{F}(m))$, so that $\widetilde{M} \cong \mathcal{F}$ ([17, II. Proposition 5.15]). We note that the map $M(-1) \to M$ given by multiplication by f as in fact (ii) above induces the following short exact sequence of sheaves

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is the cokernel. We claim that $\operatorname{Supp} \mathcal{G}$ is contained in the hyperplane $\{f = 0\}$ of our ambient projective space. Indeed, on any open affine subset $U = \operatorname{Spec} A$ of Y, we have $\operatorname{Supp} \mathcal{G} \cap U = V(\operatorname{Ann} H^0(U, \mathcal{G}|_U))$, and by assumption $f|_U$ annihilates $H^0(U, \mathcal{G}|_U)$, which proves the claim. But note that by the same reasoning, $\operatorname{Supp} \mathcal{F}$ is not contained in $\{f = 0\}$, since by assumption multiplication by f is injective, and since $\operatorname{Supp} \mathcal{G} \subseteq \operatorname{Supp} \mathcal{F}$, that means $\dim \operatorname{Supp} \mathcal{G} < r + 1$, and thus by the inductive hypothesis we know there is some $Q \in \mathbb{Q}[z]$ such that $Q(m) = \chi(\mathcal{G}(m))$ for all $m \in \mathbb{Z}$, and so the result follows from the additivity of the Euler characteristic on short exact sequences combined with fact (i) above. $\qquad \square$

Definition 3.2.5. The polynomial P above is known as the *Hilbert polynomial* of $\mathcal{F}$ with respect to $\mathcal{O}(1)$.

Example 3.2.6. Take $Y = \mathbb{P}^r$, take $\mathcal{F} = \mathcal{O}_Y$ and take $\mathcal{O}(1)$ to be the usual twisting sheaf of Serre. Observe that if $m \geq 0$ then $h^0(\mathcal{O}(m)) = \binom{m+r}{r}$, and $h^i(\mathcal{O}(m)) = 0$ for all $i > 0$, and hence

$$P = \binom{z+r}{r} := \frac{1}{r!} \prod_{i=1}^{r} (z+i) \in \mathbb{Q}[z],$$

since $P(m)$ and $\binom{m+r}{r}$ agree on infinitely many values.

Example 3.2.7. Let $\mathcal{E}$ be a vector bundle of signature (n, d) on X. By the Riemann-Roch theorem we know

$$\chi(\mathcal{E}(m)) = d + nm \deg \mathcal{O}_X(1) + n(1 - g),$$

and so $\mathcal{E}$ has Hilbert polynomial $P = d + n \deg \mathcal{O}_X(1)z + n(1 - g) \in \mathbb{Q}[z]$ with respect to $\mathcal{O}_X(1)$. Conversely, if $\mathcal{F}$ is a vector bundle with Hilbert polynomial $P = d + n \deg \mathcal{O}_X(1)z + n(1-g) \in \mathbb{Q}[z]$, then $\chi(\mathcal{F}) = d + n(1-g)$ and $\chi(\mathcal{F}(1)) = d + n \deg \mathcal{O}_X(1) + n(1-g)$, and so we know $\mathrm{rk}(\mathcal{F}) = n$ by Corollary A.1.14 and $\deg(\mathcal{F}) = d$ by Riemann-Roch. In particular, the data of the Hilbert polynomial on $\mathcal{E}$ (with respect to $\mathcal{O}_X(1)$) is equivalent to the data of the signature of $\mathcal{E}$.

A key property of the Hilbert polynomial is the following:

Theorem 3.2.8. *Let $T \to S$ be a projective morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on T, flat over S. Then the map*

$$s \mapsto \chi(\mathcal{F}_s) = \sum_{i=0}^{\infty} (-1)^i \dim_{k(s)} H^i(S, \mathcal{F}_s),$$

where $\mathcal{F}_s$ is the fibre of $\mathcal{F}$ over $s \in S$ and $k(s)$ is the residue field of s is locally constant. In particular, the Hilbert polynomials of $\mathcal{F}_s$ all agree on a connected component.

Proof. [49, p. 669] $\square$

Next, we define the following moduli problem:

Definition 3.2.9. Let $\mathcal{F}$ be a coherent sheaf on a projective variety Y with equipped with a very ample line bundle $\mathcal{O}(1)$, and let $P \in \mathbb{Q}[z]$ be a numerical polynomial. For any scheme S of finite type over k, a *family of quotients of $\mathcal{F}$ with Hilbert polynomial P parameterised by S* is a coherent sheaf $\mathcal{G}$ on $S \times Y$, flat and with proper support over S, equipped with a surjective map $\mathcal{F}_S \to \mathcal{G}$, where $\mathcal{F}_S$ is the pullback of $\mathcal{F}$ to $S \times X$ via the projection, such that all closed fibres $\mathcal{G}_p$ are coherent quotient sheaves of $\mathcal{F}$ with Hilbert polynomial P. Two families over S are *equivalent* if they have the same kernel. It is clear how families pull back, and so we have a functor

$$\mathcal{Q}uot_Y^P(\mathcal{F}) : \mathsf{FTSch}/k \to \mathsf{Sets},$$

sending a scheme S to the equivalence classes of families over S.

Theorem 3.2.10 (Grothendieck)**.** *The $\mathcal{Q}uot_Y^P(\mathcal{F})$ functor is representable by a projective variety over k.*

Definition 3.2.11. Let $P \in \mathbb{Q}[z]$ be a numerical polynomial. The *quot scheme of $\mathcal{F}$ with respect to P*, denoted $\mathrm{Quot}_Y^P(\mathcal{F})$, is the fine moduli space of $\mathcal{Q}uot_Y^P(\mathcal{F})$. A *Hilbert scheme* is a Quot scheme of the form $\mathrm{Quot}_{\mathbb{P}^n}^P(\mathcal{O}_{\mathbb{P}^n})$, which we will simply denote Hilb_n^P.

The idea of the construction, which can be found in [33], is to define an injective natural transformation from $\mathcal{Q}uot_Y^P(\mathcal{F})$ into a certain Grassmannian functor, and show that this defines a scheme structure. However, this construction is beyond the scope of this thesis, so we will be content with our examples from Chapter 1, where we showed that $\mathrm{Quot}_{\mathrm{Spec}\, k}^1(k^{n+1}) = \mathbb{P}^n$ and $\mathrm{Hilb}_2^{2z+1} = \mathbb{P}^5$.

The importance of the Quot scheme is as follows: by Lemma 3.2.3, it follows that all semistable vector bundles of signature (n,d) with $d > n(2g-1)$, (an assumption we will fix from now) on are found in the universal family over the scheme

$$Q = \operatorname{Quot}_X^P(\mathcal{O}_X^N),$$

where $N := d + n(1-g)$ and P is the polynomial $P(z) = d + nz \deg \mathcal{O}_X(1) + n(1-g)$. However, Q also parameterises other quotients, which we would like to ignore. It turns out there is an SL_N action on Q and a linearisation such that if d is large enough, the (semi)stable points are exactly locally free (semi)stable quotients where the induced map of global sections is an isomorphism, and in particular taking the stable locus of the projective GIT quotient will yield a coarse moduli space. We will give the construction below, following [20, pp. 68–84]. Various proofs below are based on ones found in the quoted citation, so we will not cite them individually.

3.2.2 The SL_N action on Q and its Stability

To define this SL_N-action, observe that since Q is a fine moduli space, a morphism $\operatorname{SL}_N \times Q \to Q$ is exactly a family of quotients of $\mathcal{O}_X^N$ parameterised by $\operatorname{SL}_N \times Q$. To this end, observe that the group $\operatorname{SL}_N(\Gamma(\operatorname{SL}_N, \mathcal{O}_{\operatorname{SL}_N}))$ of $\Gamma(\operatorname{SL}_N, \mathcal{O}_{\operatorname{SL}_N}) = k[x_{ij}, 1 \le i,j \le N]/\langle \det(x_{ij}) - 1\rangle$-valued points of SL_N is just the abstract group of automorphisms $\operatorname{SL}_N \to \operatorname{SL}_N$, and is dual to the group of automorphisms of $k[x_{ij}, 1 \le i,j \le N]/\langle \det(x_{ij}) - 1\rangle$. Now $\operatorname{SL}_N(\Gamma(\operatorname{SL}_N, \mathcal{O}_{\operatorname{SL}_N}))$ may also be seen as the group of $\Gamma(\operatorname{SL}_N, \mathcal{O}_{\operatorname{SL}_N})$-linear automorphisms of the module $\Gamma(\operatorname{SL}_N, \mathcal{O}_{\operatorname{SL}_N})^N$, and thus by extension the free sheaf of rank N on SL_N. Thus the inversion morphism $\operatorname{SL}_N \to \operatorname{SL}_N$ corresponds to an automorphism ι of $\Gamma(\operatorname{SL}_N, \mathcal{O}_{\operatorname{SL}_N})^N$, specifically given by the inverse of the matrix (x_{ij}), and thus for any k-point $(g_{ij}) \in \operatorname{SL}_N(k)$, the fibre of this morphism is exactly $(g_{ij})^{-1} \in \operatorname{SL}_N(k)$.

Now let $U : \mathcal{O}_{Q \times X}^N \to \mathcal{E}$ denote the universal family on $Q = \operatorname{Quot}_X^{d+nz \deg \mathcal{O}_X(1)+n(1-g)}(\mathcal{O}_X^N)$, and let π with subscripts denote the projection from $\operatorname{SL}_N \times Q \times X$ onto the subscripts. Now we define the action $\sigma : \operatorname{SL}_N \times Q \to Q$ as the morphism associated to the family

$$\pi_{Q \times X}^*(U) \circ \pi_{\operatorname{SL}_N}^*(\iota) : \mathcal{O}_{\operatorname{SL}_N \times Q \times X}^N \to \pi_{Q \times X}^*(\mathcal{E}).$$

One can check that this is indeed a group action.

The next question to ask is what this does to k-points. Let (g,p) be a k-point in $\operatorname{SL}_N \times Q$ (by abuse of notation, we identify p with its fibre in its universal family). Then one can check that $\sigma(g,p)$ is the quotient

$$\sigma(g,p) = p \circ g^{-1},$$

where g acts on $\mathcal{O}_{\operatorname{SL}_N \times Q \times X}^N$ in the obvious way; indeed, pulling ι back via $g : \operatorname{Spec} k \to \operatorname{SL}_N$ is just the map $g^{-1} : k^N \to k^N$, and thus pulling $\pi_{\operatorname{SL}_N}^*(\iota)$ back via $g \times \operatorname{id} \times \operatorname{id} : \operatorname{Spec} k \times Q \times X \to \operatorname{SL}_N \times Q \times X$ is just $g^{-1} : \mathcal{O}_{Q \times X}^N \to \mathcal{O}_{Q \times X}^N$. Finally, pulling the universal quotient back via p, we obtain the required

$$\sigma(g,p) = p \circ g^{-1}.$$

The key theorem to be proven is the following:

Theorem 3.2.12. *Fix n and let d be sufficiently large (depending on n). Then there exists a very ample linearisation $\mathcal{L}$ of this SL_N action such that a k-point $q : \mathcal{O}_X^N \to \mathcal{E}$ of Q is semistable if and only if $\mathcal{E}$ is a semistable vector bundle and $H^0(q)$ is an isomorphism.*

The rest of this subsection will be dedicated to its proof. To begin, we will need to study the 1-PS's of this action. So let $\lambda : \mathbb{G}_m \to \mathrm{SL}_N$ be a 1-PS, and as before, we have a weight space decomposition

$$k^N = V = \bigoplus_{r \in \mathbb{Z}} V_r.$$

Write $V_{\leq r} = \bigoplus_{s \leq r} V_s$, so that we have a filtration $V_{\leq r} \subseteq V_{\leq r+1}$. Now if $q : \mathcal{O}_X^N \to \mathcal{E}$ is a k-point in Q, write

$$\mathcal{E}_{\leq r} := \operatorname{im} q|_{V_{\leq r} \otimes_k \mathcal{O}_X}$$

and

$$\mathcal{E}_r := \mathcal{E}_{\leq r} / \mathcal{E}_{\leq r-1} .$$

We are now in a position to state:

Proposition 3.2.13. *There exists a natural number $M_0 > 0$ such that for any $M \geq M_0$, there is a very ample linearisation $\mathcal{L}_M$ of this SL_N action, depending on M, such that for a k-point $q : \mathcal{O}_X^N \to \mathcal{E}$ and 1-PS $\lambda : \mathbb{G}_m \to \mathrm{SL}_N$ inducing a weight space decomposition and filtration as above, we have*

$$\mu^{\mathcal{L}_M}(q, \lambda) = \sum_{r \in \mathbb{Z}} P_{\mathcal{E}_{\leq r}}(M) - \frac{\dim V_{\leq r}}{N} P(M),$$

where $P_{\mathcal{E}_{\leq r}}$ is the Hilbert polynomial of $\mathcal{E}_{\leq r}$ and P is the Hilbert polynomial $d + nz \deg \mathcal{O}_X(1) + n(1-g)$ of $\mathcal{E}$.

Proof. [20, p. 76]. $\qquad\square$

Thus the way forward is clear: we assume M is sufficiently large and use this expression for the weight and the Hilbert-Mumford criterion to calculate the stability of this linearised action, and eventually relate it to the usual vector bundle stability. To begin, let us investigate the sum $\sum_{r \in \mathbb{Z}} P_{\mathcal{E}_{\leq r}}(M) - \frac{\dim V_{\leq r}}{N} P(M)$. Suppose the weights of λ are labelled $r_1 < ... < r_m$. Then for any $r < r_1$, it follows $V_{\leq r} = 0$ whence $\mathcal{E}_{\leq r} = 0$ too, hence

$$P_{\mathcal{E}_{\leq r}}(M) - \frac{\dim V_{\leq r}}{N} P(M) = P_0(M) - \frac{0}{N} P(M) = 0.$$

Similarly, if $r > r_m$, then it follows $\mathcal{E}_{\leq r} = \mathcal{E}$ and $\dim V_{\leq r} = N$, and thus we also have

$$P_{\mathcal{E}_{\leq r}}(M) - \frac{\dim V_{\leq r}}{N} P(M) = P(M) - \frac{N}{N} P(M) = 0.$$

In particular, we do get a finite sum. Now for any r such that $r_i \leq r < r_{i+1}$, it follows $V_{\leq r_i} = V_{\leq r}$, hence $\mathcal{E}_{\leq r} = \mathcal{E}_{\leq r_i}$, and thus

$$\sum_{r_i \leq r \leq r_{i+1}} P_{\mathcal{E}_{\leq r}}(M) - \frac{\dim V_{\leq r}}{N} P(M) = (r_{i+1} - r_i) \left(P_{\mathcal{E}_{\leq r_i}}(M) - \frac{\dim V_{\leq r_i}}{N} P(M) \right)$$

whence

$$\mu^{\mathcal{L}_M} = \sum_{r \in \mathbb{Z}} P_{\mathcal{E}_{\leq r}}(M) - \frac{\dim V_{\leq r}}{N} P(M) = \sum_{1 \leq i \leq m-1} (r_{i+1} - r_i)\left(P_{\mathcal{E}_{\leq r_i}}(M) - \frac{\dim V_{\leq r_i}}{N} P(M)\right).$$

From this and the above proposition, we deduce:

Proposition 3.2.14. *Let* $q : \mathcal{O}_X^N \to \mathcal{E}$ *be a k-point in* Q. *Then* q *is semistable with respect to* $\mathcal{L}_M$ *if and only if for any subspace* $V' \subseteq V$ *we have*

$$P_{\mathcal{E}'}(M) - \frac{\dim V'}{N} P(M) \geq 0. \tag{3.5}$$

where $\mathcal{E}' = \operatorname{im} q|_{V' \otimes \mathcal{O}_X}$. *Stability holds if and only if the inequality is strict.*

Proof. Firstly, suppose the inequality holds. Now let λ be a 1-PS with weights $r_1 < \dots < r_m$. Now observe that for any $1 \leq i \leq m - 1$, we have

$$P_{\mathcal{E}_{\leq r_i}}(M) - \frac{\dim V_{\leq r_i}}{N} P(M) \geq 0,$$

hence

$$\mu^{\mathcal{L}_M}(q, \lambda) = \sum_{1 \leq i \leq m-1} (r_{i+1} - r_i)\left(P_{\mathcal{E}_{\leq r_i}}(M) - \frac{\dim V_{\leq r_i}}{N} P(M)\right) \geq 0,$$

and semistability follows from the Hilbert-Mumford criterion. If strict inequality holds in (3.5), then strict inequality holds above, and thus stability holds, as desired.

   Conversely, suppose there is some $V' \subseteq V$ such that the strict reverse inequality holds in (3.5). Fix a complement W such that $V = V' \oplus W$, and define the 1-PS λ to act with equal weight r_1 on V' and $r_2 > r_1$ on W (since we may scale bases as we want, this is always possible). Then

$$\mu^{\mathcal{L}_M}(q, \lambda) = (r_2 - r_1)\left(P_{\mathcal{E}_{\leq r_1}}(M) - \frac{\dim V_{\leq r_1}}{N} P(M)\right) = (r_2 - r_1)\left(P_{\mathcal{E}'}(M) - \frac{\dim V'}{N} P(M)\right) \leq 0.$$

and hence q is unstable. If no strict reverse inequality holds, but equality holds, then q is semistable, but not stable. $\qquad \square$

   Note that this is starting to look like our notion of stability already, since we are relating quantities of $\mathcal{E}$ to quantities of subsheaves of $\mathcal{E}$. For example, if $q : \mathcal{O}_X^N \to \mathcal{E}$ is a point, $\mathcal{E}$ is locally free and $\mathcal{E}'$ is one such subsheaf and $H^1(X, \mathcal{E}') = 0$, then the Riemann-Roch theorem tells us that the degree of $\mathcal{E}'$ cannot be too small. And in fact, we can already concretely deduce:

Corollary 3.2.15. *Let* $q : \mathcal{O}_X^N \to \mathcal{E}$ *be a semistable k-point in* Q *with respect to a sufficiently large* M. *Then* $\mathcal{E}$ *is a locally free sheaf.*

Proof. Choose $M \geq M_0$, and such that $P(M) > N^2$. Firstly suppose $\mathcal{E}$ is not locally free. Then $\mathcal{E}$ is not torsion free, so let $\mathcal{F}$ be a nonzero torsion subsheaf of $\mathcal{E}$. Then $\mathcal{F}$ is supported on a discrete set, since we are on a curve, and so $H^0(\mathcal{F}) \neq 0$. Now let $V' := H^0(q)^{-1}(H^0(\mathcal{F}))$ and let $\mathcal{E}' = \operatorname{im} q|_{V' \otimes \mathcal{O}_X}$. Since q is surjective, it follows that $\mathcal{E}'$ is a subsheaf of $\mathcal{F}$ and hence is also torsion. But

then $\mathcal{E}'$ has constant Hilbert polynomial (since it is supported on a discrete set, twisting does nothing), and moreover $h^1(\mathcal{E}) = 0$ for dimension reasons ([17, III, Theorem 2.7]), and so

$$0 < P_{\mathcal{E}'}(M) = h^0(\mathcal{E}') \leq N \leq \frac{P(M)}{N} < \frac{\dim V'}{N} P(M).$$

Hence $\mathcal{E}$ is unstable.   □

We now take a closer look at the sheaves $\mathcal{E}' := \operatorname{im} q|_{V' \otimes \mathcal{O}_X}$. In particular, we want to know what proportion of subsheaves they make up, and whether they have special properties among the subsheaves. To this end, we have the following result:

Proposition 3.2.16. *The set of Hilbert polynomials $\{P_{\mathcal{E}'}\}$, where $\mathcal{E}' := \operatorname{im} q|_{V' \otimes \mathcal{O}_X}$, taken across all $q : \mathcal{O}_X^N \to \mathcal{E}$ and $V' \subseteq V$ is finite.*

Proof Sketch. Let $1 \leq r \leq N$. It suffices to show that the set $\{P_{\mathcal{E}'}\}_{\dim V'=r}$ is finite. To this end, recall that the Grassmannian $\operatorname{Gr}^{r,N}$ is a fine moduli space parameterising injections $k^r \to k^N$, and so there is a universal family $\mathcal{U} \subseteq \mathcal{O}_{\operatorname{Gr}^{r,N}}^N$, where $\mathcal{U}$ is the universal bundle, which has rank r ([12, 8.4]). Pulling this back via the projection $\pi_{\operatorname{Gr}^{r,N}} : \operatorname{Gr}^{r,N} \times Q \times X \to \operatorname{Gr}^{r,N}$, and composing this with the pullback of the universal family on Q via $\pi_{Q \times X} : \operatorname{Gr}^{r,N} \times Q \times X \to Q \times X$ we have a restricted family of sheaves

$$\mathscr{E}' := \operatorname{im} \pi_{Q \times X}^* U|_{\mathcal{U} \otimes \mathcal{O}_{\operatorname{Gr}^{r,N} \times Q \times X}} \subseteq \mathscr{E},$$

and it is easy to see that given $V' \in \operatorname{Gr}^{r,N}(k)$, which we treat as a subspace of k^N, and $q : \mathcal{O}_X^N \to \mathcal{E}$, the fibre $\mathscr{E}'_{(V',q)}$ of $\mathscr{E}'$ via the projection $\pi_{\operatorname{Gr}^{r,N} \times Q} : \operatorname{Gr}^{r,N} \times Q \times X \to \operatorname{Gr}^{r,N} \times Q$ is just $\mathcal{E}' = \operatorname{im} q|_{V' \otimes \mathcal{O}_X}$. In particular, every such $\operatorname{im} q|_{V' \otimes \mathcal{O}_X}$ where $\dim V' = r$ is a fibre of this sheaf at some k-point. It thus suffices to show that the set of Hilbert polynomials of fibres over $\mathscr{E}'$ is finite.

To this end, observe that even though $\mathscr{E}'$ may not be flat, it can be shown ([36, Lemma 4.4.8]) that we can cover $\operatorname{Gr}^{r,N} \times Q$ by a finite collection $\{S_i\}$ of locally closed subvarieties such that $\mathscr{E}'|_{S_i \times X}$ is flat over S_i, and hence by Theorem 3.2.8, it follows that the $\mathscr{E}'|_s$ all have the same Hilbert polynomial on each S_i. Since there are finitely many S_i, the result follows.   □

Let $q : \mathcal{O}_X^N \to \mathcal{E}$ is a k-point in Q; let us now examine explicitly the Hilbert polynomials $P_{\mathcal{E}'}$. If $\mathcal{E}'$ is torsion-free (hence locally free) of signature (n', d') then we know

$$P_{\mathcal{E}'} = d' + n' \deg \mathcal{O}_X(1)z + n'(1-g).$$

If $\mathcal{E}'$ is not torsion-free, let $\mathcal{F}' \subseteq \mathcal{E}'$ be its torsion subsheaf, so that $\mathcal{E}'/\mathcal{F}'$ is locally free of signature (n', d'). Now the Hilbert polynomial of $\mathcal{F}'$ is constant, equal to $h^0(\mathcal{F}')$, and so the Hilbert polynomial of $\mathcal{E}'$ is

$$P_{\mathcal{E}'} = d' + h^0(\mathcal{F}') + n' \deg \mathcal{O}_X(1)z + n'(1-g).$$

In particular, the Hilbert polynomial of $\mathcal{E}'$ has degree at most one, and its leading coefficient is the rank of its torsion-free part, which is at most n. Thus given $\mathcal{E}'$, we can choose a sufficiently large M_0' such that the inequality (3.5) holds for any $M > M_0'$ if and only if

$$\operatorname{rk} \mathcal{E}' \geq \frac{n \dim V'}{N}, \tag{3.6}$$

where by abuse of notation $\operatorname{rk} \mathcal{E}'$ is the rank of the torsion-free part of $\mathcal{E}'$. Moreover, note that this M_0' depends only on $P_{\mathcal{E}'}$ and $\dim V'$, and since there are finitely many such pairs, we can choose this sufficiently large to work for all $\mathcal{E}'$. With this in mind, we can deduce the following:

Corollary 3.2.17. *Let* $M > M_0'$ *and* $q : \mathcal{O}_X^N \to \mathcal{E}$ *be a* k-valued point in Q, semistable with respect to $\mathcal{L}_M$. *Then* $H^0(q)$ *is an isomorphism.*

Proof. We begin with injectivity. Let $V' = \ker H^0(q)$. Then $\mathcal{E}' = \operatorname{im} q|_{V' \otimes \mathcal{O}_X} = 0$, and so

$$0 = P_{\mathcal{E}'}(M) \geq \frac{\dim V'}{N} P(M),$$

whence $\dim V' = 0$, as desired.

To prove surjectivity, observe that by the Riemann-Roch theorem, it suffices to show that $H^1(X, \mathcal{E}) = 0$, whence $h^0(\mathcal{E}) = N = \dim \operatorname{im} H^0(q)$. So suppose for contradiction $H^1(X, \mathcal{E}) \neq 0$; then by the Serre duality theorem there is a nonzero map $\varphi : \mathcal{E} \to \omega_X$. Let $\mathcal{F}$ denote the image of this map. We claim $h^0(\ker \varphi) \neq 0$. Indeed, since $\mathcal{F}$ injects into ω_X, it follows that $h^0(\mathcal{F}) \leq g$, and hence

$$h^0(\ker \varphi) \geq h^0(\mathcal{E}) - h^0(\mathcal{F}) \geq d + n(1-g) - g > n(2g-1) + n(1-g) - g = ng - g \geq 0,$$

as claimed. Now let $V' := H^0(q)^{-1}(H^0(X, \ker \varphi))$, and let $\mathcal{E}' := \operatorname{im} q|_{V' \otimes \mathcal{O}_X} \neq 0$, so that $\mathcal{E}' \subseteq \ker \varphi$. By the previous discussion, we have

$$\operatorname{rk} \mathcal{E}' \geq \frac{n \dim V'}{N},$$

and since $\operatorname{rk} \ker \varphi \geq \operatorname{rk} \mathcal{E}'$, we have

$$\operatorname{rk} \ker \varphi = n - 1 \geq \frac{n \dim V'}{N} \geq \frac{n(N-g)}{N} = n\left(1 - \frac{g}{N}\right).$$

But that would imply $d \leq n(2g-1)$ after a rearrangement, which contradicts our choice of d. $\square$

Our next job is to relate stability of Q as above back to vector bundle stability. In particular, we need to rephrase usual vector bundle stability (involving degree and rank invariants) in terms of cohomology. To begin, we have the following useful bound:

Lemma 3.2.18. *Let* $\mathcal{E}$ *be a semistable vector bundle of signature* (n, d) *and slope* μ. *Then*

$$\frac{h^0(\mathcal{E})}{n} \leq [\mu + 1]_+ := \sup\{0, \mu + 1\}. \tag{3.7}$$

Proof. We induct on the degree. Firstly, if $d < 0$, we claim $H^0(X, \mathcal{E}) = 0$. Indeed, if not, let $s \in H^0(X, \mathcal{E})$ be nonzero. Then by Remark A.1.11, there exists a line subbundle $\mathcal{L}$ isomorphic to the subsheaf generated by s. In particular, $\mathcal{L}$ has a nonzero global section. But by semistability, $\deg \mathcal{L} < 0$, which contradicts Lemma A.1.15. Thus $H^0(X, \mathcal{E}) = 0$ as claimed. Now, suppose $d \geq 0$ and supposing this has been proven for all smaller values, we fix some $p \in X(k)$ and as in the proof of Lemma 3.2.3, we have the following short exact sequence

$$0 \to \mathcal{L}(-p) \to \mathcal{O}_X \to k_p \to 0.$$

Tensoring with $\mathcal{E}$ and taking cohomology, we obtain

$$h^0(\mathcal{E}) \leq h^0(\mathcal{E} \otimes \mathcal{L}(-p)) + n,$$

and finally, observing that

$$\deg(\mathcal{E} \otimes \mathcal{L}(-p)) = d - n < d$$

by Corollary A.1.14 and applying the induction hypothesis, we obtain

$$\frac{h^0(\mathcal{E})}{n} \leq \frac{h^0(\mathcal{E} \otimes \mathcal{L}(-p))}{n} + 1 \leq \left[\frac{d-n}{n} + 1\right]_+ + 1 = [\mu + 1]_+$$

as desired. $\square$

We now give another criterion for vector bundle (semi)-stability.

Proposition 3.2.19 (Le Potier's Theorem). *Let $n \in \mathbb{N}$ be fixed, let $d > gn^2 + n(2g - 2)$ and let $\mathcal{E}$ be a vector bundle of signature (n, d) and slope μ. Then $\mathcal{E}$ is semistable if and only if for every subsheaf $\mathcal{F}$ of rank n' we have*

$$h^0(\mathcal{F}) \leq \frac{n'}{n} h^0(\mathcal{E}). \tag{3.8}$$

Strict inequality holds if and only if $\mathcal{E}$ is stable.

Proof. Firstly suppose $\mathcal{E}$ is unstable, and let $\mathcal{F}$ be a semistable subbundle of signature (n', d') such that $\mu(\mathcal{F}) > \mu(\mathcal{E})$ (for example, we can take $\mathcal{F}$ to be the first term in the Harder-Narasimhan filtration of $\mathcal{E}$). Now observe that

$$d' > \mu n' \geq n'(gn + 2g - 2) > n'(2g - 2),$$

so in particular $H^1(X, \mathcal{F}) = 0$ by Lemma 3.2.3. Thus by the Riemann-Roch theorem we have

$$h^0(\mathcal{F}) = d' + n'(1-g) = n'(\mu(\mathcal{F}) + 1 - g) > n'(\mu(\mathcal{E}) + 1 - g) = \frac{n'}{n}(d + n(1-g)) = \frac{n'}{n} h^0(\mathcal{E}).$$

If $\mathcal{E}$ is strictly semistable, then we repeat the argument with $\mathcal{F}$ of equal slope and deduce equality in (3.8).

Conversely, suppose $\mathcal{E}$ is semistable. Let $\mathcal{F}$ be a subsheaf of signature (n', d') and let $(\mathcal{F}_i)_{i=0}^m$ be the Harder-Narasimhan filtration of $\mathcal{F}$. Observe that

$$h^0(\mathcal{F}) = \sum_{i=1}^m h^0(\mathcal{F}_i) - h^0(\mathcal{F}_{i-1}) \leq \sum_{i=1}^m h^0(\mathcal{F}_i/\mathcal{F}_{i-1}) \leq \sum_{i=1}^m \mathrm{rk}(\mathcal{F}_i/\mathcal{F}_{i-1})[\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) + 1]_+,$$

and furthermore note that $\mu(\mathcal{F}_1) \geq \mu(\mathcal{F}_i/\mathcal{F}_{i-1})$ by the construction of the Harder-Narasimhan filtration. In particular, since $\mathcal{E}$ is semistable we have $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) \leq \mu$ and applying this to the above we have

$$h^0(\mathcal{F}) \leq \sum_{i=1}^m \mathrm{rk}(\mathcal{F}_i/\mathcal{F}_{i-1})[\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) + 1]_+ \leq (n' - 1)(\mu + 1) + [\mu(\mathcal{F}/\mathcal{F}_{m-1}) + 1]_+.$$

Now if $\mu(\mathcal{F}/\mathcal{F}_{m-1}) < \mu - gn$, then

$$h^0(\mathcal{F}) < (n'-1)(\mu+1) + (\mu - gn + 1) \leq n'(\mu+1) - gn' = \frac{n'}{n}(d + n(1-g)) = \frac{n'}{n}h^0(\mathcal{E}),$$

since $H^1(X, \mathcal{E}) = 0$ by Lemma 3.2.3. Otherwise, if $\mu(\mathcal{F}/\mathcal{F}_{m-1}) \geq \mu - gn > 2g - 2$, we claim that $H^1(X, \mathcal{F}) = 0$, whence

$$h^0(\mathcal{F}) = d' + n'(1-g) \leq \frac{n'd}{n} + n'(1-g) = \frac{n'}{n}(d + n(1-g)) = \frac{n'}{n}h^0(\mathcal{E}). \tag{3.9}$$

In fact, we will show inductively that each $H^1(X, \mathcal{F}_i) = 0$. Firstly note that $\mu(\mathcal{F}_1) \geq \mu(\mathcal{F}/\mathcal{F}_{m-1}) > 2g - 2$ (with equality holding on the left if and only if $\mathcal{F}$ is semistable) whence $H^1(X, \mathcal{F}_1) = 0$ by Lemma 3.2.3. Now supposing we have shown $H^1(X, \mathcal{F}_{i-1}) = 0$, we observe that $H^1(X, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ too, by the same reasoning as before, and thus by the long exact sequence of cohomology applied to

$$0 \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{F}_i/\mathcal{F}_{i-1} \to 0$$

we deduce that $H^1(X, \mathcal{F}_i) = 0$ too.

Finally, suppose equality in (3.8) holds for some $\mathcal{F}$. We will prove $\mathcal{E}$ is strictly semistable. Let $(\mathcal{F}_i)_{i=0}^m$ be the Harder-Narasimhan filtration of $\mathcal{F}$, and note that by the above, equality cannot hold if $\mu(\mathcal{F}/\mathcal{F}_{m-1}) < \mu - gn$, so $\mu(\mathcal{F}/\mathcal{F}_{m-1}) \geq \mu - gn$, and hence $H^1(X, \mathcal{F}) = 0$. Since equality holds in (3.8), it must be that $\mu(\mathcal{F}) = \mu = \mu(\mathcal{E})$, as can clearly be seen in (3.9). This completes the proof. $\square$

And finally, we conclude the subsection with the promised result:

Theorem 3.2.20. *Let $n \in \mathbb{N}$ be fixed, let $d > gn^2 + n^2(2g-1) \geq \sup\{n(2g-1), gn^2 + n(2g-2)\}$, write $N := d + n(1-g)$, let $P \in \mathbb{Q}[z]$ be the polynomial*

$$P := d + nz \deg \mathcal{O}_X(1) + n(1-g),$$

and finally let $Q = \mathrm{Quot}_X^P(\mathcal{O}_X^N)$. Then there exists an M_0 such that for any $M \geq M_0$, a k-point $q : \mathcal{O}_X^N \to \mathcal{E}$ of Q is semistable with respect to the linearised SL_N-action on $\mathcal{L}_M$ if and only if $\mathcal{E}$ is a semistable locally free sheaf and $H^0(q)$ is an isomorphism. Moreover, q is stable if and only if $\mathcal{E}$ is stable.

Proof. We choose M_0 sufficiently large such that:

  (i) The conclusion of Proposition 3.2.13 holds.

  (ii) The conclusion of Corollary 3.2.15 holds.

  (iii) The inequality (3.5) for $M > M_0$ holds if and only if the inequality (3.6) holds.

Then we know from Corollaries 3.2.15 and 3.2.17 that if q is semistable then $\mathcal{E}$ is locally free and $H^0(q)$ is an isomorphism. To show that $\mathcal{E}$ is in fact semistable, let $\mathcal{F}$ be a subsheaf of $\mathcal{E}$. If $h^0(\mathcal{F}) = 0$,

we are done; otherwise, let $V' := H^0(q)(H^0(\mathcal{F}))$ and let $\mathcal{E}' := \operatorname{im} q|_{V' \otimes \mathcal{O}_X}$, and suppose $\mathcal{E}'$ has signature (n', d'). By Proposition 3.2.14 combined with (iii) above, we have

$$n' \geq \frac{n \dim V'}{N},$$

and noting that $n' \leq \operatorname{rk} \mathcal{F}$, $h^0(\mathcal{E}) = N$ and $\dim V' = h^0(\mathcal{F})$, we find

$$h^0(\mathcal{F}) = \dim V' \leq \frac{n'}{n} h^0(\mathcal{E})$$

and hence $\mathcal{E}$ is semistable by Proposition 3.2.19. If q is stable, then $\mathcal{E}$ is stable too, by running the above argument, replacing the inequalities with strict ones.

Conversely, suppose $\mathcal{E}$ is semistable and $H^0(q)$ is an isomorphism. Now let $V' \subseteq k^N$ be a nonzero subspace, let $\mathcal{E}' := \operatorname{im} q|_{V' \otimes \mathcal{O}_X}$ and suppose $\mathcal{E}'$ has signature (n', d'). Then we know

$$n' \geq \frac{n h^0(\mathcal{E}')}{N} = \frac{n \dim V'}{N}$$

and so q is semistable. Now if $\mathcal{E}$ is stable, we run the argument through with strict inequality and deduce strict inequality. □

3.2.3   Putting it all together

So now we have an open subscheme $Q^{(s)s}$ whose k-points are exactly $q : \mathcal{O}_X^N \to \mathcal{E}$ (semi)-stable vector bundles of our given signature (n, d), where $d > gn^2 + n^2(2g - 2)$ (indeed, it is easy to see that tensoring with a line bundle of degree one is a natural isomorphism of functors between $\mathcal{V}_{n,d}^s$ and $\mathcal{V}_{n,d+n}^s$, so we may assume without loss of generality d is sufficiently large). In particular, we may take the projective GIT quotient $Q^{ss} \to Q^{ss} /\!\!/_{\mathcal{L}_M} \operatorname{SL}_N =: V_{n,d}^{ss}$ to obtain a categorical quotient, and moreover we have the geometric quotient $Q^s /\!\!/_{\mathcal{L}_M} \operatorname{SL}_N =: V_{n,d}^s$. What remains is to complete the argument and show that $V_{n,d}^s$ is in fact the moduli space for stable bundles of signature (n, d). To begin, we have the following:

Lemma 3.2.21. *Let* $\eta : \mathcal{V}_{n,d}^s \to \operatorname{Hom}(-, N)$ *be a natural transformation. Then*

$$\eta_{Q^s}(\mathscr{E}|_{Q^s}) : Q^s \to N$$

is SL_N*-invariant.*

Proof. Let $\sigma : \operatorname{SL}_N \times Q^s \to Q^s$ denote the action. This is just saying that the families $\sigma^*(\mathcal{E}|_U)$ and $\pi_Q^*(\mathcal{E}|_U)$ are equivalent, which is obvious because they differ by the inverted universal $\mathcal{O}_{\operatorname{SL}_N}^N$ automorphism on SL_N, which is an isomorphism. □

Lemma 3.2.22. *The scheme* Q^s *along with the family* $\mathscr{E}|_{Q^s}$ *is locally versal.*

Proof. Let $\mathcal{E}$ be a family of stable bundles over a variety S. By Grauert's theorem ([17, III Corollary 12.9]), the pushforward $\pi_{S*}\mathcal{E}$ is locally free. Moreover, by Lemma 3.2.3 and our assumption on d, for any $s \in S(k)$ we have $h^0(X, \mathcal{E}_s) = N$, and thus Grauert's theorem combined with Nakayama's lemma tells us that it is in fact locally free of rank N. Now let U be an affine open subset where $\pi_{S*}\mathcal{E}$ is free, so that there is an isomorphism $\mathcal{O}_U^N \cong \pi_{S*}\mathcal{E}|_U$. Pulling this back via $\pi_S|_U$ and composing with the counit map of the pullback-pushforward m adjunction, we have a map

$$\mathcal{O}_{U \times X}^N \xrightarrow{\cong} \pi_S^*\pi_{S*}\mathcal{E}|_U \longrightarrow \mathcal{E}|_U.$$

Since every fibre $\mathcal{E}_s$ is generated by global sections, by Nakayama's lemma applied to every stalk of $S \times X$ we find that $\mathcal{E}$ is generated by global sections too, and since $\pi_{S*}\mathcal{E} = H^0(S \times X, \mathcal{E})^\sim$, it follows the above map is surjective, and in particular determines a morphism into Q, with image in Q^s by assumption, such that $\mathcal{E}|_U$ is equal to the pullback of $\mathscr{E}|_{Q^s}$. $\square$

In particular, if we can use the fact that $\mathscr{E}|_{Q^s}$ is locally versal to build a natural transformation $\eta : \mathcal{V}_{n,d}^s \to \mathrm{Hom}(-, V_{n,d}^s)$, then we automatically get a coarse moduli space by virtue of the fact that $Q^s \to V_{n,d}^s$ is a geometric quotient. So naturally we have:

Theorem 3.2.23. *Let $\mathcal{E}$ be a family over S and let $\{U_\alpha = \mathrm{Spec}\, A_\alpha\}$ be an affine open cover, where each U_α is small enough to determine a morphism to Q^s. Let $\varphi_\alpha : U_\alpha \to Q^s$ denote the morphism determined on U_α. Let $\psi : Q^s \to V_{n,d}^s$ denote the GIT quotient. Then the $\psi \circ \varphi_\alpha$ glue to form a functorial morphism $\eta : S \to V_{n,d}^s$. In particular, $V_{n,d}^s$ is the coarse moduli space for $\mathcal{V}_{n,d}^s$.*

Proof. Let U_α, U_β as above be given, and let $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ denote the pullback of $\mathscr{E}|_U$ via φ_α and φ_β respectively. By assumption, their restriction to the overlap are equivalent, so given an affine open subset $U_{\alpha\beta\gamma} = \mathrm{Spec}\, A_{\alpha\beta\gamma}$ of $U_\alpha \cap U_\beta$, there is some $\mathcal{L}$ on $U_{\alpha\beta\gamma}$ such that $\mathcal{E}_\alpha|_{U_{\alpha\beta\gamma}} \cong \mathcal{E}_\beta|_{U_{\alpha\beta\gamma}} \otimes \pi_S^*\mathcal{L}$. Since the map $H^0(\mathcal{O}_{Q^s \times X}^N) \to H^0(\mathscr{E}|_{Q^s})$ is an isomorphism, this means that identifying $H^0(\mathcal{E}_\alpha|_{U_{\alpha\beta\gamma}})$ with $H^0(\mathcal{O}_{U_{\alpha\beta\gamma} \times X}^N)$, there is some $g^{-1} \in \mathrm{GL}_N(A_{\alpha\beta\gamma})$ that takes $H^0(\mathcal{O}_{U_{\alpha\beta\gamma}}^N) \cong H^0(\mathcal{E}_\alpha|_{U_{\alpha\beta\gamma}})$ to $H^0(\mathcal{E}_\beta|_{U_{\alpha\beta\gamma}})$. Dividing g by its determinant (and replacing $\mathcal{L}$), we may assume $g \in \mathrm{SL}_N(A_{\alpha\beta\gamma})$. But since $\mathcal{E}_\alpha|_{U_{\alpha\beta\gamma}}$ and $\mathcal{E}_\beta|_{U_{\alpha\beta\gamma}} \otimes \pi_S^*\mathcal{L}$ are actually isomorphic, this means that

$$g \cdot \mathcal{E}_\alpha|_{U_{\alpha\beta\gamma}} = \mathcal{E}_\beta|_{U_{\alpha\beta\gamma}} \otimes \pi_S^*\mathcal{L}.$$

But that means $\psi \circ \varphi_\alpha$ and $\psi \circ \varphi_\beta$ agree on $U_{\alpha\beta\gamma}$ (since φ_α and φ_β differ by g), and since the $U_{\alpha\beta\gamma}$ cover the overlap, this means $\psi \circ \varphi_\alpha$ and $\psi \circ \varphi_\beta$ agree, and so we have a well-defined morphism to $V_{n,d}^s$. The fact that it is functorial is easily checked, and the fact that it is a coarse moduli space follows from the fact that ψ is a geometric quotient. $\square$

Part II

Analytic Theory

Chapter 4

The Narasimhan Seshadri Theorem

In Part I, we have contructed the moduli spaces $V_{n,d}^s$ as a projective GIT quotient. In Part II, we will give another naïve moduli space construction for the underlying naïve problem of $\mathcal{V}_{n,0}^s$ in the special case $k = \mathbb{C}$. Specifically, if X is a compact connected Riemann surface of genus g (which may be identified with the $\mathbb{C}$-points of a nonsingular projective curve over $\mathbb{C}$) which we will fix in this chapter, one can identify stable vector bundles of degree zero on X with two other spaces, the basic result being the following:

Theorem 4.0.1 (Narasimhan-Seshadri, 1965). *Let X be a compact Riemann surface of genus g and suppose $g \geq 2$. Then there is a bijection between $V_{n,0}^s$ and irreducible representations $\pi_1(X) \to U(n)$ up to conjugation.*

This is first given as Corollary 1 of [31]. Since then, Donaldson gave a different proof in [8] by studying unitary connections, and using the *Riemann-Hilbert Correspondence* (described in §B.3). His result is stated as follows:

Theorem 4.0.2 (Donaldson, 1983). *An indecomposable holomorphic bundle $\mathcal{E}$ over X with a hermitian metric h is stable if and only if there is a unitary connection on the underlying smooth bundle E giving rise to $\mathcal{E}$ with curvature equal to a constant multiple of the volume form. Such a connection is unique up to isomorphism.*

Remark 4.0.3. Note that the degree is not mentioned in Donaldson's result. However, we will show that a connection satisfying the property in the theorem statement is flat if and only if $\mathcal{E}$ is degree zero.

In this chapter, we will use these bijections to give the space of stable bundles a topological structure, inherited from the space $\mathrm{Hom}(\pi_1(X), U(n))/ \sim$. It turns out that this second structure is homeomorphic to $V_{n,0}^s(\mathbb{C})$, the latter given the usual complex topology, but very unfortunately we will not be proving this.

Of course, since we are moving into analytic territory, some comments are in order. *Open* and *closed* will always mean with respect to the usual complex topology. We will make use of the correspondence between vector bundles and locally free sheaves (of an appropriate structure sheaf) without comment, and we will also use without comment the correspondence between holomorphic vector bundles on X and algebraic vector bundles on the nonsingular curve which defines X.

4.1   Holomorphic Structures on a Smooth Bundle

The goal of this section is to study the space of holomorphic bundles that restrict to a given smooth bundle. Indeed, as we will see, the degree of a holomorphic bundle is actually a smooth invariant, and in fact, along with the rank, completely classifies E! Thus $V_{n,d}(\mathbb{C})$, the isomorphism classes of all holomorphic bundles with signature (n, d), is equal to the set of holomorphic structures on E, up to isomorphism. It turns out that this space is, in turn, canonically identified with the space of unitary connections (to be defined) on E, and this correspondence, known as the *Chern Correspondence*, gives us a tool to turn the study of holomorphic bundles into the study of connections.

4.1.1   The Chern Correspondence

Recall some notation. Let $\Omega_X^{p,q}$ denote the sheaf of smooth (p, q)-forms on X, and let

$$\Omega_E^{p,q} := \Omega_X^{p,q} \otimes E.$$

Note that $\Omega_E^0 := \Omega_E^{0,0}$ is isomorphic to the sheaf of sections of E.

Definition 4.1.1. A *Dolbeault operator* on E is a homomorphism of abelian sheaves (in particular, **NOT** as $\mathcal{O}_X$-modules)

$$\bar{\partial}_E : \Omega_E^0 \to \Omega_E^{0,1}$$

such that for any smooth $f \in C^\infty(U)$ and local section $s \in \Omega_U^0$, we have

$$\bar{\partial}_E(fs) = \bar{\partial}(f) \otimes s + f\bar{\partial}_E(s).$$

Example 4.1.2. Let E be the trivial bundle $E = X \times \mathbb{C}^n$. Then the usual $\bar{\partial}$ operator is a Dolbeault operator.

Example 4.1.3. Let ∇ be any connection on E. Composing ∇ with the projection to its $(0, 1)$-component we obtain a Dolbeault operator. This is often known as the $(0, 1)$-*component of* ∇.

Example 4.1.4. Let $\mathcal{E}$ be a holomorphic bundle whose underlying smooth bundle is E. We may think of $\mathcal{E}$ as E equipped with a collection of distinguished local frames, which we deem to be holomorphic. Then there is a natural Dolbeault operator, known as the *canonical Dolbeault operator*, characterised by $\bar{\partial}_E(s) = 0$ for any homomorphic section s. To see that this is well-defined, let $u : U \to E$ be a smooth section. Covering U with sufficiently small open subsets $\{U_\alpha\}$ we may assume there are local holomorphic frames $\{(s_i)_\alpha\}$. Now we can write $u_\alpha := u|_{U_\alpha} = \sum a_i s_i$ where $s_\alpha = (s_i)_\alpha$ is a holomorphic frame on U_α and $a = (a_i)$ is smooth. Then we see

$$\bar{\partial}_E(u_\alpha) = \bar{\partial}_E(\sum a_i s_i) = \sum \bar{\partial}(a_i) \otimes s_i$$

Repeating on U_β with local holomorphic frame $t_\beta = (t_i)_\beta$ such that $u_\beta = \sum b_i t_i$ we have $\bar{\partial}_E(u_\beta) = \sum \bar{\partial}(b_i) \otimes t_i$. Now if $g = g_{\alpha\beta}$ is the transition map, letting g_{ij} denote the i, j-th entry of g, we observe

$$\sum_j \bar{\partial}(b_j) \otimes t_j = \sum_j \sum_i \bar{\partial}(b_j) \otimes (g_{ij}s_i) = \sum_i \sum_j \bar{\partial}(g_{ij}b_j) \otimes s_i = \sum_i \bar{\partial}(a_i) \otimes s_i$$

as desired (note that since g is holomorphic we have $\bar{\partial}(g) = 0$) and hence by the sheaf axioms this defines $\bar{\partial}_E(u)$ uniquely.

Caution 4.1.5. It is important to note that this Dolbeault operator on E is in fact dependent on the **choice** of holomorphic frames, and not just the isomorphism class of $\mathcal{E}$. Indeed, we will see very soon that there are different Dolbeault operators which can give rise to isomorphic holomorphic bundles, and in fact we will describe exactly when two distinct Dolbeault operators give rise to the same holomorphic bundle.

In fact, the converse of the above example is true for Riemann surfaces: if $\bar{\partial}_E$ is a Dolbeault operator and there exists an open cover U_α with frames s_α such that $\bar{\partial}_E(s_\alpha) = 0$, then it is not hard to show that the transition maps are holomorphic and hence the $\{s_\alpha\}$ define a holomorphic structure on E. Such a Dolbeault operator is said to be *integrable*. It can be shown ([2, p.555]) that every Dolbeault operator on a Riemann surface is integrable. Hence Dolbeault operators parameterise holomorphic structures on E (however, we will soon see that they actually over-parameterise holomorphic structures).

Next we recall the following definition:

Definition 4.1.6. A *Hermitian metric* on E, denoted h is the assignment of a complex inner product $h_p(\cdot, \cdot) : E_p^2 \to E_p$ to every $p \in X$ such that for any smooth sections s, t we have that $p \mapsto h_p(s, t)$ is smooth. A *hermitian bundle* is a vector bundle E equipped with a hermitian metric. A *morphism* of hermitian bundles is a smooth morphism of bundles $\varphi : E \to F$ such that $h_F(\varphi(s), \varphi(t)) = h_E(s, t)$ for any sections s, t if E. A hermitian automorphism is known as a *unitary gauge transformation*, and the group of all such automorphisms is known as the *unitary gauge group*.

A frame $(s_i)_\alpha$ is *unitary* if $h(s_i, s_j) = \delta_{ij}$. A *unitary connection* is a connection ∇ such that

$$dh(s, t) = h(\nabla s, t) + h(s, \nabla t)$$

for any smooth sections s, t. Given a local frame $(s_i)_\alpha$ on U_α, we define the *local matrix of h*, denoted h_α such that $(h_\alpha)_{ij} := h(s_i, s_j)$.

Lemma 4.1.7. *Let ∇ be a connection. Then ∇ is a unitary connection if and only if for any unitary frame $(s_i)_\alpha$, the local 1-form ω_α is skew-Hermitian.*

Proof. Suppose ∇ is unitary. Note that for any i, j, we have

$$0 = dh(s_i, s_j) = h\big(\sum_k (\omega_\alpha)_{ki} s_k, s_j\big) + h\big(s_i, \sum_k \omega_\alpha)_{kj} s_k\big) = (\omega_\alpha)_{ij} + (\overline{\omega}_\alpha)_{ji}$$

and thus $(\omega_\alpha)_{ij} = -(\overline{\omega}_\alpha)_{ji}$ as required.

Conversely, suppose ω_α is skew-Hermitian and let $s = \sum a_i s_i$ and $t = \sum b_i s_i$ be local sections.

Then we can check

$$
\begin{aligned}
h(\nabla(s),t) + h(s,\nabla(t)) &= h(\nabla(\sum a_i s_i), \sum b_i s_i) + h(\sum a_i s_i, \nabla(\sum b_i s_i)) \\
&= h(\sum_i da_i s_i + a_i \sum_j (\omega_\alpha)_{ij} s_j, \sum_i b_i s_i) \\
&\quad + h(\sum_i a_i s_i, \sum_i db_i + b_i \sum_j ((\omega_\alpha)_{ij}, s_j)) \\
&= (\sum_i b_i da_i + a_i db_i) + (\sum_j \sum_i a_i (\omega_\alpha)_{ij} b_j) + (\sum_j \sum_i a_j (\overline{\omega}_\alpha)_{ij} b_i) \\
&= (\sum_i b_i da_i + a_i db_i) + (\sum_j \sum_i a_i (\omega_\alpha)_{ij} b_j) - (\sum_j \sum_i a_j (\omega_\alpha)_{ji} b_i) \\
&= \sum_i b_i da_i + a_i db_i \\
&= dh(s,t)
\end{aligned}
$$

as desired.   □

It can be shown by a partition of unity argument ([51, III Theorem 1.2]) that hermitian metrics exist on any bundle. Similarly, a standard Gram-Schmidt argument will show that smooth unitary frames always exist locally, and finally, we will show that unitary connections always exist:

Theorem 4.1.8 (Chern Correspondence). *Let $\mathcal{E}$ be a holomorphic vector bundle, let $\bar{\partial}_E$ be a Dolbeault operator on E giving rise to $\mathcal{E}$, and h a hermitian metric. Then there is a unique unitary connection ∇ on $\mathcal{E}$ with $(0,1)$ component $\bar{\partial}_E$. Moreover, if $(s_i)_\alpha$ is a local holomorphic frame, then this connection is described by $\omega_\alpha = (\partial h_\alpha) h_\alpha^{-1}$*

Proof. We first prove uniqueness. Suppose ∇ is such a connection, and let $(s_i)_\alpha$ be a holomorphic frame defined on U_α. Then the corresponding matrix of 1-forms ω_α satisfies

$$
\nabla(s_i) = \sum_j (\omega_\alpha)_{ij} s_j.
$$

Observe that since all the s_i are holomorphic, all the entries of ω_α must be of type $(1,0)$. Now we compute:

$$
\begin{aligned}
dh(s_i, s_j) &= h(\nabla s_i, s_j) + h(s_i, \nabla s_j) \\
&= h(\sum_k (\omega_\alpha)_{ik} s_k, s_j) + h(s_i, \sum_k (\omega_\alpha)_{jk} s_k) \\
&= \sum_k (\omega_\alpha)_{ik} h(s_k, s_j) + \overline{(\omega_\alpha)_{jk}} h(s_i, s_k)
\end{aligned}
$$

But $dh(s_i, s_j) = \partial(h_\alpha)_{ij} + \bar{\partial}(h_\alpha)_{ij}$, and thus comparing types we must have $(\partial h_\alpha)_{ij} = \sum_k (\omega_\alpha)_{ik} (h_\alpha)_{kj}$ and letting i, j vary, we observe

$$
\omega_\alpha h_\alpha = \partial h_\alpha
$$

as required. This proves uniqueness.

To prove existence, we define the connection to be $\omega_\alpha := (\partial h_\alpha)h_\alpha^{-1}$ for any holomorphic frame $(s_i)_\alpha$, and extend by the Leibniz rule. By the proof of uniqueness, this satisfies the properties in the theorem, thus we just need to check that this is well-defined. So let $(t_i)_\beta$ be another holomorphic frame, and suppose g satisfies $t_i = \sum g_{ij}s_j$. Then it is not hard to see

$$h_\beta = gh_\alpha g^*$$

where g^* is the conjugate transpose of g. By Proposition B.2.2, it suffices to show that $\omega_\beta = (dg)g^{-1} + g\omega_\alpha g^{-1}$. We compute:

$$\partial h_\beta = \partial(gh_\alpha g^*) = (\partial g)h_\alpha g^* + g(\partial h_\alpha)g^* + gh_\alpha(\partial g^*) = (\partial g)h_\alpha g^* + g(\partial h_\alpha)g^*$$

since g is holomorphic. Thus

$$\omega_\beta = (\partial h_\beta)h_\beta^{-1} = ((\partial g)h_\alpha g^* + g(\partial h_\alpha)g^*)(g^*)^{-1}h_\alpha g^{-1} = (\partial g)g^{-1} + g(\partial h_\alpha)h_\alpha^{-1}g^{-1} = (dg)g^{-1} + g\omega_\alpha g^{-1}$$

as desired. $\square$

Definition 4.1.9. The unitary connection in the above theorem is the *Chern connection*.

Remark 4.1.10. Again, it is important to note that different unitary connections could give rise to the same holomorphic bundle. This is why we will often use the phrase "a Chern connection".

We sum up the above results as follows: as established, there is a 1-1 correspondence between holomorphic structures and Dolbeault operators. Now the natural follow-up question to that is given a Dolbeault operator $\bar\partial_E$, is there a canonical connection we can put on E with (0,1)-component $\bar\partial_E$? The Chern correspondence answers this in the affirmative, with the choice of a hermitian metric placed on E. Thus, in order to study holomorphic structures, we can study unitary connections instead.

Finally, we describe the gauge group action on the space of connections. To begin, we consider the gauge group action on the space of Dolbeault operators. Let u be a gauge transformation, and $\bar\partial_E$ a Dolbeault operator. We define

$$(u \cdot \bar\partial_E)(s) := u\bar\partial_E(u^{-1}(s)).$$

It is easy to check that this is a group action that results in a Dolbeault operator.

Proposition 4.1.11. *Let $\bar\partial_1, \bar\partial_2$ be two Dolbeault operators on E, and $\mathcal{E}_1, \mathcal{E}_2$ their associated holomorphic bundles. Then $\mathcal{E}_1 \cong \mathcal{E}_2$ if and only if there is a gauge transformation u such that $\bar\partial_2 = u \cdot \bar\partial_1$. Moreover, $u : \mathcal{E}_1 \to \mathcal{E}_2$ is one such isomorphism.*

Proof. Suppose firstly that $\bar\partial_2 = u \cdot \bar\partial_1 = u\bar\partial_1 u^{-1}$. Now let s be a holomorphic section of $\mathcal{E}_1$. Observe

$$0 = \bar\partial_1(s) = \bar\partial_1(u^{-1}u(s)) = u\bar\partial_1(u^{-1}u(s)) = \bar\partial_2(u(s))$$

and hence $u(s)$ is a holomorphic section of $\mathcal{E}_2$. By the same argument, if t is a holomorphic section of $\mathcal{E}_2$, then $u^{-1}(t)$ is holomorphic in $\mathcal{E}_1$ as desired.

Conversely, suppose $\mathcal{E}_1 \cong \mathcal{E}_2$ and let $u : \mathcal{E}_1 \to \mathcal{E}_2$ be an isomorphism. Now let $s_\alpha = (s_i)_\alpha$ be a holomorphic frame of $\mathcal{E}_1$; whence by the above calculation we have $u\bar\partial_1(u^{-1}u(s_i)) = 0$. But also $\bar\partial_2(u(s_i)) = 0$ (since $u(s_i)$) is holomorphic). Since $u\bar\partial_1 u^{-1}$ and $\bar\partial_2$ agree on a collection of frames on an open cover of X, they must be equal. $\square$

We now want to extend this to the space of all unitary connections, so let ∇ be a unitary connection with Dolbeault operator $\bar{\partial}_E$. By the Chern correspondence, the Dolbeault operator $u \cdot \bar{\partial}_E$ corresponds to a unique unitary connection; thus we simply need to find a unitary connection with $(0,1)$-component $u \cdot \bar{\partial}_E$. To this end, observe that as in the space of all connections, the space of unitary connections is an affine space. The underlying vector space is the subspace of $H^0(X, \Omega^1_{\mathcal{E}nd(E)})$ consisting of 1-forms with values in a skew-hermitian endomorphism; that is, an endomorphism F such that

$$h(F(s), t) + h(s, F(t)) = 0$$

Observe then, that

$$-(\bar{\partial}_{\mathcal{E}nd\,E} u) u^{-1} + ((\bar{\partial}_{\mathcal{E}nd\,E} u) u^{-1})^*$$

is clearly skew-hermitian, (where $(\bar{\partial}_{\mathcal{E}nd\,E} u)(s) := \bar{\partial}_E(u(s)) - u\bar{\partial}_E(s)$ is the *induced Dolbeault operator*) and u^* satisfies $h(us, t) = h(s, u^* t))$ and has $(0,1)$-component equal to

$$(-(\bar{\partial}_{\mathcal{E}nd\,E} u) u^{-1} + (\bar{\partial}_{\mathcal{E}nd\,E} u) u^{-1})^*)^{0,1} = -(\bar{\partial}_{\mathcal{E}nd\,E} u) u^{-1}$$

since $((\bar{\partial}_{\mathcal{E}nd\,E} u) u^{-1})^*$ is of type (1,0). Hence $u \cdot \nabla$ defined by

$$(u \cdot \nabla)(s) := \nabla(s) - (\bar{\partial}_{\mathcal{E}nd\,E} u) u^{-1}(s) + ((\bar{\partial}_{\mathcal{E}nd\,E} u) u^{-1})^*(s) \tag{4.1}$$

is a unitary connection, and its associated Dolbeault operator is equal to

$$\bar{\partial}_E(s) - (\bar{\partial}_{\mathcal{E}nd\,E} u) u^{-1}(s) = \bar{\partial}_E(s) - \bar{\partial}_E(uu^{-1}(s)) + u\bar{\partial}_E(u^{-1}(s)) = u\bar{\partial}_E(u^{-1}(s))$$

as desired. Hence defining the action of the gauge group by the formula in (4.1) (and it is easy to check that this is indeed a group action) extends the gauge group action on Dolbeault operators, and in particular two unitary connections induce isomorphic holomorphic structures if and only if they lie in the same gauge orbit. In summary, we have the following bijection:

{Holomorphic structures on E}/isomorphism $\leftrightarrow$ {unitary connections on E}/gauge equivalence.

Remark 4.1.12. It is important to note that the above bijection extends to greater generality; more specifically one can define a $(1,2)$-Sobolev norm on the space of p-forms, and define a completion of this space. The closure of the affine space of unitary connections is the space of $W^{1,2}$-unitary connections, and one can show that the Chern correspondence extends to $W^{1,2}$-connections ([2, Lemma 14.8]). This will become relevant in the next section

4.1.2 Degree of a Smooth Bundle

In this section, we will give a complete classification of smooth vector bundles on X. In particular, we will show that the degree of a bundle is actually a **smooth** invariant, and in fact, along with the rank, the only smooth invariant there is! Thus for each signature $(n, d) \in \mathbb{N} \times \mathbb{Z}$, there is a unique smooth bundle with that signature. The vehicle for showing this is the invariant known as the *first Chern class*. To define it, we fix a smooth bundle E. We begin with a result:

Lemma 4.1.13. *Let ∇_1, ∇_2 be connections on E with curvature forms Θ_1, Θ_2 respectively. Then $\operatorname{tr}\Theta_1$ and $\operatorname{tr}\Theta_2$ are cohomologous.*

Proof. Since X is a Riemann surface, clearly $\operatorname{tr}\Theta_i$, being a two-form, is closed, thus the statement makes sense. Now $\nabla_1 - \nabla_2$ is a global $\mathcal{E}\mathrm{nd}(E)$-valued 1-form; call it A. Let ω_1, ω_2 be respective local 1-forms of ∇_1 and ∇_2 on a local frame. Then locally, we have

$$\Theta_2 = d\omega_2 + \omega_2 \wedge \omega_2 = d(\omega_1 + A) + (\omega_1 + A) \wedge (\omega_1 + A) = \Theta_1 + dA + A \wedge A + \omega_1 \wedge A + A \wedge \omega_1$$

hence

$$\operatorname{tr}(\Theta_2) = \operatorname{tr}(\Theta_1) + \operatorname{tr}(dA) + \operatorname{tr}(A \wedge A) + \operatorname{tr}(\omega_1 \wedge A + A \wedge \omega_1).$$

It is not hard to show that $\operatorname{tr}(A \wedge A)$ and $\operatorname{tr}(\omega_1 \wedge A + A \wedge \omega_1)$ are both zero, from the antisymmetry of the wedge product. Since A is a global form, the dA glue to a global exact 2-form and hence the result follows. $\square$

Definition 4.1.14. The *first Chern class* of E is defined to be the cohomology class

$$c_1(E) := \left[\operatorname{tr}(\frac{i}{2\pi}\Theta)\right] \in H^2_{\mathrm{DR}}(X)$$

where Θ is the curvature form for any connection. By the above lemma, this does not depend on the connection.

The key theorem we will be proving in this section is the following:

Theorem 4.1.15. *For any holomorphic bundle $\mathcal{E}$ with underlying smooth bundle E, we have*

$$\int_X c_1(E) = \deg(\mathcal{E}),$$

where X is given the standard orientation $i\,dz \wedge d\bar{z}$ for any holomorphic coordinate z.

Let us take a moment to appreciate this result. We defined the degree of a line bundle $\mathcal{L}$ on a curve X over an algebraically closed field k to be the degree of its corresponding divisor class, and we defined the degree of a vector bundle $\mathcal{E}$ to be the degree of $\det(\mathcal{E})$. Note that this is purely algebraic. The theorem states that in the case $k = \mathbb{C}$, where we have access to analytic tools, when we equip $\mathcal{E}$ with a connection (any connection, in fact), take its curvature, take the trace of the curvature, multiply by $i/2\pi$ and integrate it, we get, not only an integer, but the same integer representing the degree of the divisor associated to its determinant bundle!

The way to prove this is to reduce to the case of line bundles, and to do this, we present the following well-known result:

Theorem 4.1.16 (Structure Theorem of Smooth Vector Bundles)**.** *There is a diffeomorphism*

$$E \cong \det E \oplus \mathcal{O}_X^{\mathrm{rk}\,E-1}.$$

Before we give the proof, we first recall it can be shown ([5, II, Theorem 15.3]) that every section s has a section s' which intersects s transversally (i.e. if s and s' intersect at $P \in X$, then $\operatorname{im} s_{*,P} \oplus \operatorname{im} s'_{*,P} = T_{s(P)}E$; in other words the images of the differential of s, s' at P generate the tangent space of $s(P)$ in E).

Proof. We proceed by induction on the rank, with the rank 1 case being trivial. Now supposing $\operatorname{rk} E > 1$, we observe that by the above there is a section s which intersects the zero section transversally. However, the real dimension of E is $\dim_{\mathbb{R}} E = 2 + 2 \operatorname{rk} E > 4$ and thus s and the zero section cannot intersect at all (otherwise the space spanned by the images of their differential is at most 4), or in other words s is nonvanishing. In particular, the bundle spanned by s, which is a trivial line bundle, is a subbundle of E, and thus we may write $E = \mathcal{O}_X \oplus E'$ for some complement E' (for example, placing a hermitian metric h on E and taking the orthogonal complement of s with respect to h), and by the inductive hypothesis the result follows. $\qquad\square$

Before we proceed, we review the Snake Lemma; in particular how the "snake" is constructed. We recall the statement:

Proposition 4.1.17 (Snake Lemma)**.** *Suppose we have the following diagram of A-modules with exact rows:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A^0 & \longrightarrow & B^0 & \longrightarrow & C^0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^A} & & \downarrow{\scriptstyle d^B} & & \downarrow{\scriptstyle d^C} & & \\
0 & \longrightarrow & A^1 & \longrightarrow & B^1 & \xrightarrow{\ f\ } & C^1 & \longrightarrow & 0
\end{array}
$$

Then there is a map $\delta : \ker d^C \to \operatorname{coker} d^A$ such that the following sequence is exact:

$$
0 \longrightarrow \ker d^A \longrightarrow \ker d^B \longrightarrow \ker d^C \rceil
$$

$$
\lfloor\to \operatorname{coker} d^A \to \operatorname{coker} d^B \to \operatorname{coker} d^C \to 0
$$

The δ above is constructed as follows: Take $c \in \ker d^C$. Since the top row is exact, there exists some $b \in B^0$ that maps to c. Now observe that since $d^C(c) = 0$, it must follow that the image of $d^B(b) \in \ker f$, by the commutativity of the diagram. Since the bottom row is exact, this pulls back to some unique $a \in A^1$, and moreover it can be shown that the image of a in $\operatorname{coker} d^A$ does not depend on our choice of b, and in particular it is well-defined. Hence we define $\delta(c) := b$, and one can show that the resulting sequence is exact.

Now we continue our investigation. We clearly have the short exact sequence of abelian groups:

$$
0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\ \exp\ } \mathbb{C}^* \longrightarrow 0
$$

where $\exp$ above is the map $x \mapsto \exp(2\pi i x)$. Now let $\mathscr{P}$ denote the property of smoothness or holomorphicity (in particular, statements made about $\mathscr{P}$ will be valid in both the smooth and holomorphic

settings). Taking the sheaf of $\mathscr{P}$-functions with values in the above groups, we have the following short exact sequence of sheaves, known as the *exponential sheaf sequence*:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0 \tag{4.2}$$

where $\mathcal{O}_X$ is the sheaf of $\mathscr{P}$-functions on X, and $\underline{\mathbb{Z}}$ is the constant sheaf $\mathbb{Z}$. Recall that $H^i(X, \underline{\mathbb{Z}}) \cong H^i_{\text{sing}}(X, \mathbb{Z})$ ([13, pp. 42-43]). Now taking cohomology of (4.2), we have:

$$0 \rightarrow H^0(X, \mathbb{Z}) \rightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X^*)$$
$$\tag{4.3}$$
$$\hookrightarrow H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X^*)\dots$$

and in particular, we have a map $\delta : H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z})$. Now recall that $H^1(X, \mathcal{O}_X^*)$ is the Picard group of X; in particular it parameterises the isomorphism classes of $\mathscr{P}$-line bundles on X.

Proposition 4.1.18. *Under the inclusion $H^2(X, \mathbb{Z}) \subseteq H^2(X, \mathbb{R}) \cong H^2_{\text{DR}}(X)$, we have*

$$\delta(L) = -c_1(L)$$

for any $\mathscr{P}$-line bundle L

Proof. This follows the proof given in [13, pp. 141-142]. Let $\{U_\alpha\}$ be a sufficiently fine open cover (in particular, one where we can take local logarithms on overlaps) and let $\{g_{\alpha\beta}\}$ be a Čech cocycle L. We define local inverses

$$h_{\alpha\beta} := \frac{1}{2\pi i} \log g_{\alpha\beta}$$

and by the construction of the snake map of the Snake Lemma, it follows that $\{h_{\alpha\beta} - h_{\alpha\gamma} + h_{\beta\gamma}\}$ is a 2-cocycle of $\underline{\mathbb{Z}}$ representing $\delta(L)$.

Next we look at $c_1(L)$. Fix a connection and let $\{\omega_\alpha\}$ be the associated 1-forms. By Proposition B.2.2, we have

$$\omega_\beta = (dg_{\alpha\beta})g_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\alpha g_{\alpha\beta}^{-1} = g_{\alpha\beta}dg_{\alpha\beta} + \omega_\alpha$$

hence

$$\omega_\beta - \omega_\alpha = g_{\alpha\beta}^{-1}dg_{\alpha\beta} = d\log g_{\alpha\beta}$$

and in particular $d\omega_\alpha = d\omega_\beta$. Now we compute the curvature. Since we are working with a line bundle, ω is just a usual 1-form; it follows $\omega \wedge \omega = 0$, hence

$$\Theta = d\omega_\alpha = d\omega_\beta.$$

Finally, we reconcile de Rham and Čech cohomology. Note that by the Poincare Lemma we have the following short exact sequences of sheaves (recall exactness of sheaves is measured locally):

$$0 \rightarrow \underline{\mathbb{R}} \rightarrow \Omega_X^0 \rightarrow \mathcal{Z}^1 \rightarrow 0 \tag{4.4}$$

and

$$0 \to \mathcal{Z}^1 \to \Omega_X^1 \to \mathcal{Z}^2 \to 0 \tag{4.5}$$

where $\mathcal{Z}^p$ denotes the sheaf of closed p-forms on X. Now it can be shown [13, p. 42] that Ω_X^p is acyclic, hence we have

$$H^0(X, \mathcal{Z}^2)/H^0(X, \Omega_X^1) \cong H^1(X, \mathcal{Z}^1) \tag{4.6}$$

by the long exact sequence of (4.5) and

$$H^1(X, \mathcal{Z}^1) \cong H^2(X, \mathbb{R}) \tag{4.7}$$

by the long exact sequence of (4.4). Using (4.6), and the construction of the snake map, the image of Θ in $H^1(X, \mathcal{Z})$ is represented by the Čech 1-cocycle $\{i(\omega_\beta - \omega_\alpha)/2\pi\}$, and using (4.7), the image of Θ in $H^2(X, \mathbb{R})$ is

$$\{\frac{i}{2\pi}(\log g_{\alpha\beta} - \log g_{\alpha\gamma} + \log g_{\beta\gamma})\} = -\{h_{\alpha\beta} - h_{\alpha\gamma} + h_{\beta\gamma}\} = -\delta(L)$$

as desired. $\square$

Corollary 4.1.19. *In the smooth category, or in the holomorphic category with $g = 0$, every line bundle is uniquely determined by its first Chern class.*

Proof. In both cases $\mathcal{O}_X$ is acyclic. $\square$

Next we prove Theorem 4.1.15 for line bundles:

Theorem 4.1.20. *If $\mathcal{L}$ is a holomorphic line bundle with underlying smooth bundle L, then we have:*

$$\int_X c_1(L) = \deg \mathcal{L}. \tag{4.8}$$

Proof. We first prove the case where $\mathcal{L}$ is the line bundle of a prime divisor. Let D be a prime divisor, supported on $P \in X$, suppose $\mathcal{L} = \mathcal{L}(D)$. Since D is effective, by Proposition A.1.6, it is the divisor of zeroes of some global section $s \in H^0(X, \mathcal{L})$ that vanishes exactly once, at P. In particular, s is a frame on $X \smallsetminus \{P\}$. Now fix a hermitian metric h. By the Chern correspondence, a Chern connection of $(\mathcal{E}, h)$ is given locally by $h_\alpha^{-1}\partial h_\alpha$ with respect to a holomorphic frame, and the curvature by $dh_\alpha^{-1}\partial h_\alpha = d\partial \log h_\alpha$ (here we may take logarithms because $h_\alpha > 0$). Now for an $\varepsilon > 0$, write U_ε for the open set $\{p \in X \mid h(s(p), s(p)) > \varepsilon\}$ and $\overline{U}_\varepsilon$ for its closure, define s_ε to be the frame $s|_{U_\varepsilon}$ and finally write $h_\varepsilon := h(s_\varepsilon, s_\varepsilon) > 0$. We compute:

$$\int_X c_1(L) = \lim_{\varepsilon \to 0} \int_{\overline{U}_\varepsilon} c_1(L) = \lim_{\varepsilon \to 0} \int_{\overline{U}_\varepsilon} \frac{i}{2\pi} d\partial \log h_\varepsilon = \lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon} \frac{i}{2\pi} \partial \log h_\varepsilon$$

where the final equality follows from Stoke's theorem. Now since $s|_{X \smallsetminus \overline{U}_\varepsilon}$ vanishes exactly at P, we may pick a holomorphic coordinate chart centred at P such that $s = z$. Hence

$$\int_{\partial U_\varepsilon} \partial \log h_\varepsilon = - \int_{|z|=\varepsilon} \partial(\log z + \log \bar{z} + \log h_z(1,1)) \tag{4.9}$$

integrating anticlockwise as usual. The negative sign is there due to our choice of orientation. Now observe that since $\log h_z(1,1)$ is smooth, it follows $\partial \log(h_z(1,1))/\partial z$ is continuous, and since $\{|z| \leq \delta\}$ is compact, for some sufficiently small δ, it follows $\partial \log(h_z(1,1))/\partial z$ is bounded on $\{|z| \leq \delta\}$. Thus

$$\lim_{\varepsilon \to 0} \left| \int_{|z|=\varepsilon} \partial \log h(1,1) \right| = \lim_{\varepsilon \to 0} \left| \int_{|z|=\varepsilon} \frac{\partial \log h_z(1,1)}{\partial z} dz \right| \leq \lim_{\varepsilon \to 0} 2\pi\varepsilon \sup\left\{ \left| \frac{\partial \log h_z(1,1)}{\partial z} \right| \mid |z| \leq \delta \right\} = 0.$$

Finally, we have, by Cauchy's integration formula,

$$\int_{|z|=\varepsilon} \partial(\log z + \log \bar{z}) = \int_{|z|=\varepsilon} \frac{dz}{z} = 2\pi i$$

as desired.

Finally, we observe that $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, and by Proposition 4.1.18, it follows

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_2(L_2)$$

Hence if $D = \sum n_i P_i$ is any divisor and $\mathcal{L} = \mathcal{L}(D)$, it follows $\mathcal{L} = \otimes \mathcal{L}(P_i)^{\otimes n_i}$ hence

$$\int_X c_1(L) = \int_X c_1(\otimes \mathcal{L}(P_i)^{\otimes n_i}) = \int_X \sum n_i c_1(\mathcal{L}(P_i)) = \sum n_i = \deg \mathcal{L}$$

as desired. $\qquad \square$

Example 4.1.21. Suppose $X = \mathbb{P}^1$. We will compute the Chern class of $\mathcal{O}_X(n)$. Suppose the homogeneous coordinates on X are $[x_0 : x_1]$, and let $U_0 = \{x_0 \neq 0\}$ and similarly with U_1. We define $z_0 := x_1/x_0$ to be the affine coordinate on $U_0 \cong \mathbb{A}^1$ and similarly for z_1. Now by definition of $\mathcal{O}_X(n)$, we may find frames s_0, s_1 on the corresponding affine patches such that

$$g_{0,1} := \frac{s_1}{s_0} = z_0^n = z_1^{-n}$$

We define the hermitian metric on $\mathcal{O}_X(n)$ to be

$$h_0 := h(s_0, s_0) = (1 + |z_0|^2)^{-n}$$

on U_0, and

$$h_1 := h(s_1, s_1) = (1 + |z_1|^2)^{-n}.$$

Of course, we need to check this is well-defined, that is

$$h_0 = h(s_0, s_0) = h(z_1^n s_1, z_1^n s_1) = |z_1|^{2n} h_1.$$

To this end observe

$$h_0 = (1 + |z_0|^2)^{-n} = \left(1 + \frac{1}{|z_1|^2}\right)^{-n} = \left(\frac{|z_1|^2 + 1}{|z_1|^2}\right)^{-n} = |z_1|^{2n}(|z_1|^2 + 1)^{-n} = |z_1|^{2n} h_1$$

as claimed. Hence for brevity, we will simply write

$$h = (1 + |z|^2)^{-n}$$

understanding that this works for any affine patch, and finally observe that both affine patches have complement measure zero, and thus we may ultimately just work on one affine patch.

By the Chern correspondence, the unitary connection ω is given by

$$\omega = h^{-1}\partial h = -\frac{n\bar{z}}{1 + |z|^2}dz$$

and now computing the curvature Θ:

$$\Theta = d\omega = \frac{-n}{(1 + |z|^2)^2}d\bar{z} \wedge dz = \frac{n}{(1 + |z|^2)^2}dz \wedge d\bar{z}$$

and hence

$$c_1(\mathcal{O}_X(n)) = \frac{ni}{2\pi(1 + |z|^2)^2}dz \wedge d\bar{z}$$

Finally, we integrate. Write $z = x + iy$ and we interpret the real coordinates x, y as the standard coordinates of $\mathbb{R}^2$. Then $idz \wedge d\bar{z} = 2dx \wedge dy$, and letting r, θ denote the polar coordinates, we deduce

$$\int_X c_1(\mathcal{O}_X(n)) = \int_U \frac{n}{\pi(1 + |z|^2)^2}dx \wedge dy = \int_0^\infty \int_0^{2\pi} \frac{nr}{\pi(1 + r^2)^2}d\theta dr = n$$

as desired.

Proof of Theorem 4.1.15. By the structure theorem we know $E = \det E \oplus \mathcal{O}_X^{\mathrm{rk}\,E-1}$. Then given connections on $\det E$ and $\mathcal{O}_X^{rkE-1}$, we can build a connection on E with a diagonal matrix of associated 1-forms. More precisely, if ∇ is a connection on $\det E$ with local 1-forms $\{\omega_\alpha\}$, then $\{\mathrm{diag}(\omega_\alpha, 0, ..., 0)\}$ is the matrix of 1-forms for a connection on $\det E \oplus \mathcal{O}_X^{\mathrm{rk}\,E-1} = E$. Thus

$$\int_X c_1(E) = \int_X c_1(\det E) = \deg(E)$$

as desired. $\square$

Corollary 4.1.22. *For each $(n, d) \in \mathbb{N} \times \mathbb{Z} \cong \mathbb{N} \times H^2(X, \mathbb{Z})$, there exists a unique smooth bundle E over X with signature (n, d).*

To summarise, we have given an interpretation of $V_{n,d}$, the set of isomorphism classes of holomorphic vector bundles of signature (n, d) as the set of holomorphic structures on the unique smooth bundle E of signature (n, d). Our earlier work on the Chern correspondence in turn describes this set as equal to the affine space of unitary connections on E modulo gauge equivalence. Now the next question is, where does stability fit in all of this?

4.2 An Overview of Donaldson's Proof

In this section, we conduct an exposition of Donaldson's paper [8], which interprets stability of a given holomorphic bundle $\mathcal{E}$ in terms of the type of Chern connection on the underlying smooth bundle E that gives rise to $\mathcal{E}$. This builds on earlier work by Atiyah and Bott in [2], and provides a short proof of the theorem of Narasimhan and Seshadri, as we will see shortly, and it is this correspondence which will allow us to topologise our moduli space.

To begin, observe that since X is a compact Riemann surface, it is Kähler, and we make the further assumption that the volume of X is 1 (that is, we fix a volume form such that $\int_X \text{vol} = 1$). The result is the following:

Theorem 4.2.1 (Donaldson-Narasimhan-Seshadri)**.** *Let $\mathcal{E}$ be a indecomposable holomorphic bundle. Then $\mathcal{E}$ is stable if and only if there is some Chern connection ∇ on E giving rise to $\mathcal{E}$ with curvature $\Theta \in H^0(\Omega_X^2) \otimes \text{End}\, E$ satisfying*

$$\Theta = -2\pi i \mu \,\text{vol} \otimes \text{id}_E \tag{4.10}$$

Moreover, ∇ is unique up to the action of the unitary gauge group.

Note that if $\deg \mathcal{E} = 0$, this means Θ will be flat.

Example 4.2.2. Of course, over $\mathbb{P}^1$ the only stable bundles are line bundles. So let $\mathcal{O}_X(n)$ be a line bundle. In Example 4.1.21, we defined a hermitian metric and computed the Chern class, Chern connection and curvature. Now we will need to compute the volume form. Of course, X is obviously Kähler with its Fubini-Study metric (which can be realised as the metric of $\mathcal{T}_X = \mathcal{O}_X(2)$ or its dual $\mathcal{O}_X(-2) = \Omega_X^{1,0}$ described in Example 4.1.21), and so by taking the real part of this complex inner product, we have a natural Riemannian structure. Locally, if we pick an affine patch with holomorphic coordinate $z = x + iy$, and frame $(\partial x, \partial y)$ of T_X (the real smooth tangent space) the metric is given by

$$g = \begin{pmatrix} \frac{1}{\sqrt{\pi}(1+x^2+y^2)^2} & 0 \\ 0 & \frac{1}{\sqrt{\pi}(1+x^2+y^2)^2} \end{pmatrix}$$

The $\sqrt{\pi}$ is there so that the resulting volume is 1. An orthonormal frame is given by $(\sqrt{\pi}(1 + x^2 + y^2)\partial x, \sqrt{\pi}(1 + x^2 + y^2)\partial y)$, and hence the volume form is

$$\text{vol} = \frac{dx \wedge dy}{\pi(1 + x^2 + y^2)^2} = \frac{i dz \wedge d\bar{z}}{2\pi(1 + |z|^2)^2}.$$

Now we computed the curvature of the Chern connection on $\mathcal{O}_X(n)$ to be

$$\Theta = \frac{n}{(1 + |z|^2)^2} dz \wedge d\bar{z} = -2\pi i \deg(\mathcal{O}_X(n)) \,\text{vol}$$

as expected. Hence the theorem is verified for $\mathbb{P}^1$.

In fact, we will first prove the theorem for line bundles in general:

Theorem 4.2.3. *The Donaldson-Narasimhan-Seshadri theorem is true for line bundles.*

Proof. Of course, if $\mathcal{L}$ is a line bundle with hermitian metric h, underlying smooth bundle L and Chern connection ∇, then it is already stable. Thus we reduce to showing that a connection ∇' in the orbit of ∇ with curvature in the form (4.10) exists.

To this end, we observe that the curvature Θ of ∇ is just an imaginary global (1,1)-form (since $\mathcal{E}\mathrm{nd}\,\mathcal{L}$ is trivial; the identity endomorphism is a global frame), so $i\Theta$ differs from its harmonic representative $i\Theta_0$ by a real exact 1-form, say $i\Theta - d\eta = i\Theta_0$. Now observe that since Θ_0 is harmonic, $d \star \Theta_0 = 0$, and so $\star \Theta_0$ is a constant; necessarily equal to $-2\pi i \mu$. So we reduce once again to showing that there is a gauge transformation g such that Θ_0 is the curvature of $g \cdot \nabla$.

Observe that $d\eta$ is real and closed and therefore the following Poisson equation has a real solution ([3, Theorem 4.7]):
$$2\bar{\partial}\partial f = \Delta f = i d\eta.$$

Now write $g := \exp f$, let $\nabla' = g \cdot \nabla$, and write Θ' for the curvature of ∇'. Firstly, observe that since $\mathcal{E}\mathrm{nd}\,\mathcal{L}$ is trivial, the operators induced by the connection, $\partial_{\mathcal{E}\mathrm{nd}\,\mathcal{L}}$ and $\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{L}}$, are just the usual ∂ and $\bar{\partial}$ operators. Hence

$$\Theta' = \Theta - d(\bar{\partial}g)g^{-1} + \overline{d(\bar{\partial}g)g^{-1}} = \Theta - \partial\bar{\partial}f + \bar{\partial}\partial\bar{f} = \Theta + 2\bar{\partial}\partial f = \Theta + i d\eta = \Theta_0$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Observe that the condition (4.10) is a little awkward to work with, so we introduce the *Donaldson J-functional* on the space of $W^{1,2}$-unitary connections (the $W^{1,2}$-condition is required to use Uhlenbeck's compactness theorem, which will be stated below), which satisfies the property that $J(\nabla) = 0$ if and only if ∇ satisfies (4.10). It is defined as follows: Firstly recall that the *trace norm* (which despite its name, is not a norm in general) of a square matrix $M \in \mathbb{C}^{r \times r}$ is defined to be

$$\nu(M) := \mathrm{tr}((MM^*)^{\frac{1}{2}}),$$

where $(MM^*)^{\frac{1}{2}}$ is the unique positive semidefinite matrix B such that $B^2 = MM^*$, which exists since MM^* is hermitian (and hence diagonalisable) and positive semidefinite. In fact, if M is diagonalisable, it is easy to see that

$$\nu(M) = \sum |\lambda_i|,$$

where the sum is taken across all eigenvalues of M, counting multiplicity. The key property is the following:

Lemma 4.2.4. *For any hermitian matrix M, we have*

$$\nu(M) = \sup_{\{s_i\}} \sum_{i=1}^{n} |\langle Ms_i, s_i \rangle|,$$

where the supremum is taken across all unitary bases $\{s_i\}$ of $\mathbb{C}^n$.

Proof. We first observe that M has a unitary basis of eigenvectors, say $\{v_i\}$ and letting $\{s_i\} = \{v_i\}$ we deduce
$$\nu(M) = \sum |\lambda_i| = \sum_{i=1}^{n} |\langle Mv_i, v_i \rangle| \leq \sup_{\{s_i\}} \sum_{i=1}^{n} |\langle Ms_i, s_i \rangle|.$$

For the reverse inequality, let $\{s_i\}$ be a unitary basis, and let $(g_{ij}) \in U(n)$ denote the matrix taking $\{v_i\}$ to $\{s_i\}$; that is, $s_i = \sum g_{ij}v_j$. We compute:

$$
\begin{aligned}
\sum_{i=1}^n |\langle Ms_i, s_i \rangle| &= \sum_{i=1}^n |\langle M \sum_{j=1}^n g_{ij}v_j, \sum_{k=1}^n g_{ik}v_k \rangle| \\
&= \sum_{i=1}^n |\sum_{j=1}^n g_{ij}\langle Mv_j, \sum_{k=1}^n g_{ik}v_k \rangle| \\
&= \sum_{i=1}^n |\sum_{j=1}^n g_{ij}\langle Mv_j, g_{ij}v_j \rangle| \\
&= \sum_{i=1}^n |\sum_{j=1}^n \lambda_j \langle g_{ij}v_j, g_{ij}v_j \rangle| \\
&\leq \sum_{i=1}^n \sum_{j=1}^n |\lambda_j||\langle g_{ij}v_j, g_{ij}v_j \rangle| \\
&= \sum_{i=1}^n \sum_{j=1}^n |\lambda_j||g_{ij}|^2 = \sum_{j=1}^n |\lambda_j|
\end{aligned}
$$

as desired. $\square$

Of course, this in itself is not particularly interesting or useful, but it does give us two very important corollaries:

Corollary 4.2.5. *Let $H(n)$ denote the vector space of hermitian n-by-n matrices.*

(i) *ν is a norm on $H(n)$.*

(ii) *If $M \in H(n)$ can be written in the form*

$$
M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},
$$

then $\nu(M) \geq |\operatorname{tr} A| + |\operatorname{tr} C|$.

Proof. To prove (i), we need only check the triangle inequality. So suppose $M, N \in H(n)$ are given. Then

$$
\nu(M + N) = \sup_{\{e_i\}} \sum |\langle (M+N)e_i, e_i \rangle| =\leq \sup_{\{e_i\}} \sum |\langle Me_i, e_i \rangle| + |\langle Ne_i, e_i \rangle| \leq \nu(M) + \nu(N)
$$

as desired. To prove (ii), let $\{e_i\}$ denote the standard basis of $\mathbb{C}^n$. Then

$$
\nu(M) \geq \sum_{i=1}^n |\langle Me_i, e_i \rangle| \geq |\sum_{i=1}^{\operatorname{rk} A} \langle Me_i, e_i \rangle| + |\sum_{i=\operatorname{rk} A+1}^n \langle Me_i, e_i \rangle| = |\operatorname{tr} A| + |\operatorname{tr} C|
$$

as desired. $\square$

With this in mind, we define the *N-norm* on the space of self-adjoint smooth endomorphisms of E, as

$$N(s) := \left(\int_X \nu^2(s) \operatorname{vol} \right)^{\frac{1}{2}}.$$

By the above corollary, this is a norm.

Now let ∇ be a unitary $W^{1,2}$-connection with curvature $\Theta \in H^0(\Omega^2_{\mathcal{E}\mathrm{nd}\, E})$. Since the matrix of a unitary connection with respect to a unitary frame is skew-hermitian, its curvature Θ is also skew-hermitian, and since the volume form is real, it follows that $\star\Theta$ is also skew-hermitian. In particular, it follows that $\frac{\star\Theta}{2\pi i}$ is actually **hermitian**. Thus we define the *Donaldson J-functional* as

$$J(\nabla) := N(\frac{\star\Theta}{2\pi i} + \operatorname{diag}(\mu)) = \left(\int_X \nu^2 \left(\frac{\star\Theta}{2\pi i} + \operatorname{diag}(\mu) \right) \operatorname{vol} \right)^{\frac{1}{2}},$$

where $\nu^2(s) := (\nu(s))^2$. Observe that $J = 0$ if and only if ∇ is a unitary connection of the type we want (known as, *projectively flat*, or *Yang-Mills connections*). Thus we have turned our problem into one of finding zeroes of J.

The rough idea of Donaldson's proof is as follows: let $\mathcal{E}$ be an indecomposable bundle of signature (n, d). We fix a reference Chern connection ∇_0 of $\mathcal{E}$, and use gauge transformations to find our desired ∇. Denote the $W^{2,2}$-gauge orbit of ∇_0 by O_{∇_0}. We will show that if $\mathcal{E}$ is stable, then the infimum of $J(O_{\nabla_0})$ is attained; that is there is some $\nabla \in O_{\nabla_0}$ such that $J(\nabla) = \inf J(O_{\nabla_0})$. One then deduces that the infimum must be zero, by looking near ∇. In order to deduce that the infimum is attained, we take a minimising sequence (that is, a sequence ∇_i such that $J(\nabla_i) \to \inf J(O_{\nabla_0})$) in O_{∇_0} and extract, using Uhlenbeck's weak compactness theorem (to be stated), a weakly convergent subsequence that converges to some ∇_∞. Now ∇_∞ defines a holomorphic bundle, say $\mathcal{F}$, and the key property is that $\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$. So we take a nonzero $\varphi : \mathcal{E} \to \mathcal{F}$, and apply Proposition A.1.17 to get a factorisation of φ through two exact rows, and apply estimates to these rows to deduce that $\mathcal{E}$ is stable if and only if $\mathcal{E} \cong \mathcal{F}$. The converse (that if there is some connection annihilating J then $\mathcal{E}$ is stable) also follows from these estimates.

Now we begin with a statement of Uhlenbeck's compactness theorem:

Theorem 4.2.6 (Uhlenbeck's weak compactness). *Let (∇_i) be a sequence of $W^{1,2}$-connections with curvatures (Θ_i), and suppose the sequence $(\|\Theta_i\|_{L^2} := \int_X \operatorname{tr}(\Theta_i) \wedge \operatorname{tr}(\star\overline{\Theta_i}))$ is bounded. Then there is a sequence of $W^{2,2}$-gauge transformations (g_i) and a subsequence (∇_{i_k}) such that $(g_{i_k} \cdot \nabla_{i_k})$ weakly converges to some ∇_∞ (that is, $\int_X \operatorname{tr}(g_{i_k} \cdot \Theta_{i_k}) \wedge \operatorname{tr}(\star A) \to \int_X \operatorname{tr}(\Theta_\infty) \wedge \operatorname{tr}(\star A)$ for all $W^{1,2}$-connections A).*

Proof. [47, p. 41].   □

Let (∇_i) be a sequence in O_{∇_0} with curvatures (Θ_i), such that $J(\nabla_i) \to \inf J(\mathcal{O}_{\nabla_0})$. In order to use the theorem, we need to check that $\|\Theta_i\|_{L^2}$ is bounded. To this end, we first observe that $N(\star\Theta_i)$ is bounded, since $J(\nabla_i)$ is and N is a norm. Note that

$$\nu^2(\star\Theta_i) \operatorname{vol} = \operatorname{tr}(\sqrt{(\star\Theta_i)(\star\Theta_i)^*})^2 \operatorname{vol},$$

and similarly,

$$\text{tr}(\Theta_i) \wedge \star \text{tr}(\overline{\Theta_i}) = \text{tr}(\star\Theta)(\star\Theta)^* \text{vol}.$$

Since all norms are equivalent in finite dimensions, it follows that there is some $m, M > 0$ such that for any matrix A we have

$$m \, \text{tr}(AA^*) \leq \text{tr} \sqrt{AA^*}^2 \leq M \, \text{tr}(AA^*),$$

thus since $\{N(\star\Theta_i)\}$ is bounded, it follows that $\{\|\Theta_i\|_{L^2}\}$ is also bounded. Thus (replacing ∇_i with $g \cdot \nabla_i$, and replacing the sequence with a weakly convergent subsequence) we may assume without loss of generality ∇_i converges weakly to some ∇_∞, which is a unitary connection and hence defines a holomorphic bundle, say $\mathcal{F}$ with signature (n, d).

Proposition 4.2.7. *Let $\mathcal{E}, \mathcal{F}$ be as above.*

(i) *Then $J(\nabla_\infty) \leq \inf J(O_{\nabla_0})$.*

(ii) *The group $\text{Hom}(\mathcal{E}, \mathcal{F})$ is nonzero.*

Proof Sketch. To prove (i), we first observe that for any $\varepsilon > 0$ the set $C_\varepsilon = \{\alpha \in \text{End} \, E \mid N(\alpha + \text{diag}(\mu)) < J(\nabla_\infty) - \varepsilon\}$ is convex and closed, and thus by the Hahn-Banach separation theorem, we can separate $\frac{\star\Theta}{2\pi i}$ from C_ε by a hyperplane. Now if

$$J(\nabla_\infty) > \inf J(O_{\nabla_0}) = \liminf_{n \to \infty} J(\nabla_i),$$

then picking some ε_0 such that $J(\nabla_\infty) - \varepsilon_0 > \inf J(O_{\nabla_0})$, we find that infinitely many $\frac{\star\Theta_i}{2\pi i}$ lie in C_{ε_0}. But that means Θ_i cannot converge weakly to Θ_∞ in L^2, and since the curvature is a bounded linear operator, it follows that weak convergence is preserved, and thus we have a contradiction. This proves (i).

To prove (ii), we first observe that an element of $\text{Hom}(\mathcal{E}, \mathcal{F})$ is just a global section of $\mathcal{H}om(\mathcal{E}, \mathcal{F}) = \mathcal{E}^\vee \otimes \mathcal{F}$. Now the underlying smooth bundle of $\mathcal{E}^\vee \otimes \mathcal{F}$ is just $\mathcal{E}nd(E) = E^\vee \otimes E$, and it is easy to see that given any Dolbeault operators $\bar\partial_\mathcal{E}$ and $\bar\partial_\mathcal{F}$ giving rise to the holomorphic structures on $\mathcal{E}$ and $\mathcal{F}$, the operator

$$\bar\partial_{\mathcal{E}^\vee \otimes \mathcal{F}} := 1 \otimes \bar\partial_\mathcal{F} - \bar\partial_\mathcal{E} \otimes 1$$

is a Dolbeault operator for $\mathcal{E}^\vee \otimes \mathcal{F}$. Since $\mathcal{E}$ and $\mathcal{F}$ have Chern connections ∇_0 and ∇_∞ respectively, and since the ∇_i for $i \geq 0$ all give rise to the same (more precisely isomorphic) holomorphic structures, we can take the $(0, 1)$ part of these connections to build the Dolbeault operators $\bar\partial_{i,\infty} := (1 \otimes \nabla_\infty - \nabla_i \otimes 1)_{0,1}$. Now to say $\text{Hom}(\mathcal{E}, \mathcal{F}) = 0$ is to say that the Dolbeault operators $\bar\partial_{i,\infty}$ for $i \geq 0$, considered as maps $\text{End} \, E \to \text{End} \, E \otimes H^0(X, \Omega^{0,1})$ have trivial kernel. One can then apply the theory of elliptic operators and the Sobolev embedding theorem to deduce that $\bar\partial_{0,i} := 1 \otimes \nabla_i - \nabla_0 \otimes 1$ also has no kernel. But that would imply $\text{End} \, \mathcal{E} = 0$, a clear contradiction. $\square$

With this result in hand, we fix a nonzero homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ and apply Proposition A.1.17, so that we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0 \\
& & & & \downarrow{\varphi} & & \downarrow & & \\
0 & \longleftarrow & \mathcal{F}'' & \longleftarrow & \mathcal{F} & \longleftarrow & \mathcal{F}' & \longleftarrow & 0
\end{array}
\tag{4.11}
$$

with exact rows, $\mathcal{E}' \cong \ker \varphi$, $\mathcal{E}'' \cong \operatorname{im} \varphi$ and $\operatorname{rk} \mathcal{E}'' = \operatorname{rk} \mathcal{F}'$, $\deg \mathcal{E}'' \leq \deg \mathcal{F}'$. The key now is to apply estimates to these rows.

Proposition 4.2.8 (First Estimate)**.** *Consider the following short exact sequence of vector bundles:*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

and suppose $\mu(\mathcal{F}') \geq \mu(\mathcal{F})$. Then if $\nabla_\mathcal{F}$ is a unitary connection on E giving rise to the holomorphic structure of $\mathcal{F}$, we have

$$J(\nabla_\mathcal{F}) \geq \operatorname{rk} \mathcal{F}'(\mu(\mathcal{F}') - \mu(\mathcal{F})) + \operatorname{rk} \mathcal{F}''(\mu(\mathcal{F}) - \mu(\mathcal{F}'')).$$

Equality holds only if the sequence splits.

Proof. Firstly, we fix a local unitary frame s_α compatible with a local holomorphic splitting and consider the matrix of one-forms of $\nabla_\mathcal{F}$, which is skew-hermitian. One can show that it has the shape

$$\omega_\alpha = \begin{pmatrix} \omega'_\alpha & \beta_\alpha \\ -\beta^*_\alpha & \omega''_\alpha \end{pmatrix},$$

where the β_α glue to the second fundamental form (c.f. Remark 3.1.4) β, and the ω'_α are the 1-forms of a Chern connection ∇' on $\mathcal{F}'$, and similarly with ω''_α. If we compute the curvature, we see that it is of the form

$$\Theta_\mathcal{F} = \begin{pmatrix} \Theta' - \beta \wedge \beta^* & \nabla_{\mathcal{F}'',\mathcal{F}}\beta \\ -\nabla_{\mathcal{F}'',\mathcal{F}}\beta^* & \Theta'' - \beta^* \wedge \beta \end{pmatrix},$$

where Θ' and Θ'' are the curvatures of ∇' and ∇'' respectively and $\nabla_{\mathcal{F}'',\mathcal{F}} : \Omega^1(\mathcal{H}om(\mathcal{F}'', \mathcal{F})) \to \Omega^2(\mathcal{H}om(\mathcal{F}'', \mathcal{F}))$ is built from the connections ∇', ∇'' (see [13, p. 78] for details). Now by Corollary 4.2.5, it follows that

$$\nu\left(\frac{\star\Theta_\mathcal{F}}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{F}}(\mu)\right) \geq \left|\operatorname{tr}\left(\frac{\star(\Theta' - \beta \wedge \beta^*)}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{F}'}(\mu)\right)\right| + \left|\operatorname{tr}\left(\frac{\star(\Theta'' - \beta^* \wedge \beta)}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{F}''}(\mu)\right)\right|,$$

where $\mu = \mu(\mathcal{F})$. Applying Hölder's inequality, we deduce

$$J(\nabla_\mathcal{F}) \geq \int_X \nu\left(\frac{\star\Theta_\mathcal{F}}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{F}}(\mu)\right)\operatorname{vol}$$

$$= \left|\int_X \operatorname{tr}\left(\frac{\star(\Theta' - \beta \wedge \beta^*)}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{F}'}(\mu)\right)\operatorname{vol}\right| + \left|\int_X \operatorname{tr}\left(\frac{\star(\Theta'' - \beta^* \wedge \beta)}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{F}''}(\mu)\right)\operatorname{vol}\right|$$

Let us consider the term $\star(\beta \wedge \beta^*)$. Observe that β is a $(0,1)$-form and so $\beta \wedge \beta^*$ has entries of the form $|f|d\bar{z} \wedge dz$ for any holomorphic coordinate z. Since, by our conventions, our orientation is $idz \wedge d\bar{z}$, this means that $-i\operatorname{tr}\star(\beta \wedge \beta^*)$ will be nonnegative.

Next we observe that by Theorem 4.1.15, we have

$$\int_X \operatorname{tr}\left(\frac{\star\Theta'}{2\pi i}\right)\operatorname{vol} = -\deg \mathcal{F}' \leq -\operatorname{rk}\mathcal{F}'\mu(\mathcal{F}) = -\operatorname{tr}\operatorname{diag}_{\operatorname{rk}\mathcal{F}}(\mu) = -\int_X \operatorname{tr}\operatorname{diag}_{\operatorname{rk}\mathcal{F}'}(\mu)\operatorname{vol},$$

where the last equality follows from the assumption $\int_X \text{vol} = 1$. Hence

$$\left| \int_X \text{tr} \left(\frac{\star(\Theta' - \beta \wedge \beta^*)}{2\pi i} + \text{diag}_{\text{rk}\,\mathcal{F}'}(\mu) \right) \text{vol} \right| = - \int_X \text{tr} \left(\frac{\star(\Theta' - \beta \wedge \beta^*)}{2\pi i} + \text{diag}_{\text{rk}\,\mathcal{F}'}(\mu) \right) \text{vol}$$

$$= \text{rk}\,\mathcal{F}'(\mu(\mathcal{F}') - \mu(\mathcal{F})) + \frac{1}{2\pi i} \text{tr} \star (\beta \wedge \beta^*)$$

and note that $\frac{1}{2\pi i} \text{tr} \star (\beta \wedge \beta^*) \geq 0$ by the above discussion. Similarly, note that

$$\left| \int_X \text{tr} \left(\frac{\star(\Theta'' - \beta^* \wedge \beta)}{2\pi i} + \text{diag}_{\text{rk}\,\mathcal{F}''}(\mu) \right) \text{vol} \right| = \text{rk}\,\mathcal{F}''(\mu(\mathcal{F}) - \mu(\mathcal{F}'')) + \frac{1}{2\pi i} \text{tr} \star (\beta \wedge \beta^*)$$

And putting it all together we get

$$J(\nabla_{\mathcal{F}}) \geq \text{rk}\,\mathcal{F}'(\mu(\mathcal{F}') - \mu(\mathcal{F})) + \text{rk}\,\mathcal{F}''(\mu(\mathcal{F}) - \mu(\mathcal{F}'')) + \frac{1}{\pi i} \text{tr} \star (\beta \wedge \beta^*)$$

$$\geq \text{rk}\,\mathcal{F}'(\mu(\mathcal{F}') - \mu(\mathcal{F})) + \text{rk}\,\mathcal{F}''(\mu(\mathcal{F}) - \mu(\mathcal{F}'')).$$

as desired. Finally, if equality occurs, that means $\beta = 0$, but β defines an element of $\text{Ext}^1(\mathcal{F}'', \mathcal{F}')$ via the Dolbeault cohomology representation of sheaf cohomology (Remark 3.1.4), and in particular if it is zero then the sequence splits. $\qquad \square$

And in fact, from this we may already deduce one direction of the Donaldson-Narasimhan-Seshadri theorem:

Corollary 4.2.9. *Suppose $\mathcal{E}$ is indecomposable, and there is a Chern connection ∇ giving rise to $\mathcal{E}$ such that $J(\nabla) = 0$. Then $\mathcal{E}$ is stable.*

Proof. Suppose for contradiction $\mathcal{E}$ is not stable. Then there is some subbundle $\mathcal{E}'$ such that $\mu(\mathcal{E}') \geq \mu(\mathcal{E})$, whence $\mu(\mathcal{E}) \geq \mu(\mathcal{E}/\mathcal{E}')$. Then

$$0 = J(\nabla) \geq \text{rk}\,\mathcal{E}'(\mu(\mathcal{E}') - \mu(\mathcal{E})) + \text{rk}(\mathcal{E}/\mathcal{E}')(\mu(\mathcal{E}) - \mu(\mathcal{E}/\mathcal{E}')) \geq 0,$$

which means the sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}/\mathcal{E}' \to 0$$

splits, by the above proposition, contradicting the indecomposability of $\mathcal{E}$. $\qquad \square$

Remark 4.2.10. In fact, we can deduce that if $\mathcal{E}$ has a Chern connection which is a zero of J, then $\mathcal{E}$ must be polystable, since the proposition tells us that $\mathcal{E}$ can be written as a direct sum of two subbundles of equal slope.

Our next estimate applies to the top row. However, it is more technical and requires the stronger hypothesis that the Donaldson-Narasimhan-Seshadri theorem has been proven for bundles of smaller rank:

Proposition 4.2.11 (Second Estimate)**.** *Consider the following short exact sequence of vector bundles:*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0.$$

Suppose this exension is proper, that $\mathcal{E}$ is stable and the Donaldson-Narasimhan-Seshadri theorem has been proven for bundles of rank less than $\mathrm{rk}\,\mathcal{E}$. Then there exists a unitary connection $\nabla_{\mathcal{E}}$ on E giving rise to $\mathcal{E}$ such that

$$J(\nabla_{\mathcal{E}}) < \mathrm{rk}\,\mathcal{E}'(\mu(\mathcal{E}) - \mu(\mathcal{E}')) + \mathrm{rk}\,\mathcal{E}''(\mu(\mathcal{E}'') - \mu(\mathcal{E})).$$

Proof Sketch. The idea here is to use the Harder-Narasimhan and Jordan-Hölder filtrations and the inductive hypothesis to build this $\nabla_{\mathcal{E}}$. Let $(\mathcal{E}'_i)$ be the Harder-Narasimhan filtration of $\mathcal{E}'$, and for each i let $(\mathcal{E}'_{ij})$ denote the Jordan-Hölder filtration of $\mathcal{E}'_i / \mathcal{E}'_{i-1}$. Since $\mathrm{rk}\,\mathcal{E}'_{i,j} / \mathcal{E}_{i,j-1} < \mathrm{rk}\,\mathcal{E}$, by assumption we know that there is a projectively flat Chern connection ∇'_{ij} on $\mathcal{E}'_{i,j} / \mathcal{E}_{i,j}$. Now given any

$$0 \to \mathcal{E}'_{i,j} \to \mathcal{E}'_{i,j+1} \to \mathcal{E}'_{i,j+1} / \mathcal{E}'_{i,j} \to 0$$

with second fundamental form $B_{i,j}$, one can inductively (starting with $j = 0$) build a connection on $\mathcal{E}'_{i,j+1}$ from the one on $\mathcal{E}'_{i,j}$ and the one on $\mathcal{E}'_{i,j+1} / \mathcal{E}'_{i,j}$ given to us, and letting the i vary we can build a connection on each $\mathcal{E}'_i$. Now given any short exact sequence of vector bundles

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

with second fundamental form B, one can scale B by any nonzero constant $t \in \mathbb{C} \setminus \{0\}$ and the resulting bundle in the middle is isomorphic to $\mathcal{F}$, by the proof of Theorem 3.1.3. In particular, given any short exact sequence from any of our filtrations above (which either looks like

$$0 \to \mathcal{E}'_{i,j} \to \mathcal{E}'_{i,j+1} \to \mathcal{E}'_{i,j+1} / \mathcal{E}'_{i,j} \to 0$$

or

$$0 \to \mathcal{E}'_i \to \mathcal{E}'_{i+1} \to \mathcal{E}'_{i+1} / \mathcal{E}'_i \to 0),$$

the connection built in the middle is of the form

$$\begin{pmatrix} \nabla_1, B \\ -B^*, \nabla_2 \end{pmatrix}$$

where ∇_1 and ∇_2 are connections on the left and right respectively and B is the second fundamental form. Now as mentioned, we may scale B by any constant $t > 0$ and retain the same isomorphism class, so the matrix

$$\begin{pmatrix} \nabla_1, tB \\ -tB^*, \nabla_2 \end{pmatrix}$$

gives rise another Chern connection on the middle bundle. Doing this (with a fixed $t > 0$) for every step of both filtrations, we have a collection of Chern connections $\{\nabla'_t\}$ on $\mathcal{E}'$, but their limit ∇'_0 is a Chern connection for $\bigoplus_{i,j} \mathcal{E}'_{ij}$, and moreover by construction we have

$$\star \Theta'_0 = -2\pi i \,\mathrm{diag}(\mu(\mathcal{E}'_{ij})),$$

where Θ'_t is the curvature of ∇'_t. Similarly, we can build a collection of Chern connections ∇''_t on $\mathcal{E}''$ that converge to a connection ∇''_0 on some $\bigoplus_{i',j'} \mathcal{E}''_{i'j'}$ and $\star\Theta''_0 = -2\pi i \operatorname{diag}(\mu(\mathcal{E}''_{i',j'}))$.

Let $[\beta] \in \operatorname{Ext}^1(\mathcal{F}'', \mathcal{F}')$ denote the extension class of $\mathcal{E}$. For each ∇'_t, ∇''_t, one can build a connection $\nabla^t_{\mathcal{E}'',\mathcal{E}'}$ on $\mathcal{H}om(\mathcal{E}'', \mathcal{E}')$, and for each $\nabla^t_{\mathcal{E}'',\mathcal{E}'}$, standard arguments from Hodge theory tell us that there is a representative of $[\beta]$, call it β_t, such that $\nabla^t_{\mathcal{E}'',\mathcal{E}'}(\beta_t) = 0$. Letting $s > 0$ be another variable, we have connections depending on s and t

$$\nabla_{s,t} = \begin{pmatrix} \nabla'_t & s\beta_t \\ -s\beta^*_t & \nabla''_t \end{pmatrix}$$

with curvature

$$\Theta_{s,t} = \begin{pmatrix} \Theta'_t - s^2\beta_t \wedge \beta^*_t & 0 \\ 0 & \Theta''_t - s^2\beta^*_t \wedge \beta_t \end{pmatrix}$$

that converge to $\nabla_{0,0}$ with curvature $\Theta_{0,0} = \operatorname{diag}(\Theta'_0, \Theta''_0)$. Now we observe

$$\operatorname{tr}\left(\frac{\star\Theta'_0}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{E}'}(\mu(\mathcal{E}))\right) = \sum(\mu(\mathcal{E}) - \mu(\mathcal{E}'_{ij})) = \operatorname{rk}\mathcal{E}'(\mu(\mathcal{E}) - \mu(\mathcal{E}')) > 0$$

and similarly

$$\operatorname{tr}\left(\frac{\star\Theta''_0}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{E}''}(\mu(\mathcal{E}))\right) = \operatorname{rk}\mathcal{E}''(\mu(\mathcal{E}) - \mu(\mathcal{E}'')) < 0,$$

and put together this tells us

$$J(\nabla_{0,0}) = \operatorname{rk}\mathcal{E}'(\mu(\mathcal{E}) - \mu(\mathcal{E}')) + \operatorname{rk}\mathcal{E}''(\mu(\mathcal{E}'') - \mu(\mathcal{E})).$$

Our next task is to show that for sufficiently small s, t we have $J(\nabla_{s,t}) < J(\nabla_{0,0})$. To this end, we first note that since $A' := \operatorname{diag}(\mu - \mu(\mathcal{E}'_{ij}))$ is a diagonal matrix with positive entries and hence has negative eigenvalues, it follows that $\nu(A') = \operatorname{tr} A'$, and hence the same is true for matrices sufficiently close to A'. Now it can be shown that the $i \operatorname{tr} \star(\beta^*_t \wedge \beta_t)$ are uniformly bounded, and hence for sufficiently small s, t, it follows

$$\nu\left(\frac{\star(\Theta'_t - s^2\beta_t \wedge \beta^*_t)}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{E}'}(\mu)\right) = \operatorname{tr}\left(\frac{\star\Theta'_t - s^2\beta_t \wedge \beta^*_t}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{E}'}(\mu)\right)$$

$$= \operatorname{rk}\mathcal{E}'(\mu(\mathcal{E}) - \mu(\mathcal{E}')) - s^2 \operatorname{tr}\star\left(\frac{\beta_t \wedge \beta^*_t}{2\pi i}\right) + \varepsilon_1(t),$$

where $\varepsilon_1(t)$ is some error term that vanishes as $t \to 0$, and the uniform bound is used to control the $|\operatorname{tr}\star(\frac{\beta_t \wedge \beta^*_t}{2\pi i})|$, so that the matrix does not deviate from A' too much. Similarly,

$$\nu\left(\frac{\star(\Theta'_t + s^2\beta_t \wedge \beta^*_t)}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{E}'}(\mu)\right) = -\operatorname{tr}\left(\frac{\star\Theta''_t + s^2\beta_t \wedge \beta^*_t}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{E}'}(\mu)\right)$$

$$= \operatorname{rk}\mathcal{E}''(\mu(\mathcal{E}'') - \mu(\mathcal{E})) - s^2 \operatorname{tr}\star\left(\frac{\beta_t \wedge \beta^*_t}{2\pi i}\right) + \varepsilon_2(t),$$

and hence

$$\nu\left(\frac{\star\Theta_{s,t}}{2\pi i} + \operatorname{diag}_{\operatorname{rk}\mathcal{E}}(\mu)\right) = J(\nabla_{0,0}) - s^2 \operatorname{tr}\star\left(\frac{\beta_t \wedge \beta^*_t}{\pi i}\right) + \varepsilon(t).$$

Integrating, we find

$$
\begin{aligned}
J(\nabla_{s,t})^2 &= \int_X \nu^2 \left(\frac{\star\Theta}{2\pi i} + \mathrm{diag}(\mu) \right) \mathrm{vol} \\
&= \int_X \left(J(\nabla_{0,0}) - s^2 \, \mathrm{tr} \star (\frac{\beta_t \wedge \beta_t^*}{\pi i}) + \varepsilon(t) \right)^2 \mathrm{vol} \\
&= J(\nabla_{0,0})^2 + \varepsilon'(s,t) + \int_X \left(s^4 \, \mathrm{tr} \star (\frac{\beta_t \wedge \beta_t^*}{\pi i})^4 - C_t s^2 \, \mathrm{tr} \star (\frac{\beta_t \wedge \beta_t^*}{\pi i})^2 \right) \mathrm{vol}
\end{aligned}
$$

where C_t is some term depending on t which is positive and bounded for sufficiently small t and ε' is some error term depending on s and t which goes to zero. In particular, one can choose an s, t so small that the term in the integral is negative (since s^4 is much smaller than s^2 for sufficiently small s and the $i \, \mathrm{tr} \star (\beta_t^* \wedge \beta_t)$ are uniformly bounded) and ε' is negligible, whence $J(\nabla_{s,t}) < J(\nabla_{0,0})$, as desired. $\square$

Corollary 4.2.12. *Suppose the Donaldson-Narasimhan-Seshadri theorem has been proven for lower-rank bundles, and let $\mathcal{E}$ and $\mathcal{F}$ be as in Proposition 4.2.7. If $\mathcal{E}$ is stable, we have $\mathcal{E} \cong \mathcal{F}$. In particular, $J(\nabla_\infty) = \inf J(O_{\nabla_0})$.*

Proof. Suppose for contradiction $\mathcal{E}$ is not isomorphic to $\mathcal{F}$. Then there is a nonzero homomorphism $\varphi : \mathcal{E} \to \mathcal{F}$ by Proposition 4.2.7 which is not an isomorphism. Recalling Proposition A.1.17, φ factors through

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\
0 & \longleftarrow & \mathcal{F}'' & \longleftarrow & \mathcal{F} & \longleftarrow & \mathcal{F}' & \longleftarrow & 0,
\end{array}
$$

where $\mathcal{E}' = \ker \varphi$ and $\mathcal{E}'' = \mathrm{im} \, \varphi$. We first claim that φ cannot be injective. Indeed, if it is, then $\mathcal{E}' = 0$ and it is easy to see $\mathcal{F}'' = 0$ too. But that means $\mathcal{E}'' = \mathcal{E}$ and $\mathcal{F}' = \mathcal{F}$, but $\deg \mathcal{E} = \deg \mathcal{E}'' < \deg \mathcal{F}' = \deg \mathcal{F}$ (c.f. Lemma A.1.16), contradicting the fact that $\mathcal{E}$ and $\mathcal{F}$ have the same signature. Hence we may assume the top row is proper.

Now recall $\deg \mathcal{E}'' \leq \deg \mathcal{F}'$, and the ranks of each column are equal (i.e. $\mathrm{rk} \, \mathcal{E}' = \mathrm{rk} \, \mathcal{F}''$ etc.). Applying the first estimate to the bottom row, we find that

$$
J(\nabla_\infty) \geq \mathrm{rk} \, \mathcal{F}'(\mu(\mathcal{F}') - \mu(\mathcal{F})) + \mathrm{rk} \, \mathcal{F}''(\mu(\mathcal{F}) - \mu(\mathcal{F}'')),
$$

and similarly, by the second estimate (which requires the top row to be a proper extension), there is some Chern connection $\nabla_\mathcal{E}$ on $\mathcal{E}$ such that

$$
J(\nabla_\mathcal{E}) < \mathrm{rk} \, \mathcal{E}'(\mu(\mathcal{E}) - \mu(\mathcal{E}')) + \mathrm{rk} \, \mathcal{E}''(\mu(\mathcal{E}'') - \mu(\mathcal{E})).
$$

But by assumption, $J(\nabla_\infty) \leq \inf J(O_{\nabla_0}) \leq J(\nabla_\mathcal{E})$, and so

$$
\mathrm{rk} \, \mathcal{F}'(\mu(\mathcal{F}') - \mu(\mathcal{F})) + \mathrm{rk} \, \mathcal{F}''(\mu(\mathcal{F}) - \mu(\mathcal{F}'')) < \mathrm{rk} \, \mathcal{E}'(\mu(\mathcal{E}) - \mu(\mathcal{E}')) + \mathrm{rk} \, \mathcal{E}''(\mu(\mathcal{E}'') - \mu(\mathcal{E})).
$$

Using the fact that the columns have the same rank and the fact that $\mathcal{E}$ and $\mathcal{F}$ have the same signature (n, d), we deduce

$$\deg \mathcal{F}' - \deg \mathcal{E}'' < \deg \mathcal{F}'' - \deg \mathcal{E}'. \tag{4.12}$$

Now $\deg \mathcal{F}' \geq \deg \mathcal{E}''$, so the left hand side of (4.12) is nonnegative. But $\deg \mathcal{F}'' = d - \deg \mathcal{F}'$ and $\deg \mathcal{E}' = d - \deg \mathcal{E}''$, and so the right hand side of (4.12) is $-(\deg \mathcal{F}' - \deg \mathcal{E}'')$, so in summary, we have

$$0 \leq \deg \mathcal{F}' - \deg \mathcal{E}'' < -(\deg \mathcal{F}' - \deg \mathcal{E}''),$$

which is absurd. $\qquad\square$

Proof of Theorem 4.2.1. We have $J(\nabla_\infty) = \inf J(O_{\nabla_0})$, so in particular, the infimum of $J(O_{\nabla_0})$ is attained. For brevity, we will henceforth denote ∇_∞ by just ∇, and we will show that this is the connection we are interested in. The rest of this section will be dedicated to showing this.

The connection ∇ splits into its (0,1) and (1,0) components $\bar{\partial}_{\mathcal{E}}, \partial_{\mathcal{E}}$ respectively, and these operators define an associated operators on $\mathcal{E}\mathrm{nd}(E)$, for example $\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}} : \Omega^0_{\mathcal{E}\mathrm{nd}\,\mathcal{E}} \to \Omega^{0,1}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}$ is defined by

$$\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}(u)(s) = \bar{\partial}_{\mathcal{E}}(u(s)) - u(\bar{\partial}_{\mathcal{E}}(s)),$$

for any local section u of $\mathcal{E}\mathrm{nd}\,E$ and s of E, and similarly with $\partial_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}$. We thus have an operator

$$\star i(\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}\partial_{\mathcal{E}\mathrm{nd}\,\mathcal{E}} - \partial_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}) : \mathcal{E}\mathrm{nd}\,E \to \mathcal{E}\mathrm{nd}\,E.$$

Denote this operator by D. It can be shown that there exists some self-adjoint section h such that

$$Dh = 2\pi\mu - \star i\Theta,$$

where Θ is the curvature of ∇. Using h and sufficiently small t, we define the gauge transformation $g_t := 1 + th$ (observe that this is a gauge transformation, since g_t is close to 1 for sufficiently small t and hence never vanishes). Recall the gauge group action is given by

$$\nabla_t := g_t \cdot \nabla = \nabla - (\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}g_t)g_t^{-1} + ((\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}g_t)g_t^{-1})^*.$$

For brevity, write $\alpha := (\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}g_t)g_t^{-1}$, and observe that $\alpha = t\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}h + O(t^2)$, where by abuse of notation $O(t^2)$ means the L^2-norm of the error term (and hence our N-norm, if our error term is self-adjoint) is $O(t^2)$ (since $g_t^{-1} = \sum_{n=0}^\infty (-1)^n (th)^n$ and so $\alpha = t\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}h(1 + O(t))$). Now we compute the curvature Θ_t of ∇_t:

$$\Theta_t = \Theta - \partial_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}\alpha + \bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}\alpha^* - \alpha \wedge \alpha^* - \alpha^* \wedge \alpha = \Theta - t(\partial_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}} - \bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}\partial_{\mathcal{E}\mathrm{nd}\,\mathcal{E}})h + O(t^2),$$

where $(t\bar{\partial}_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}h)^* = t\partial_{\mathcal{E}\mathrm{nd}\,\mathcal{E}}h$ since h is self-adjoint. But taking the Hodge star, we find

$$\star\Theta_t = \star\Theta - itDh + O(t^2) = \star\Theta(1 - t) - 2\pi it\mu + O(t^2),$$

and finally, note that

$$J(\nabla_t) = N\left(\frac{\star\Theta_t}{2\pi i} + \mu\right) = N\left(\frac{\star\Theta}{2\pi i} + \mu\right)(1 - t) + O(t^2) = (1 - t)J(\nabla) + O(t^2),$$

and thus if $J(\nabla)$ is minimal, it follows that we must have $J(\nabla) = 0$, as desired.

$\qquad\square$

And finally, Donaldson shows that this ∇ is in fact smooth and unique up to the action of the unitary gauge group. Combining with the Riemann-Hilbert correspondence, we have:

Corollary 4.2.13. *There is a canonical bijection between polystable bundles of signature $(n, 0)$ and $U(n)$ representations of the fundamental group $\pi_1(X)$ up to conjugation. Moreover, stable bundles coincide with irreducible representations under this bijection.*

Proof. Let E be the unique bundle of signature $(n, 0)$ and fix a hermitian metric h on E. Firstly, Theorem 4.2.1 tells us that an indecomposable bundle of signature $(n, 0)$ is stable if and only if it has a flat Chern connection, which must be unique. Now given a general bundle $\mathcal{E}$, Remark 4.2.10 tells us that if $\mathcal{E}$ has a flat Chern connection, then it is polystable and by considering its irreducible components, this connection must be unique. Conversely, if $\mathcal{E}$ is polystable then by considering its irreducible (necessarily stable) components, we can build a flat connection on $\mathcal{E}$. Hence we have a bijection between flat unitary connections on (E, h), and polystable bundles with signature $(n, 0)$.

Now let P denote the unitary frame bundle of (E, h), which is a principal $U(n)$-bundle. Now a flat unitary connection on E induces a flat connection on P, and taking its holonomy representation, we get a $U(n)$-representation of $\pi_1(X)$, by the Riemann-Hilbert correspondence. Conversely, given a $U(n)$-representation of $\pi_1(X)$, the Riemann-Hilbert correspondence tells us it is the holonomy representation of some flat ω on some P'. Now it is not hard to see the associated bundles of P and P' are isomorphic, and hence we may assume $P = P'$, and thus ω induces a unitary connection ∇ on (E, h). Moreover, it is not hard to see that flatness and gauge equivalence between connections on P and E agree, and hence this ∇ is flat and unique up to gauge equivalence, and this gives us the bijection between polystable bundles $\mathcal{E}$ of signature $(n, 0)$ and $U(n)$-representations of $\pi_1(X)$.

Finally, we need to show that irreducible representations coincide with stable bundles. But this simply follows from the fact that the flat Chern connection on a polystable bundle $\mathcal{E}$ is built from the flat connections on its irreducible components, and a simple induction argument. $\qquad\square$

4.3 Topologising the Moduli Space

Our final task is to use the correspondences to topologise the moduli space of $V_{n,0}^s$. In particular, we need to topology on either the space of flat unitary connections, or the space of representations of $\pi_1(X)$. We will choose the latter. To begin, we have the following result:

Proposition 4.3.1. *Let G be a Lie group. Then the multiplication map $\mu : G \times G \to G$ is a submersion.*

Before we give the proof, we extract the following theorem:

Theorem 4.3.2 ([25], Theorem 4.26). *Let $f : M \to N$ be a smooth map between manifolds. Then f is a submersion if and only if every $m \in M$ is in the image of a smooth local section.*

Proof of Proposition 4.3.1. Let $h \in G$. Then $\mu(-, h) : G \to G$ is a diffeomorphism, and hence its inverse, which is given by $x \mapsto (xh^{-1}, h)$ is also smooth, and clearly a section of μ, and so every $(g, h) \in G$ is contained in the image of $x \mapsto (xh^{-1}, h)$, and the result follows from the above theorem. $\qquad\square$

Now we use this to topologise $\text{Hom}(\pi_1(X), U(n))$. To this end, observe that $\pi_1(X)$ is a finitely generated group, in particular it has the presentation

$$\pi_1(X) = \langle a_1, b_1, ..., a_g, b_g \mid \prod [a_i, b_i] = 1 \rangle,$$

where g is the genus of X ([19, p. 51]). Now to give a representation $\pi_1(X) \to U(n)$ is equivalent to giving $A_i, B_i \in U(n)$ subject to $\prod [A_i, B_i] = 1$, so in particular, $\text{Hom}(\pi_1(X), U(n))$ may be identified with the subset of $U(n)^{2g}$ subject to the constraint $\prod [A_i, B_i] = 1$, which, by Proposition 4.3.1 is actually a smooth manifold. This is known as the $U(n)$-*representation variety* of $\pi_1(X)$, which, despite its name, is not an algebraic variety. The *character variety* is then defined to be the (topological) quotient of $\text{Hom}(\pi_1(X), U(n))$ by $U(n)$, where the latter acts via conjugation. Note that this is a Hausdorff space, since $U(n)$ is compact, but not a manifold, as the action is not free (indeed, the centre of $U(n)$ is S^1). This gives a topology to the space of polystable bundles, and hence to the space of stable bundles too. This concludes the second construction of $V_{n,0}^s$, the moduli space of stable bundles.

4.4 Further Discussion

Many generalisations and analogues of the Narasimhan-Seshadri theorem are now known. For example, the Kobayashi-Hitchin correspondence ([48]), also known as the Donaldson-Uhlenbeck-Yau theorem describes a correspondence between stable bundles on a complex projective variety and a certain class of connections, known as *Hermitian-Einstein* connections, which generalise Yang-Mills connections. The non-abelian Hodge correspondence also describe homeomorphisms between three moduli spaces: the space of so-called semistable Higgs bundles with certain vanishing Chern classes (which generalise degree zero vector bundles) on a complex projective variety X, the space of vector bundles with integrable connections (generalising flat connections) and the space of complex representations of the fundamental group of X ([43]). Finally, there is a correspondence between semistable vector bundles of slope zero on the Fargues-Fontaine curve and certain p-adic Galois representations, in analogy with the Narasimhan-Seshadri theorem. More precisely, given a perfectoid field K of characteristic $p > 0$ and L a complete nonarchimedean field with residue field $\mathbb{F}_p$, one can associate a curve over $\mathbb{Q}_p$ known as the *Fargues-Fontaine curve*, denoted $X_{K,L}$, and there is an equivalence of categories between semistable vector bundles of slope zero on $X_{K,L}$ and L-representations of $G_K = \text{Gal}(\bar{K}/K)$, the absolute Galois group of K ([10]) (which may be thought of as the étale fundamental group of $\text{Spec} \, K$).

However, one direction that has not been explored is towards the recent development of "twisted" analogues of these objects. For example, given a reductive Lie group G and compact Riemann surface X, one can define the notion of a *twisted principal $\mathcal{G}$-bundle*, where $\mathcal{G}$ is a so-called *twisted bundle of groups*, which generalises the notion of a principal G-bundle (and hence the notion of a vector bundle). Now *twisted* character varieties and the analogue of flat connections on twisted $\mathcal{G}$-bundles (known as $\mathcal{G}$-*local systems*) have been defined and studied in [4], and an analogue of the Riemann-Hilbert has been proven ([4, Proposition 19]). One can then ask, is there an analogue of stability for principal $\mathcal{G}$-bundles, and if so, does the analogy of the Narasimhan-Seshadri theorem hold?

Appendix A

Preliminaries to Part I

Let X be a nonsingular quasiprojective curve over an algebraically closed field k.

A.1 Divisors and Line Bundles

In this section, we will review the relation between divisors and line bundles, which will ultimately help us define the degree of a vector bundle. Our exposition roughly follows the one found in [17, II, Section 6].

Definition A.1.1. We define the *group of Weil divisors* on X, denoted $\mathrm{Div}(X)$ to be the free abelian group on the k-points of X. The elements of $\mathrm{Div}\, X$ are known as *Weil divisors*. The *support* of a Weil divisor $D = \sum n_i p_i$, denoted $\mathrm{Supp}\, D$, is the set $\{p_i \mid n_i \neq 0\}$, and the *degree* of D is defined to be $\deg D := \sum n_i$. We say D is *effective* if $n_i > 0$ for all $p_i \in \mathrm{Supp}\, D$. A *prime divisor* is an effective Weil divisor of degree 1.

Since X is nonsingular, for any k-point p, the local ring $\mathcal{O}_{X,p}$ is regular of dimension 1, and in particular it is a DVR, with fraction field $K(X)$, the function field of X. Denote the valuation v_p. For any $f \in K(X)^*$, we define the *divisor of f*, denoted $\mathrm{div}(f)$ to be

$$\mathrm{div}\, f := \sum_{p \in X(k)} v_p(f)p.$$

We can show ([17, II Lemma 6.1]) that all but finitely many of the $v_p(f)$ vanish, hence we get a Weil divisor. A divisor of the form $\mathrm{div}\, f$ is known as a *principal divisor*; clearly the principal divisors form a subgroup. Two Weil divisors are *linearly equivalent* if their difference is a principal divisor. The group of Weil divisors modulo linear equivalence is the *divisor class group*, denoted $\mathrm{Cl}\, X$. A *divisor class*, is an element of $\mathrm{Cl}\, X$. By a *divisor*, we will abuse language and refer to either a Weil divisor or its divisor class; it will either be clear from context or unimportant which is meant.

Example A.1.2. If X is affine, equal to $\mathrm{Spec}\, A$ where A is necessarily a Dedekind domain ([32, I, Proposition 11.5]), then $\mathrm{Cl}\, X = 0$ if and only if A is a PID. Indeed, if A is a PID, then if $D = \sum n_p p$ is a divisor, then for each $p \in \mathrm{Supp}\, D$, the maximal ideal $\mathfrak{m}_p$ of p is principal, generated by, say ϖ_p, which must be a uniformiser of $\mathcal{O}_{X,p}$. Now let

$$f = \prod_{p \in \mathrm{Supp}\, D} \varpi^{n_p} \in \mathrm{Frac}\, A = K(X)$$

then clearly $D = \mathrm{div}\, f$.

Conversely, if A is not a PID, then there is some maximal ideal $\mathfrak{m}_p$ corresponding to some $p \in X(k)$ generated by two elements ([32, I, 3, Ex 6.]), say $\mathfrak{m}_p = \langle f, g \rangle$. Now localising at p, it is clear that either f or g must be the uniformiser of $\mathcal{O}_{X,p}$; suppose it is f without loss of generality. Now if p was principal, then clearly p must be the divisor of some associate of f in $\mathcal{O}_{X,p}$, say F. But F is not prime, and by the unique ideal factorisation property of Dedekind domains ([32, I, Theroem 3.3]), we may write

$$\langle F \rangle = \mathfrak{m}_p \prod \mathfrak{m}_{p_i}^{n_i}$$

where at least one n_i is nonzero, but all but finitely many of them are. But that means

$$\operatorname{div} F = p + \sum n_i p_i$$

which is a contradiction.

More concretely, if $X = \mathbb{A}^1 = \operatorname{Spec} k[x]$, then any $D = \sum n_i a_i$ can be written $D = \operatorname{div} \prod (x - a_i)$, but if $X = \operatorname{Spec} k[x, y]/\langle y^2 = x^3 - x \rangle$ and $\operatorname{char} k \neq 2$, then we claim $p = (0, 0)$ is not principal. Indeed, $\mathfrak{m}_p = \langle x, y \rangle$ and clearly $y^2 = x(x-1)(x+1)$, so $v_p(x) = 2$, so y is a uniformiser of $\mathcal{O}_{X,p}$. But

$$\operatorname{div} y = (0, 0) + (1, 0) + (-1, 0) \neq p,$$

and similarly for any other uniformiser.

We now state a key result, which will allow us to define the degree of a line bundle:

Proposition A.1.3. *The degree map* $\deg : \operatorname{Div} X \to \mathbb{Z}$ *descends to a map* $\operatorname{Cl} \to \mathbb{Z}$.

Proof. [17, p. 138]. □

Example A.1.4. Let $X = \mathbb{P}^1 = \operatorname{Proj} k[x_0, x_1]$. We claim that the degree map is an isomorphism; in particular its kernel is trivial. So suppose $D = \sum n_p p$ is a degree zero divisor. Now each p can be written $[p_0 : p_1]$, corresponding to $\langle x_0 p_1 - x_1 p_0 \rangle$. Write

$$f = \prod_{p \in \operatorname{Supp} D} (x_0 p_1 - x_1 p_0)^{n_p} \in k(x_0, x_1)$$

Now observe that

$$K(X) = k\left(\frac{x_0}{x_1}\right)$$

in particular, $K(X)$ is the subfield of $k(x_0, x_1)$ consisting of degree 0 elements. Since $\sum n_p = 0$, it follows $\deg f = 0$ too, hence $D = \operatorname{div} f$ as desired.

Next we describe the relation between divisors and line bundles: Let $[D]$ be a divisor class. We define the *line bundle associated to* $[D]$, denoted $\mathcal{L}(D)$ as follows: let $D = \sum n_p p$ be a divisor in the class of $[D]$. For any open set U, we define

$$\mathcal{L}(D)(U) := \{ f \in K(X) \mid v_p(f) + n_p \geq 0 \text{ for all } p \in U(k) \}$$

Firstly, we need to check that this is indeed a line bundle. To see this, observe the following: if $U \cap \operatorname{Supp} D = \varnothing$ then $\mathcal{L}(D)(U) = \mathcal{O}_X(U)$. Now for any $p \in \operatorname{Supp} D$, we choose some uniformiser

ϖ_p of $\mathcal{O}_{X,p}$, and we may assume $\varpi_p \in \mathcal{O}_X(U_p)$ for some open set U_p containing p. We may pick U_p sufficiently small that $(\operatorname{div} \varpi_p^{n_p})|_U = n_p p$ (in other words, $U_p \cap \operatorname{Supp}(\operatorname{div} \varpi_p) = \{p\}$). Now observe $\mathcal{L}(D)(U_p) = \varpi_p^{-1} \mathcal{O}_X(U)$, and in particular they are isomorphic. Now covering X with the open sets U_p and open sets which do not intersect $\operatorname{Supp} D$, we deduce that $\mathcal{L}(D)$ is a line bundle.

Finally, we need to check that $\mathcal{L}(D)$ does not depend on our choice of D. So let $D' = \sum n_p' p$ be a divisor linearly equivalent to D, so that $D - D' = \operatorname{div} F$. Observe that then $v_p(F) = n_p - n_p'$. Now we define the isomorphism $\varphi : \mathcal{L}(D) \to \mathcal{L}(D')$ to be $f \mapsto Ff$. To see that this is an isomorphism, observe that given an open set U, we have

$$\mathcal{L}(D)(U) = \{f \in K(X) \mid v_p(f) + n_p \geq 0 \text{ for all } p \in U(k)\}$$

and

$$\mathcal{L}(D')(U) := \{f \in K(X) \mid v_p(f) + n_p' \geq 0 \text{ for all } p \in U(k)\}$$

now if $f \in \mathcal{L}(D)(U)$, then

$$v_p(fF) + n_p' = v_p(f) + v_p(F) + n_p' \geq -n_p + v_p(F) + n_p' = 0$$

hence we have a well-defined morphism of sheaves, and moreover, clearly division by F is its inverse.

Before we state our next result, we recall that the *Picard group*, denoted $\operatorname{Pic} X$ is the group of line bundles on X, with the group operation given by tensor product, and inversion given by dualising.

Theorem A.1.5. *The map $D \mapsto \mathcal{L}(D)$ is an isomorphism* $\operatorname{Cl} X \to \operatorname{Pic} X$.

Proof. This follows the proof given in [17, pp. 144, 145]. First of all, let $\mathcal{K}$ denote the constant sheaf $K(X)$, which is an $\mathcal{O}_X$-module. We claim $\mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$. To see this, we consider the base of the topology $\{U_\alpha = \operatorname{Spec} A_\alpha\}$ on X consisting of the affine open sets where $\mathcal{L}$ is trivial. Then $K(X) = \operatorname{Frac} A_\alpha$ for each α, so locally we have the natural isomorphism $A_\alpha \otimes K(X) \cong K(X)$, given by $a \otimes b \mapsto b$, and it is very easy to see that these agree on overlaps, and since the U_α form a base, by [9, Proposition I-12], this extends to an isomorphisms of sheaves. It thus follows that $\mathcal{L}$ can be embedded inside $\mathcal{K}$.

First we prove surjectivity. By our discussion above, we may consider $\mathcal{L}$ as a subsheaf of $\mathcal{K}$; in particular, every local section is an element of $K(X)$. Since $\mathcal{L}$ is invertible, it follows that $\mathcal{L}(U)$ is a free $\mathcal{O}_X(U)$-submodule of $K(X) = \operatorname{Frac} \mathcal{O}_X(U)$, for sufficiently small U, and in particular it is generated by some $\varpi_U^{-1} \in K(X)$. Now we define the divisor D locally as $\operatorname{div} \varpi_U$, and let U vary across a cover. To see this is well-defined, note that if ϖ_U^{-1} generates $\mathcal{L}(U)$ and ϖ_V^{-1} generates $\mathcal{L}(V)$, then both ϖ_U^{-1} and ϖ_V^{-1} generate $\mathcal{L}(U \cap V)$, and in particular they are associates in $\mathcal{O}_X(U \cap V)$, hence they generate the same Weil divisor on $U \cap V$. It is then clear that $\mathcal{L}(D) = \mathcal{L}$.

Next we prove the homomorphism property. But this is obvious since if $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$ are generated locally by $\{\varpi_1^{-1}\}$ and $\{\varpi_2^{-1}\}$ respectively, then $\mathcal{L}(D_1 + D_2)$ are generated locally by $\{\varpi_1^{-1} \varpi_2^{-1}\}$. But $\mathcal{L}(D_1) \otimes \mathcal{L}(D_2)$ is also generated locally by $\{\varpi_1^{-1} \varpi_2^{-1}\}$, and hence they are isomorphic.

Finally, we prove injectivity. Since we have shown this is a group homomorphism, it suffices to show that the kernel is trivial. So suppose $\mathcal{L}(D) \cong \mathcal{O}_X$, and fix an isomorphism $\varphi : \mathcal{O}_X \to \mathcal{L}(D)$. We claim $D = \operatorname{div} \varphi(1)^{-1}$. Indeed, choosing a sufficiently fine open affine cover $\{U_\alpha\}$ such that

$\#(U_\alpha \cap \operatorname{Supp} D) \le 1$, we suppose $D|_{U_\alpha}$ is the divisor of ϖ_α. Then ϖ_α^{-1} generates $\mathcal{L}(D)(U_\alpha)$. But $\varphi(1)$ also generates $\mathcal{L}(D)(U_\alpha)$, hence $\varphi(1)$ and ϖ_α^{-1}, and hence both generate the same Weil divisor. In particular, $\varphi(1)^{-1}$ generates D, as desired.   □

Now let $\mathcal{L}$ be a line bundle. We will take a look at $H^0(X, \mathcal{L})$, the space of global sections. For each nonzero $s \in H^0(X, \mathcal{L})$, we define the *divisor of zeroes* of s, denoted $\operatorname{div} s$ as follows: on any open subset U on which $\mathcal{L}$ is trivial, we let $\Phi_U : \mathcal{L}|_U \to \mathcal{O}_U$ be an isomorphism, and define

$$\operatorname{div} s|_U := \operatorname{div} \Phi_U(s).$$

Of course, this is well-defined: let $p \in U(k)$, and let $\Phi_V : \mathcal{L}|_V \to \mathcal{O}_V$ be another trivialisation, with $p \in V(k)$. Then at p, $\Phi_V(s)$ and $\Phi_U(s)$ differ by an invertible element of $\mathcal{O}_{X,p}$, and thus they have the same valuation, and since p is arbitrary, this means we get a well-defined Weil divisor. Furthermore, observe that $\operatorname{div} s$ is effective, since it is locally the divisor of some section of $\mathcal{O}_X$, which must have nonnegative valuation at any k-point. We are now in a position to state our result:

Proposition A.1.6. *Let $\mathcal{L} = \mathcal{L}(D)$ be the line bundle associated to a divisor D. Then:*

(i) *The divisor of zeroes of any nonzero $s \in H^0(X, \mathcal{L})$ is linearly equivalent to D.*

(ii) *Any effective divisor D_0 linearly equivalent to D is the divisor of zeroes of some nonzero $s \in H^0(X, \mathcal{L})$.*

Proof. [17, p. 157].   □

We now come to the definition of the degree:

Definition A.1.7. Let $\mathcal{E}$ be a line bundle. We define the *degree* of $\mathcal{E}$, denoted $\deg \mathcal{E}$ as follows: if $\mathcal{E}$ is a line bundle, we define $\deg \mathcal{E}$ to be the degree of the divisor corresponding to $\mathcal{E}$ via the isomorphism $\operatorname{Pic} X \cong \operatorname{Cl} X$. This is well-defined, by Proposition A.1.3. In general, we define the *determinant line bundle* of a rank n vector bundle $\mathcal{E}$ to be the line bundle

$$\det \mathcal{E} := \wedge^n \mathcal{E}$$

and we define $\deg \mathcal{E} := \deg(\det \mathcal{E})$. Note that $\det \mathcal{L} = \mathcal{L}$ if $\mathcal{L}$ is a line bundle, hence our definition is consistent. We define the *signature* of $\mathcal{E}$ to be the pair $(\operatorname{rk} \mathcal{E}, \deg \mathcal{E})$.

Example A.1.8. Let $X = \mathbb{P}^1 = \operatorname{Proj} k[x_0, x_1]$. We will show $\deg \mathcal{O}_X(n) = n$. Let $D = n[0:1]$. We compute $\mathcal{L}(D)$ as follows: on $U_1 = \{x_1 \ne 0\} = \operatorname{Spec} k[x_0/x_1]$, we have

$$D|_{U_1} = \operatorname{div}(\frac{x_0}{x_1})^n$$

and on $U_0 = \{x_0 \ne 0\} = \operatorname{Spec} k[x_1/x_0]$, we have $D|_{U_0} = 0$, hence

$$\mathcal{L}(D)(U_0) = \mathcal{O}_X(U_0) = k[x_1/x_0]$$

and

$$\mathcal{L}(D)(U_1) = (\frac{x_1}{x_0})^n k[x_0/x_1]$$

Meanwhile,
$$\mathcal{O}_X(n)(U_0) = x_0^n k[x_1/x_0]$$
and
$$\mathcal{O}_X(n)(U_1) = x_1^n k[x_0/x_1]$$
hence we define the isomorphism $\mathcal{L}(D) \to \mathcal{O}_X(n)$ by $f \mapsto x_0^n f$. It is easy to check that this is well-defined and agrees on overlaps (which is just localising), hence we get an isomorphism of line bundles.

We also observe that D is the divisor of zeroes of $x_0^n \in H^0(X, \mathcal{O}_X(n))$, as expected.

Example A.1.9. Let X be the elliptic curve $X = \operatorname{Proj} k[x, y, z]/\langle y^2 z = x^3 - xz^2 \rangle$, and let $\mathcal{O}_X(1)$ be the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ induced by the embedding (or equivalently the twisting sheaf of Serre). We will show $\deg \mathcal{O}_X(1) = 3$ and is isomorphic to $\mathcal{L} = \mathcal{L}(3[0:1:0])$. Note that $x \neq 0$ implies $z \neq 0$, hence we can cover X by two open sets $U_z = \{z \neq 0\}$ and $U_y = \{y \neq 0\}$. Thus we have
$$\mathcal{O}_X(1)(U_z) = zk[y/x, z/x]/(y^2 z/x^3 = 1 - z^2/x^2) = z\mathcal{O}_X(U_z)$$
and
$$\mathcal{O}_X(1)(U_y) = yk[x/y, z/y]/(z/y = (x/y)^3 - xz^2/y^3) = y\mathcal{O}_X(U_y).$$
Now observe that
$$\mathcal{L}(U_z) = \mathcal{O}_X(U_z)$$
but
$$\mathcal{L}(U_y) = (y/z)\mathcal{O}_X(U_y)$$
since $D = 3[0:1:0]$ is the divisor of z/y on U_y. It then follows we have a global isomorphism $\mathcal{L} \to \mathcal{O}_X(1)$ defined by $f \mapsto zf$.

We conclude this section with some technical results about vector bundles. This collection of results will be used in various parts of chapters 3 and 4..

Proposition A.1.10. *Let $\mathcal{E}$ be a vector bundle. Then $\mathcal{E}$ has a line subbundle (a **subbundle** is a subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that $\mathcal{E}/\mathcal{F}$ is also locally free; note that this is a property of the inclusion $\mathcal{F} \subseteq \mathcal{E}$, not of $\mathcal{F}$ itself).*

Proof. Tensoring with a sufficiently high power of an ample line bundle (and then tensoring with its dual at the end), we may assume without loss of generality $H^0(X, \mathcal{E}) \neq 0$. Let $s \in H^0(X, \mathcal{E})$ be a nonzero global section and let $\mathcal{L}$ be the subsheaf generated by s, so that $\mathcal{L}$ is locally free of rank 1. Then s is a global section of $\mathcal{L}$ and thus if D is the divisor of zeroes of s, we know $\mathcal{L} = \mathcal{L}(D)$. Now tensoring with $\mathcal{L}(-D)$, we see
$$\mathcal{O}_X \cong \mathcal{L} \otimes \mathcal{L}(-D) \subseteq \mathcal{E} \otimes \mathcal{L}(-D),$$
and in particular there is a nonzero global section of $\mathcal{E} \otimes \mathcal{L}(-D)$, hence $\mathcal{E} \otimes \mathcal{L}(-D)/\mathcal{O}_X$ is locally free (since we are locally annihilating a free generator), and we have the following short exact sequence of vector bundles
$$0 \to \mathcal{O}_X \to \mathcal{E} \otimes \mathcal{L}(-D) \to \mathcal{E} \otimes \mathcal{L}(-D)/\mathcal{O}_X \to 0.$$
Tensoring with $\mathcal{L}(D)$ then proves the result. $\qquad \square$

Remark A.1.11. In fact, we have proven that if s is a global section of $\mathcal{E}$, then there is a line subbundle of $\mathcal{E}$ isomorphic to the subsheaf generated by s.

Proposition A.1.12. *Let*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be a short exact sequence of vector bundles. Then

$$\deg \mathcal{F} = \deg \mathcal{E} + \deg \mathcal{G}$$

Proof. If $\{g_{\alpha\beta} \in \mathrm{GL}_n(\mathcal{O}_X(U_\alpha \cap U_\beta))\}$ is the set of transition morphisms on a sufficiently fine cover representing $\mathcal{F}$, then it is not difficult to check $\{\det g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{G}_m\}$ is the set of transition functions of $\det \mathcal{F}$. Now it can be shown ([13, p. 68]) that each $g_{\alpha\beta}$ has the shape

$$\begin{pmatrix} h_{\alpha\beta} & k_{\alpha\beta} \\ 0 & j_{\alpha\beta} \end{pmatrix}$$

where $\{h_{\alpha\beta}\}$ and $\{j_{\alpha\beta}\}$ are the transition morphisms representing $\mathcal{E}$ and $\mathcal{F}$ respectively. Hence

$$\det g_{\alpha\beta} = \det h_{\alpha\beta} \det j_{\alpha\beta}$$

But $\{\det h_{\alpha\beta} \det j_{\alpha\beta}\}$ is the set of transition functions for $\det \mathcal{E} \otimes \det \mathcal{G}$, which means $\det \mathcal{F} = \det \mathcal{E} \otimes \det \mathcal{G}$, and thus the result follows. $\square$

Corollary A.1.13. *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles. then $\deg(\mathcal{E} \oplus \mathcal{F}) = \deg \mathcal{E} + \deg \mathcal{F}$.*

Proof. Apply the above proposition to

$$0 \to \mathcal{E} \to \mathcal{E} \oplus \mathcal{F} \to \mathcal{F} \to 0$$

and the result follows immediately. $\square$

Corollary A.1.14. *Let $\mathcal{E}$ be a vector bundle and $\mathcal{L}$ a line bundle. Then*

$$\deg(\mathcal{E} \otimes \mathcal{L}) = \deg \mathcal{E} + \deg \mathcal{L} \operatorname{rk} \mathcal{E}$$

Proof. As in the proof of the proposition, we look at the transition functions, and the result immediately follows. $\square$

Lemma A.1.15. *Let $\mathcal{L}$ be a line bundle of rank n and suppose $H^0(X, \mathcal{L}) \neq 0$. Then $\deg \mathcal{L} \geq 0$, with equality holding if and only if $\mathcal{L} = \mathcal{O}_X$.*

Proof. Suppose $\mathcal{L}$ is the line bundle associated to a divisor D and suppose $s \in H^0(X, \mathcal{L})$ is a nonzero global section. Then by Proposition A.1.6, the divisor of zeroes of s, say D_0, is effective, and linearly equivalent to D. But since D_0 is effective, it has nonnegative degree, and thus $\mathcal{L} = \mathcal{L}(D) = \mathcal{L}(D_0)$ has nonnegative degree. Moreover, if the degree of D_0 is zero, then it is trivial, and thus so is $\mathcal{L}$. $\square$

Lemma A.1.16. *Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles of rank n, and suppose there is a homomorphism $\mathcal{E} \to \mathcal{F}$ with nonzero determinant (i.e. the determinant is not identically zero). Then $\deg \mathcal{E} \leq \deg \mathcal{F}$. Equality holds if and only if the map is an isomorphism.*

Proof. Taking the determinant, we get a nonzero homomorphism $\det \mathcal{E} \to \det \mathcal{F}$. Now a nonzero homomorphism $\det \mathcal{E} \to \det \mathcal{F}$ is nothing more than a global section of $\mathcal{H}om(\det \mathcal{E}, \det \mathcal{F}) = (\det \mathcal{E})^{\vee} \otimes \det \mathcal{F}$, and since $\deg((\det \mathcal{E})^{\vee} \otimes \det \mathcal{F}) = \deg \mathcal{F} - \deg \mathcal{E}$, the result follows from Lemma A.1.15. Moreover, if equality holds, then $\det \mathcal{E} \cong \det \mathcal{F}$, and moreover the determinant never vanishes (being a global section of $\mathcal{O}_X$), and hence is invertible on every stalk, which means the map is an isomorphism. $\qquad\square$

Proposition A.1.17. *Let $\mathcal{E} \to \mathcal{F}$ be a nonzero homomorphism of vector bundles. Then there is a factorisation:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0 \\
& & & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\
0 & \longleftarrow & \mathcal{F}'' & \longleftarrow & \mathcal{F} & \longleftarrow & \mathcal{F}' & \longleftarrow & 0
\end{array}
\tag{A.1}
$$

where each sheaf above is locally free, the rows are exact, and $\mathcal{E}' \cong \ker \varphi$, $\mathcal{E}'' \cong \operatorname{im} \varphi$ and $\operatorname{rk} \mathcal{E}'' = \operatorname{rk} \mathcal{F}'$, $\deg \mathcal{E}'' \leq \deg \mathcal{F}'$.

Proof. Define $\mathcal{E}'$ and $\mathcal{E}''$ as in the theorem statement, so that the top row is exact. Since $\mathcal{F}$ is locally free and since X is covered by the spectra of Dedekind domains (which are hereditary), it follows that $\mathcal{E}'' = \operatorname{im} \varphi \subseteq \mathcal{F}$ is locally free. Now the sheaf $\operatorname{coker} \varphi$ is coherent, and hence if $U = \operatorname{Spec} A$ is an affine open subset of X, where A is a Dedekind domain, then $\operatorname{coker} \varphi$ is isomorphic to $\widetilde{M}$ for some finitely-generated A-module M. Let M' be the torsion submodule of M (these M' glue to form the *torsion subsheaf* of $\operatorname{coker} \varphi$) and we define $\mathcal{F}''$ to locally be the sheaf $(M/M')^{\sim}$. It is not hard to see that this is well-defined. Observe that since A is a Dedekind domain and M/M' is a torsion-free module, it is also a projective module, and hence $\mathcal{F}''$ is locally free, by the Serre-Swan Theorem. We then define $\mathcal{F}'$ to be the kernel of $\mathcal{F} \to \mathcal{F}''$. Now since the map $\mathcal{E}'' \to \mathcal{F}''$ defined by composing the obvious maps is zero, by the universal property of kernels there is unique homomorphism $\mathcal{E}'' \to \mathcal{F}'$ making everything commute. This unique homomorphism has nonzero determinant, because φ is nonzero, and $\mathcal{E}'' \to \mathcal{F}$ is the inclusion of the image. Finally, in light of the above lemma, it suffices to show that $\operatorname{rk} \mathcal{E}'' = \operatorname{rk} \mathcal{F}'$. But this follows directly from the observation that the local ring at any k-point is a DVR, and thus a PID, and so any finitely generated module splits into its torsion and free parts. $\qquad\square$

Appendix B

Preliminaries to Part II

Since notations and conventions vary between sources, the purpose of this section is to collect the basic definitions and results which will be used throughout Part II.

B.1 Smooth and Holomorphic Vector Bundles

In this section, we will recall basic definitions and results. Let X be a complex manifold, which may also be regarded as a smooth manifold.

Definition B.1.1. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $\mathbb{K} = \mathbb{R}$, let $\mathscr{P}$ = smooth, and if $\mathbb{K} = \mathbb{C}$, let $\mathscr{P} \in \{\text{smooth, holomorphic}\}$. A $\mathscr{P}$-*vector bundle*, or just $\mathscr{P}$-bundle of rank n over X is a $\mathscr{P}$-manifold E equipped with a surjective $\mathscr{P}$-map $\pi : E \to X$ such that at each $p \in X$, the fibre $E_p := \pi^{-1}(p)$ has the structure of an n-dimensional $\mathbb{K}$-vector space, and there is an open (in the usual topology) cover $\{U_\alpha\}$ and $\mathscr{P}$-isomorphisms $\{\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n\}$ such that at every $p \in U_\alpha$ the induced map $\Phi_\alpha|_p : E_p \to \mathbb{K}^n$ is a linear isomorphism and the following diagram commutes:

$$
\begin{array}{ccc}
\pi^{-1}(U_\alpha) & \xrightarrow{\;\Phi_\alpha\;} & U_\alpha \times \mathbb{K}^n \\
{\scriptstyle \pi}\downarrow & \swarrow & \\
U_\alpha & &
\end{array}
$$

where the arrow from $U_\alpha \times \mathbb{K}^n$ to U_α denotes projection onto the first factor.

A $\mathscr{P}$-*(local) section* over an open subset U is a $\mathscr{P}$-map $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$. A $\mathscr{P}$-*global section* is a section over X. A $\mathscr{P}$-*frame* is a tuple of sections $(s_1, ..., s_n)$ over U such that for all $p \in U$ the set $\{s_1(p), ..., s_n(p)\}$ is linearly independent.

A $\mathscr{P}$-*morphism* of $\mathscr{P}$-vector bundles $\pi_E : E \to X$ and $\pi_F : F \to X$ is a $\mathscr{P}$-map $f : E \to F$ such that the following diagram commutes:

$$
\begin{array}{ccc}
E & \xrightarrow{\;f\;} & F \\
{\scriptstyle \pi_E}\downarrow & \swarrow {\scriptstyle \pi_F} & \\
X & &
\end{array}
$$

and for all $p \in X$ we have that $f|_{E_p}$ is a linear map. A $\mathscr{P}$-*isomorphism* of $\mathscr{P}$-bundles is a morphism with a two-sided inverse. E and F are *isomorphic* if there is an isomorphism between them. If E

is smooth, then an automorphism of E is known as a *gauge transformation*. The group of smooth automorphisms is known as the *gauge group*.

For now we give only the most basic example:

Example B.1.2. The most basic example is $E = X \times \mathbb{K}^n$ with π being projection onto the first factor, known as the *trivial bundle*. A bundle is *trivial* if it is isomorphic to the trivial bundle.

Lemma B.1.3. *A bundle is trivial if and only if there is a global frame.*

Proof. Clearly $((x, e_i))_{i=1}^n$ is a frame for $X \times \mathbb{K}^n$. Conversely, suppose $(s_i)_{i=1}^n$ is a frame for E. Then every $p \in E$ can be written uniquely as $\sum a_i s_i(\pi(p))$ where the a_i are smooth. It is not hard to check that

$$\sum a_i s_i(\pi(p)) \mapsto (\pi(p), \sum a_i e_i)$$

is a $\mathscr{P}$-isomorphism. $\square$

In fact, the notions of local frame and local trivialisation are equivalent: given a local trivialisation $\Phi_i : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$, we can define $s_i(p) := (p, e_i)$, and conversely, given a local frame $(s_i)_\alpha$, we can define $\Phi_i(\sum a_i s_i(p)) := (p, \sum a_i e_i)$. We will be using this equivalence without further comment.

Definition B.1.4. Let E be a $\mathscr{P}$-vector bundle and let $s_\alpha = (s_i)_\alpha$ and $t_\beta = (t_i)_\beta$ be frames on U_α and U_β. We define the *transition function* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{K})$ equal to be the $\mathscr{P}$-map

$$g_{\alpha\beta}(x) := \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1}$$

Observe that if $t_j = \sum g_{ij} s_i$, then

$$g_{\alpha\beta}(e_j) = \Phi_\alpha \circ \Phi_\beta^{-1}(e_j) = \Phi_\alpha(t_j) = \Phi_\alpha(\sum g_{ij} s_i) = \sum g_{ij} e_i$$

and hence $g_{\alpha\beta} = (g_{ij})$.

If $\{U_\alpha\}$ is an open cover of X with local frames s_α, then it is not hard to check that the transition functions satisfy the following conditions:

(i) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$.

(ii) $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$

These are known as the *cocycle conditions*. Conversely:

Lemma B.1.5 (Clutching Construction). *Let $\{U_\alpha\}$ be an open cover of X and suppose for any α, β we have a $\mathscr{P}$-map*

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{K})$$

satisfying the cocycle conditions. Then there exists a unique bundle $E \to X$ trivial on each U_α with transition functions $g_{\alpha\beta}$.

Proof. We define

$$E^\sharp := \coprod_\alpha U_\alpha \times \mathbb{K}^n$$

with the induced $\mathscr{P}$-structure. Now we put an equivalence relation $\sim$ on $E^\sharp$ by declaring $(x, u)_\alpha \sim (y, v)_\beta$ if and only if $x = y$ and $u = g_{\alpha\beta}(x)v$. The cocycle conditions guarantee that this is an equivalence relation. We then define $E := E^\sharp / \sim$. Since $\mathscr{P}$-ness is local, we obtain a $\mathscr{P}$-vector bundle. Now for each α define the local frame $(s_i)_\alpha$ on U_α by

$$s_i(p) := (p, e_i)_\alpha \mod \sim$$

and observe that with respect to the frame $(s_i)_\alpha$, we have $(t_i)_\beta = g_{\alpha\beta}$ as desired. To check uniqueness, suppose F is another vector bundle with local frames $\{(s_i)_\alpha\}$ that satisfy the same transition functions. We then define an isomorphism $E \to F$ given by $(x, e_i) \mapsto s_i(x)$ and extend by linearity. It is not hard to check that this is well defined and an isomorphism. $\square$

Example B.1.6. We define the $\mathscr{P}$-*tangent bundle* $\pi : T_X \to X$ of X as follows: let $\{(U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{K}^n)\}$ be a $\mathscr{P}$-chart for X. Then the tangent bundle at X is the unique bundle that is trivial on each U_α and has transition function $g_{\alpha\beta}$ equal to the Jacobian of $\varphi_\alpha \circ \varphi_\beta$. This may be interpreted as follows: on $\mathbb{K}^n$, the tangent space is spanned by $\partial_i = \partial/\partial x_i$, where x_i are the $\mathscr{P}$-coordinates of $\mathbb{K}^n$. Thus on U_α, we define the local frame $(s_i)_\alpha$ by $s_i := \varphi_\alpha^{-1}(\partial_i)$, and hence we obtain a local trivialisation $\Phi_\alpha(s_i) := e_i$. Identifying e_i with ∂_i, we obtain

$$\Phi_\alpha \circ \Phi_\beta = \varphi_\alpha \circ \varphi_\beta$$

as desired.

Similarly, we may define the *cotangent bundle* T_X^* to be the unique bundle with transition function $\varphi_\beta \circ \varphi_\alpha$; we may interpret the fibre at each point to be the set of linear functionals from $T_p X$. Locally, we may find a basis dx_i dual to ∂_i, and on overlaps these satisfy the required transition map.

In future, we will often denote the smooth tangent bundle by T_X and the holomorphic tangent bundle by $\mathcal{T}_X$.

Our next theorem connects us back to algebraic geometry:

Theorem B.1.7. *Let $\mathcal{O}_X$ denote the sheaf of complex $\mathscr{P}$-functions on X, and let E be a vector bundle of rank n. Then:*

(i) *The presheaf $\mathcal{E}$ given by*
$$\mathcal{E}(U) := \{\text{sections of } E \text{ over } U\}$$
*is a locally free $\mathcal{O}_X$-module of rank n. Call this the **sheaf of sections** of E.*

(ii) *Any locally free $\mathcal{O}_X$-module of rank n is isomorphic to the sheaf of sections of some unique vector bundle of rank n.*

(iii) *The association $E \mapsto \mathcal{E}$ is an equivalence of categories between the category of vector bundles and locally free sheaves.*

Proof. It is not hard to check that $\mathcal{E}$ is indeed a sheaf (indeed, a section over U is a function satisfying a local property), and since each fibre E_p is a vector space, this defines the $\mathcal{O}_X$-module structure. To see that it is locally free, suppose E is trivial on U. We define an isomorphism

$$\varphi : \bigoplus_{i=1}^n \mathcal{O}_X|_U \to \mathcal{E}|_U$$

as follows: let V be an open subset of U. Then given $(f_i) \in \bigoplus_{i=1}^n \mathcal{O}_X(V)$, we interpret this as a map $V \to \mathbb{K}^n$. This naturally defines a section $s : V \to E$ given by

$$p \mapsto \Phi^{-1}(p, f_1(p), ..., f_n(p))$$

where $\Phi : \pi^{-1}(V) \to V \times \mathbb{K}^n$ is a local trivialisation. We then define $\varphi_V(f_i) := s$. It is easy to check that this is a homomorphism of $\mathcal{O}_X(V)$-modules and that it commutes with restriction, hence we have a morphism of sheaves. Conversely, given a section $s : V \to E$, composing with $\Phi : \pi^{-1}(V) \to V \times \mathbb{K}^n$ and projecting onto $\mathbb{K}^n$ we have a map $V \to \mathbb{K}^n$, which is exactly an element of $\bigoplus \mathcal{O}_X(V)$. It is clear that this is the inverse to φ, hence this proves (i).

Now let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module. Cover X with open subsets $\{U_\alpha\}$ on which $\mathcal{E}$ is free. For any α, fix an isomorphism of $\mathcal{O}_X(U_\alpha)$-modules $\Phi_\alpha : \Gamma(U_\alpha, \mathcal{E}) \to \mathcal{O}_X(U_\alpha)^n$. Now we define

$$g_{\alpha\beta} := \Phi_\alpha \circ \Phi_\beta^{-1}|_{U_\alpha \cap U_\beta} \in \mathrm{GL}_n(\mathcal{O}_X(U_\alpha \cap U_\beta))$$

In other words, $g_{\alpha\beta}$ is a matrix of $\mathbb{K}$-valued $\mathscr{P}$-functions, which may also be interpreted as a $\mathscr{P}$-map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{K})$$

It is clear they satisfy the cocycle condition, so by the Clutching Construction this gives us a unique vector bundle F. Now let $\mathcal{F}$ denote the sheaf of sections of F, so that $\mathcal{F}$ is trivial on each U_α, and fix a frame $(s_i)_\alpha$ for F on each U_α. We define a morphism $\varphi : \mathcal{F} \to \mathcal{E}$ given by

$$\varphi_{U_\alpha}(s_i) := \Phi_\alpha^{-1}(e_i)$$

where $e_i \in \mathcal{O}_X(U_\alpha)^n$ is the obvious constant function, and extend by linearity. It is then not hard to check that these are isomorphisms of $\mathcal{O}_X(U_\alpha)$-modules and that they glue on overlaps, hence we have a morphism of sheaves. Now these are isomorphisms locally, hence φ is an isomorphism of sheaves. This proves (ii).

To prove 3, it suffices to show that the embedding is fully faithful. Let $\varphi : E \to F$ denote a morphism of vector bundles. Now given a section s of E over U, this induces a section $\varphi^\sharp(s)$ of F by defining

$$\varphi^\sharp(s)(p) := \varphi(s(p))$$

and it is not hard to check that this is an $\mathcal{O}_X(U)$-module homomorphism. Since it clearly commutes with restriction, we have a morphism of sheaves $\varphi^\sharp : \mathcal{E} \to \mathcal{F}$. It is clear that $\varphi \mapsto \varphi^\sharp$ is functorial, and that it is faithful. To show that it is full, suppose $\varphi^\sharp : \mathcal{E} \to \mathcal{F}$ is a homomorphism of sheaves. Now we

define a morphism $\varphi : E \to F$ as follows: let $p \in E$ be a point, and suppose s is a section such that $s(\pi(p)) = p$. We then define $\varphi(p) := \varphi^\sharp(s)(\pi(p))$. To see this is well-defined, suppose s' also goes through p. Then

$$0 = \varphi(0)(p) = \varphi^\sharp(s - s')(p) = \varphi^\sharp(s)(p) - \varphi^\sharp(s')(p)$$

and hence φ is well-defined. It is not hard to check that φ is $\mathscr{P}$ and linear on each fibre, and is hence a morphism of vector bundles as desired. This proves (iii). $\qquad\square$

Example B.1.8. Given two bundles E, F, we may interpret these as locally free sheaves $\mathcal{E}, \mathcal{F}$. Then the *tensor product* of E and F, denoted $E \otimes F$ is the bundle associated to $\mathcal{E} \otimes \mathcal{F}$. Similarly, we may define the *direct sum* of E and F to be the bundle associated to $\mathcal{E} \oplus \mathcal{F}$, and the *exterior powers* of E, denoted $\wedge^p E$ to be the bundle associated to the sheaf $\wedge^p \mathcal{E}$. We may also define $\mathcal{H}om(E, F)$ to be the bundle associated to the sheaf-hom $\mathcal{H}om(\mathcal{E}, \mathcal{F})$, and similarly define $\mathcal{E}nd(E)$ to be $\mathcal{H}om(E, E)$. Note that these differ from the **groups** $\mathrm{Hom}(E, F)$ and $\mathrm{End}(E)$.

Henceforth, we will make very little distinction between locally free sheaves and vector bundles.

Example B.1.9. In this example, we extend the constructions in Example B.1.6. Let T_X and T_X^* be the **smooth** tangent and cotangent bundles. We define the *bundle of p-forms*, to be the p-th exterior power of T_X^*. The sections of this bundle will be called *differential p-forms*.

We will conclude this section with a study of the interplay between the smooth and holomorphic tangent bundles. Let $X^\flat$ denote the underlying smooth manifold of X. Picking a local holomorphic chart for X, we have a local diffeomorphism $\mathbb{C}^n \to \mathbb{R}^{2n}$ given by

$$(z_1 = x_1 + iy_1, ..., z_n = x_n + iy_n) \mapsto (x_1, y_1, ..., x_n, y_n).$$

The holomorphic and smooth tangent bundles are then related as follows: observe that the symbols $\partial x_i, \partial y_i$ act on real-valued functions. Tensoring $T_{X^\flat}$ with the trivial bundle $X^\flat \times \mathbb{C}$, these symbols may be interpreted as acting on complex-valued smooth functions, given by

$$\frac{\partial}{\partial x_i}(f + ig) = \frac{\partial f}{\partial x_i} + i\frac{\partial g}{\partial x_i} := \frac{\partial f}{\partial x_i} \otimes 1 + \frac{\partial g}{\partial x_i} \otimes i$$

where f, g are real-valued. Then by the chain rule we have

$$\frac{\partial}{\partial z_i} = \frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}.$$

We thus conclude that the holomorphic tangent space is spanned by $\partial x_i - i\partial y_i$. However, we also see that

$$\frac{\partial}{\partial \bar{z}_i} = \frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}.$$

We call the vector bundle spanned locally by the $\partial \bar{z}_i$ the *antiholomorphic tangent bundle*. Now observe that $T_{X^\flat} \otimes \mathbb{C}$ is the direct sum of the holomorphic and antiholomorphic tangent spaces. We will often write

$$T_{X^\flat} \otimes \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}$$

for this decomposition.

In fact, this extends to differential forms. A *complex (p,q)-form*, or a *complex form of type (p,q)*, or simply (p,q)-form is a section of $\Omega_X^{p,q} := (\wedge^p T_X^{1,0}) \oplus (\wedge^q T_X^{0,1})$. Locally, a (p,q)-form looks like

$$f_1 dz_{i_1} + \ldots + f_p dz_{i_p} + g_1 d\bar{z}_{j_1} + \ldots + g_q \bar{z}_{j_q}$$

where the f_i, g_i are smooth complex-valued functions. A *complex r-form* is a complex (p,q)-form such that $p + q = r$. We will denote the bundle of complex r-forms by Ω_X^r. Observe that we have a decomposition

$$\Omega_X^r = \bigoplus_{p+q=r} \Omega_X^{p,q}.$$

Finally, we define the operators $\partial^{p,q} : \Omega_X^{p,q} \to \Omega_X^{p+1,q}$ and $\bar{\partial}^{p,q} : \Omega_X^{p,q} \to \Omega_X^{p,q+1}$ to be

$$\partial^{p,q} := \pi_{p+1,q} \circ d$$

$$\bar{\partial}^{p,q} := \pi_{p,q+1} \circ d,$$

where $d^{p+q} : \Omega_X^{p+q} \to \Omega_X^{p+q+1}$ is the usual exterior derivative, and the projections are the obvious projections. Note that

$$d^0 = \partial^0 + \bar{\partial}^0.$$

We will often omit the superscripts.

B.2   Connections

In this section, we fix a smooth complex bundle E. We define the *bundle of complex E-valued p-forms*, denoted Ω_E^p, to be $E \otimes \Omega_X^p$, where Ω_X^p is the sheaf of complex p-forms. Note that $\Omega_E^0 \cong E$.

Definition B.2.1. A *connection* on E is a morphism of abelian sheaves (**NOT** as sheaves of modules) $\nabla : \Omega_E^0 \to \Omega_E^1$ that satisfies the following *Leibniz rule* for any $f \in C^\infty(U)$ and local seciton s:

$$\nabla(fs) = df \otimes s + f \nabla(s)$$

Now let $\{U_\alpha\}$ be a trivialising open cover, and let $(s_i)_\alpha$ be a collection of local frames on U_α. We define the *local connection 1-form*, denoted ω_α to be the matrix such that

$$\nabla(s_i) = \sum (\omega_\alpha)_{ij} s_j$$

Observe that if $s = \sum a_i s_i$ is a local section, then

$$\nabla(s) = \sum_j da_j \otimes s_j + \sum_i a_i (\omega_\alpha)_{ij} s_j$$

and hence the local 1-form carries the data of the entire connection.

Observe that the set of connections is an affine space with underlying vector space $H^0(X, \Omega^1_{\mathcal{E}nd(E)})$. In other words, any two connections differ by an $\mathrm{End}(E)$-valued global 1-form, and conversely, if ∇ is a connection and $L \in H^0(X, \Omega^1_{\mathcal{E}nd(E)})$, then $\nabla + L$ is a connection.

Proposition B.2.2. *Let ∇ be a connection, and s_α, t_β two frames and suppose $g : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{C})$ satisfies $t_i = \sum g_{ij} s_j$. Then if ω_α and ω_β are the respective local 1-forms, then we have*

$$\omega_\beta = (dg)g^{-1} + g\omega_\alpha g^{-1} \tag{B.1}$$

Conversely, if $\{U_\alpha\}$ is a trivialising open cover with local frames $\{s_\alpha\}$, and for each α we have a local 1-form ω_α that satisfy (B.1), then there exists a unique connection with local 1-forms ω_α.

Proof. [13, p. 72] □

A connection ∇ induces an operator $\Omega^p_E \to \Omega^{p+1}_E$, by asserting, for any $\eta \in \Omega^p_E$ and $s \in \Omega^0_E$ that

$$\nabla(\eta s) := d\eta \otimes s + \eta \wedge \nabla(s)$$

This allows us to make the following definition:

Definition B.2.3. Let ∇ be a connection. We define the *curvature* of ∇ to be

$$\nabla^2 : \Omega^0_E \to \Omega^2_E$$

Remark B.2.4. This is **not** the Laplacian.

Let us compute the curvature locally. Let $s_\alpha = (s_i)_\alpha$ be a local frame with local 1-form $\omega = \omega_\alpha$. We define the *local curvature matrix* Θ_α such that

$$\nabla^2(s_i) = (\Theta_\alpha)_{ij} s_j$$

Let us compute the curvature locally:

$$\nabla^2(s_i) = \nabla(\sum_j \omega_{ij} s_j) = \sum_j d\omega_{ij} s_j + \omega_{ij} \nabla(s_j) = \sum_j d\omega_{ij} s_j + \sum_j \sum_k \omega_{ij} \wedge \omega_{jk} s_k = \sum_j (d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj}) s_j$$

and hence

$$(\Theta_\alpha)_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj}$$

we commonly just write

$$\Theta_\alpha = d\omega + \omega \wedge \omega$$

Proposition B.2.5. *The curvature operator is an $\mathcal{O}_X$-module homomorphism, where $\mathcal{O}_X$ is the sheaf of smooth functions on X. In particular, the Θ_α glue together to a global $\mathcal{E}\mathrm{nd}(E)$-valued 2-form Θ.*

Proof. One simply checks that ∇^2 is C^∞-linear. □

B.3   Principal Bundles and Connections

In this section, we will discuss principal bundles, with the goal of proving the Riemann-Hilbert correspondence. Our exposition follows [21, pp. 1-15] very closely. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

Definition B.3.1. A *principal G-bundle*, or simply *principal bundle* over X is a fibre bundle $\pi : P \to X$ with fibre G equipped with a right free action $P \times G \to G$ such that the local trivialisations $\Phi_U : U \times G \to \pi^{-1}(U)$ are equivariant, where we define the action of G on $U \times G$ by $(x, g) \cdot h := (x, gh)$.

We see from the definition that the action is fibre-preserving; indeed if $p = \Phi_U(x, h) \in P$ is in the fibre over x, then $p \cdot g = \Phi_U(x, h) \cdot g = \Phi_U(x, hg)$ for any $g \in G$.

Example B.3.2. The trivial bundle $X \times G$ is a principal G-bundle, where the G-action is simply given by right multiplication.

Example B.3.3. Let X be a connected manifold. Then the universal covering map $p : \widetilde{X} \to X$ is a principal $\pi_1(X)$-bundle (where $\pi_1(X)$)) is equipped with the discrete topology.

Example B.3.4. Let $P \to X$ be a principal bundle over X. Then given local trivialisations $\{\Phi_\alpha : U_\alpha \times G \to \pi^{-1}(U_\alpha)\}$, it is easy to see that we can similarly get transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ which satisfy the cocycle condition. Moreover, it is easy to see that the transition functions define P up to isomorphism. In particular, if $G = \mathrm{GL}_n(\mathbb{K})$, then the data of a vector bundle on X is equivalent to the data of a principal $\mathrm{GL}_n(\mathbb{K})$-bundle; in particular there is a canonical bijection between their isomorphism classes. Thus given a vector bundle E, we can define its *frame bundle* $\mathrm{Fr}(E)$ as the principal $\mathrm{GL}_n(\mathbb{K})$-bundle with the same transition functions as E. Conversely, given a principal $\mathrm{GL}_n(\mathbb{K})$-bundle P we can define its *associated vector bundle* $P(\mathbb{K}^n)$ with the same transition functions as P

Extending the previous example, if g is a Riemannian metric on a real vector bundle E of rank n over a real manifold X, then taking local **orthonormal** frames gives us a collection of transition functions with values in $O(n)$. In particular, this defines a principal $O(n)$-bundle, known as its *orthogonal frame bundle*. Conversely, given a principal $O(n)$-bundle P, we can build a real vector bundle E of rank n by considering the transition functions of P as elements of $\mathrm{GL}_n(\mathbb{R})$, and we can define a Riemannian metric on E since we can define orthogonal frames of E via P. Thus the data of a principal $O(n)$-bundle is equivalent to the data of a real vector bundle with a metric. Similarly, a principal $U(n)$-bundle over a complex manifold X is equivalent to a smooth complex vector bundle E equipped with a hermitian metric. In particular, we see that vector bundles and principal bundles are intimately connected. However, their correspondence is not functorial, as we will see now:

Definition B.3.5. If $\pi_P : P \to X$ and $\pi_Q : Q \to Y$ are principal G-bundles, then a *morphism* of principal G-bundles is a pair $(\bar{f}, f)$ where $\bar{f} : P \to Q$ and $f : X \to Y$ are smooth and the following diagram commutes:

$$
\begin{array}{ccc}
P & \xrightarrow{\ \bar{f}\ } & Q \\
\downarrow{\scriptstyle \pi_P} & & \downarrow{\scriptstyle \pi_Q} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

A *gauge transformation* is a morphism of principal G-bundles as defined above where $X = Y$ and f is the identity.

Lemma B.3.6. *Any gauge transformation is an isomorphism.*

Proof. Let $\bar{f} : P \to Q$ be a gauge transformation. On each $x \in X$, $\bar{f}$ induces a map $\bar{f}_x : \pi_P^{-1}(x) \cong G \to \pi_Q^{-1}(x) \cong G$. By the discussion above, this map is simply left multiplication by $\bar{f}_x(1)$, and hence is bijective. Now in a trivialisable open set $U \times G \subseteq P$, the tangent space is canonically isomorphic to $TU \oplus TG$, and on each $(x, g) \in U \times G$ the pushforward is simply $(l_{\bar{f}_x(1)})_*$, which is nonvanishing. Thus by the inverse function theorem, it follows that $\bar{f}$ is an isomorphism. $\qquad\square$

Any element $\eta \in T_e G = \mathfrak{g}$ determines a vector field $\rho(\eta)$ on P given by $\rho_p(\eta) := (j_p)_*(\eta)$, where $j_p : G \to P$ is given by $g \mapsto pg$. This is known as the *fundamental vector field* generated by η. Note that by the properties of pushforwards, if $\eta, \theta \in \mathfrak{g}$ then $\rho([\eta, \theta]) = [\rho(\eta), \rho(\theta)]$.

In the case $P = G$, and the action is simply right multiplication, this vector field is known as the *left-invariant vector field* generated by η, it is called this because it is invariant under $(l_h)_*$ for any $h \in G$; indeed

$$(l_h)_*(\rho_g(\eta)) = ((l_h)_* \circ (p_g)_*)(\eta) = ((l_h)_* \circ (l_g)_*)(\eta) = (l_{hg})_*(\eta) = \rho_{hg}(\eta)$$

Conversely, any vector field that has this property is generated by its value at the identity, thus the space of left-invariant vector fields is in bijection with $\mathfrak{g}$.

There is an alternative characterisation of the fundamental vector field:

Proposition B.3.7.

$$\rho_p(\eta) = \frac{d}{dt}\bigg|_{t=0} p \cdot \exp(t\eta)$$

Proof. Define the curve $\gamma : [0, 1] \to G$ given by $\gamma(t) := \exp(t\eta)$ and observe $\gamma'(0) = \eta$. Thus

$$\begin{aligned}
\rho_p(\eta) &= (j_p)_*(\eta) \\
&= (j_p)_*(\gamma'(0)) \\
&= \frac{d}{dt}\bigg|_{t=0} j_p(\gamma(t)) \\
&= \frac{d}{dt}\bigg|_{t=0} p \cdot \exp(t\eta)
\end{aligned}$$

as desired. $\qquad\square$

B.3.1 Connections on Principal G-Bundles

Let $\pi : P \to X$ be a principal G-bundle. The pushforward π_* induces a map $TP \to \pi^* TX$ given by $u_p \mapsto (p, \pi_*(u_p))$. By local triviality, the this is surjective at every point.

Definition B.3.8. We define *vertical tangent bundle* $T^v P$ to be the kernel of the map $TP \to \pi^* TX$. The elements of $T^v P$ are called *vertical vectors*. A vector field U on P is *vertical* if U_p is vertical for all $p \in P$.

Proposition B.3.9. *For any $p \in P$, the map $\rho_p : \mathfrak{g} \to T_p P$ is an isomorphism onto $T_p^v P$.*

Proof. We first claim that ρ_p is injective. To see this, suppose $\eta \in \ker \rho_p$. Now consider the curve $\gamma_1 : [0, 1] \to P$ given by $\gamma_1(t) := p$. Then γ_1 satisfies $\gamma_1'(t) = \rho_{\gamma_1(t)}(\eta)$ for all $t \in [0, 1]$ with initial condition $\gamma_1(0) = p$. Now define $\gamma_2 : [0, 1] \to P$ given by $\gamma_2(t) := p \cdot \exp(t\eta)$. By the chain rule we have

$$\gamma_2'(t) = \frac{d}{ds}\Big|_{s=0} \gamma_2(t + s) = \frac{d}{ds}\Big|_{s=0} p \cdot \exp(t\eta) \exp(s\eta) = \rho_{\gamma_2(t)}(\eta)$$

and since $\gamma_2'(0) = p$, it follows that γ_1 and γ_2 satisfy the same ODE and have the same initial conditions, and hence are equal. In particular, $\exp(t\eta) = e$ for all t. Since the exponential map is locally bijective, it follows that $\eta = 0$.

Now for any $\eta \in \mathfrak{g}$, define $\gamma : [0, 1] \to P$ given by $\gamma(t) := p \cdot \exp(t\eta)$. We have

$$\pi_*(\rho_p(\eta)) = (\pi \circ \gamma)_*\Big(\frac{d}{dt}\Big|_{t=0}\Big) = 0$$

since $\pi \circ \gamma$ is constant. Thus ρ_p maps into the vertical tangent space. The result then follows from the fact that $\dim T_p^v P = \dim \mathfrak{g}$, which can easily be checked. $\qquad\square$

Corollary B.3.10. *Let U, V be fundamental vector fields on P. Then $[U, V]$ is also a fundamental vector field, and in particular it is vertical.*

We have a short exact sequence:

$$0 \longrightarrow T^v P \longrightarrow TP \longrightarrow \pi^* TX \longrightarrow 0 \tag{B.2}$$

Definition B.3.11. A *connection* on P is a smooth splitting $TP \cong T^v P \oplus T^h P$ of (B.2) such that $(r_g)_*(T_p^h P) = T_{pg}^h P$ for all $p \in P, g \in G$ (this is sometimes also known as *equivariance*). The elements of $T_p^h P$ are known as *horizontal vectors*. A vector field U on P is *horizontal* if U_p is horizontal for all $p \in P$.

Example B.3.12. Let $P = X \times G$ be the trivial bundle. Let $\pi_1 : P \to X$ and $\pi_2 : P \to G$ denote the projections onto the first and second factors respectively. Then one can show that $TP \cong \ker \pi_{1*} \oplus \ker \pi_{2*}$. We claim that this splitting is a connection. It is clearly smooth, thus it suffices to check equivariance, that is $(\ker \pi_{2*})_p = (\ker \pi_{2*})_{pg}$ for any $p = (x, h) \in P$ and $g \in G$. To this end, observe $(r_g \circ \pi_2)(p) = hg = (\pi_2 \circ r_g)(p)$, where by abuse of notation we use r_g to denote right multiplication by g in both P and G. Thus by the chain rule

$$(r_g)_* \circ (\pi_2)_* = (r_g \circ \pi_2)_* = (\pi_2 \circ r_g)_* = (\pi_2)_* \circ (r_g)_*$$

since $(r_g)_*$ is bijective the result follows. This connection is known as the *trivial connection*.

Let $TP \cong T^v P \oplus T^h P$ be a connection, and for any point p let $\omega_p : T_p P \to \mathfrak{g}$ denote the natural projection map composed with ρ_p^{-1}. As p run through P, ω may be considered as a $\mathfrak{g}$-valued 1-form, that is a section of $T^* X \otimes (\mathfrak{g} \times X)$.

Proposition B.3.13. *The form ω satisfies the following two properties:*

(i) *For any $\eta \in \mathfrak{g}$ and $p \in P$, we have $\omega_p(\rho_p(\eta)) = \eta$.*

(ii) *For all $g \in G$ we have $(r_g)^*\omega = (\operatorname{Ad} g)\omega$.*

Proof. [46, p. 255]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Clearly we can recover $T^h P$ from ω by defining $T_p^h P := \ker \omega_p$ for every $p \in P$. The next theorem shows the converse to this:

Theorem B.3.14. *Let ω be a $\mathfrak{g}$-valued 1-form that satisfies the two properties of Proposition B.3.13. Then $T_p P \cong T^v P \oplus \ker \omega_p$ is a connection.*

Proof. [46, pp. 257-258]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Thus the notion of a connection and a 1-form satisfying the two properties of Proposition B.3.13 are equivalent. For this reason, such a 1-form is sometimes known as a *connection 1-form*, or simply a *connection form*.

Example B.3.15. Consider the trivial connection on the trivial bundle as in Example B.3.12. Then the associated 1-form is simply $\omega = \rho^{-1} \circ (\pi_2)_*$. Let U, V be horizontal vector fields on P, and fix some $g \in G$. Then we have an inclusion map $i_g : X \to P$ given by $x \mapsto (x, g)$. Then $U_{(x,g)} = (i_g \circ \pi_1)(U_{(x,g)})$ for any $x \in X$ and similarly with V, hence

$$[U, V]_{(x,g)} = [(i_g \circ \pi_1)(U), (i_g \circ \pi_1)(V)]_{(x,g)} = (i_g \circ \pi_1)([U, V]_{(x,g)})$$

and hence $[U, V]$ is horizontal (we can also see this using coordinates) and ω is flat.

Example B.3.16. In the special case where X has zero dimension, ω above is known as the *Maurer-Cartan* form, and it is denoted θ. An alternative description is

$$\theta_g = (l_{g^{-1}})_* : T_g G \to T_e G \cong \mathfrak{g}$$

Definition B.3.17. Let ω be a connection 1-form on a principal bundle $P \to X$. Then the *curvature form* of ω is the 2-form given by

$$F_\omega := d\omega + [\omega, \omega]$$

where $[\omega, \omega](u, v) := [\omega(u), \omega(v)]$. We say ω is *flat* if $F_\omega = 0$. A connection is said to be *flat* if its associated 1-form is flat.

Theorem B.3.18. *A connection ω is flat if and only if $[U, V]$ is horizontal for any horizontal vector fields U and V.*

Proof. We first claim any two vertical vectors annihilate F. Indeed, fix some $p \in P$, suppose $u, v \in T_p^v P$ and let U, V denote the fundamental vector fields generated by $\rho_p^{-1}(u) = \omega_p(u)$ and $\rho^{-1}(v)$

respectively. Then it is easy to see that $\omega(U) = u$ and $\omega(V) = v$ globally, and in particular it has zero directional derivative. Then

$$
\begin{aligned}
F_{\omega,p}(u,v) &= d\omega_p(u,v) + [\omega(u),\omega(v)] \\
&= U(\omega(V))_p - V(\omega(U))_p - \omega_p([U,V]) + [\omega(u),\omega(v)] \\
&= -\omega_p([U,V]) + [\omega(u),\omega(v)] \\
&= -[u,v] + [u,v] \\
&= 0
\end{aligned}
$$

Where the equality $\omega_p([U,V]) = [u,v]$ comes from Corollary B.3.10. This proves the claim.

Now let U, V be two horizontal vector fields. Then $\omega(U) = \omega(V) = 0$ by definition, hence

$$
\begin{aligned}
F_{\omega,p} &= d\omega_p(u,v) + [\omega(u),\omega(v)] \\
&= -\omega_p([U,V])
\end{aligned}
$$

It thus follows that F_ω is 0 constantly if and only if $\omega_p([U,V]) = 0$ for all p as desired. $\qquad\square$

Finally, we will give a local description of a connection. Let ω be a connection 1-form on a (smooth) principal G-bundle $P \to X$, let $\{U_\alpha\}$ be an open cover of X and let $s_\alpha : U_\alpha \to P$ be local smooth sections for each α. Then define $\omega_\alpha := s_\alpha^* \omega$.

Proposition B.3.19. *If s_β is another section and $s_\beta = s_\alpha g_{\alpha\beta}$, then we have*

$$
\omega_\beta = \mathrm{Ad}_{g_{\alpha\beta}} \omega_\alpha + g_{\alpha\beta}^* \theta \tag{B.3}
$$

where θ is the Maurer-Cartan form;

Proof. We begin by computing the pushforward of the map s_β. We do this by decomposing s_β into a map $U \to P \times G \to P$ given by

$$
x \mapsto (s_\alpha(x), g_{\alpha\beta}(x)) \mapsto s_\alpha(x) g_{\alpha\beta}(x) = s_\beta(x)
$$

Now fix one such x and write $g := g_{\alpha\beta}(x)$ and $p := s_\alpha(x)$. Applying the chain rule, we obtain, for any $u \in T_x(U_\alpha \cap U_\beta)$:

$$
(s_\beta)_{*,x}(u) = (r_g)_{*,p}((s_\alpha)_{*,x}(u)) + (j_p)_{*,g}((g_{\alpha\beta})_{*,x}(u))
$$

And now we compute:

$$
\begin{aligned}
\omega_\beta(u) &= (s_\alpha g_{\alpha\beta})^*(\omega)_x(u) \\
&= \omega_{pg}((r_g)_{*,p}((s_\alpha)_{*,x}(u)) + (j_p)_{*,g}((g_{\alpha\beta})_{*,x}(u))) \\
&= \omega_{r_g(p)}((r_g)_{*,p}((s_\alpha)_{*,x}(u))) + \omega_{pg}((j_p)_{*,g}((g_{\alpha\beta})_{*,x}(u))) \\
&= (r_g)^*(\omega)_p((s_\alpha)_{*,x}(u)) + \omega_{pg}(j_{pg})_{*,e}(\ell_{g^{-1}})_{*,g}((g_{\alpha\beta})_{*,x}(u))) \\
&= \mathrm{Ad}_g \omega_\alpha(u) + (\theta_{g_{\alpha\beta}(x)})((g_{\alpha\beta})_{*,x}(u)) \\
&= \mathrm{Ad}_g \omega_\alpha(u) + g_{\alpha\beta}^* \theta(u)
\end{aligned}
$$

as desired. $\qquad\square$

Conversely, a collection of local $\mathfrak{g}$-valued 1-forms ω_α and local sections s_α that satisfy (B.3) determine a connection uniquely.

Example B.3.20. In the case $G = \mathrm{GL}(V)$, the form ω_α is simply an $n \times n$ matrix of 1-forms (where $n := \dim V$). It can also be seen that

$$\mathrm{Ad}_{g_{\alpha\beta}} \omega_\alpha = g_{\alpha\beta}^{-1} \omega_\alpha g_{\alpha\beta}$$

and a calculation will show that

$$g_{\alpha\beta}^* \theta(u_x) = \theta_{g_{\alpha\beta}(x)}((g_{\alpha\beta})_*(u_x)) = g_{\alpha\beta}^{-1} dg_{\alpha\beta}(u_x)$$

for any $u_x \in T_x X$ and hence the ω_α are nothing more or less than a collection of matrices that satisfy (B.2.2); in other words they define a unique connection ∇ on the associated bundle $P(V)$. The frame $(s_i)_\alpha$ upon which ω_α is the associated matrix is given by $s_i = (s_\alpha, e_i)$, where (e_i) is the standard basis of V. Conversely, a connection ∇ on E uniquely defines a connection on $\mathrm{Fr}(E)$ in exactly the same way. Thus the notions of a connection on principal $\mathrm{GL}(V)$ bundles and vector bundles are equivalent.

Example B.3.21. Similarly, given $G = U(n)$, the local 1-forms are valued in $\mathfrak{u}(n)$, the algebra of skew-hermitian n-by-n matrices. But that is exactly the data that defines a unitary connection on the associated bundle.

B.3.2 Parallel Transport and Holonomy

Let $\pi : P \to X$ be a principal G-bundle with connection ω.

Definition B.3.22. A curve $\gamma : [0, 1] \to P$ is said to be *horizontal* if $\gamma'(t)$ is horizontal for all t.

Lemma B.3.23. *Let $\gamma : [0, 1] \to P$ be a horizontal curve and suppose $g \in G$. Then $r_g \circ \gamma$ is also horizontal.*

Proof. This is simply a consequence of the fact that $(r_g)_*$ preserves the horizontal tangent space. $\square$

Theorem B.3.24. *Let $\gamma : [0, 1] \to X$ be a piecewise smooth curve. Then for $p \in P$ such that $\pi(p) = \gamma(0)$, there exists a unique horizontal $\gamma_p^\sharp : [0, 1] \to P$ such that $\gamma_p^\sharp(0) = p$ and $\pi \circ \gamma_p^\sharp = \gamma$.*

Proof. This is just a first order ODE with an initial condition. $\square$

Definition B.3.25. Let $\gamma : [0, 1] \to X$ be a piecewise smooth curve in X. Then for $p \in \pi^{-1}(\gamma(0))$, we define the *parallel transport of p along γ* as

$$P_\gamma^\omega(p) := \gamma_p^\sharp(1)$$

For the rest of this section, any curve will be assumed to be piecewise smooth. Before we proceed further, we present two elementary properties of parallel transport:

Lemma B.3.26. *Suppose $x \in X$ and let $\gamma_1, \gamma_2 : [0,1] \to X$ be two curves such that $\gamma_1(0) = x$ and $\gamma_2(0) = \gamma_1(1)$. If $p \in \pi^{-1}(x)$, then*

$$(\gamma_2 \cdot \gamma_1)^{\sharp}_p = (\gamma_2)^{\sharp}_{(\gamma_1)^{\sharp}_p(1)} \cdot (\gamma_1)^{\sharp}_p$$

$$P^{\omega}_{\gamma_2 \cdot \gamma_1}(p) = P^{\omega}_{\gamma_2}(P^{\omega}_{\gamma_1}(p))$$

Proof. This follows immediately from the definitions. $\qquad\square$

Lemma B.3.27. *Let $\gamma : [0,1] \to X$ be a curve. Then for any $g \in G$ and $p \in \pi^{-1}(x)$ we have*

$$(r_g)_* \circ \gamma^{\sharp}_p = \gamma^{\sharp}_{pg}$$

and consequently

$$P^{\omega}_{\gamma}(p)g = P^{\omega}_{\gamma}(pg)$$

.

Proof. This is a consequence of Lemma B.3.23 and Theorem B.3.24. $\qquad\square$

Definition B.3.28. Let $\gamma : [0,1] \to X$ be a loop in X and let $x := \gamma(0) = \gamma(1)$. Then for $p \in \pi^{-1}(x)$, we define the *holonomy* of γ with respect to p, denoted $\mathrm{Hol}_p(\omega, \gamma)$ to be the unique $g \in G$ such that $p \cdot g = \gamma^{\sharp}_p(1)$.

We list some properties of holonomy:

Lemma B.3.29. *Let $\gamma_1, \gamma_2 : [0,1] \to X$ be loops, let $x := \gamma_1(0)$ and let $\delta : [0,1] \to X$ be a path from x to $y := \delta(1)$. Then for $p \in \pi^{-1}(x)$ and $g \in G$, we have the following:*

(i) $\mathrm{Hol}_{pg}(\omega, \gamma_1) = \mathrm{Hol}_p(\omega, \gamma_1)^g$.

(ii) $\mathrm{Hol}_p(\omega, \gamma_1) = \mathrm{Hol}_{P^{\omega}_{\delta}(p)}(\omega, \delta \cdot \gamma_1 \cdot \delta^{-1})$.

(iii) $\mathrm{Hol}_p(\omega, \gamma_1 \cdot \gamma_2) = \mathrm{Hol}_p(\omega, \gamma_2) \, \mathrm{Hol}_p(\omega, \gamma_1)$.

Recall that $\gamma_1 \cdot \gamma_2$ means γ_1 is traversed after γ_2.

Proof. Observe

$$pg \cdot (\mathrm{Hol}_p(\omega, \gamma_1)^g) = pgg^{-1} \mathrm{Hol}_p(\omega, \gamma_1)g = p \, \mathrm{Hol}_p(\omega, \gamma_1)g = P_{\gamma_1}(pg)$$

this proves (1). To prove (2), observe

$$(\delta \cdot \gamma_1 \cdot \delta^{-1})^{\sharp}_{P^{\omega}_{\delta}(p)} = \delta^{\sharp}_{P^{\omega}_{\gamma_1}(p)} \cdot (\gamma_1)^{\sharp}_p \cdot (\delta^{-1})^{\sharp}_{P^{\omega}_{\delta}(p)}$$

Hence

$$
\begin{aligned}
P_\delta^\omega(p) \cdot \mathrm{Hol}_{P_\delta^\omega(p)}(\omega, \delta \cdot \gamma_1 \cdot \delta^{-1}) &= P_{\delta \cdot \gamma_1 \cdot \delta^{-1}}^\omega(P_\delta^\omega(p)) \\
&= P_{\delta \cdot \gamma_1}^\omega(p) \\
&= P_\delta^\omega(P_{\gamma_1}^\omega(p)) \\
&= P_\delta^\omega(p\,\mathrm{Hol}_p(\omega, \gamma_1)) \\
&= P_\delta^\omega(p)\,\mathrm{Hol}_p(\omega, \gamma_1)
\end{aligned}
$$

This proves (2). Finally, we have

$$
\begin{aligned}
p \cdot \mathrm{Hol}_p(\omega, \gamma_1 \cdot \gamma_2) &= P_{\gamma_1 \cdot \gamma_2}^\omega(p) \\
&= P_{\gamma_1}^\omega(P_{\gamma_2}^\omega(p)) \\
&= P_{\gamma_1}^\omega(p\,\mathrm{Hol}_p(\omega, \gamma_2)) \\
&= P_{\gamma_1}^\omega(p)\,\mathrm{Hol}_p(\omega, \gamma_2) \\
&= p\,\mathrm{Hol}_p(\omega, \gamma_2)\,\mathrm{Hol}_p(\omega, \gamma_2)
\end{aligned}
$$

which proves (3). $\qquad\square$

Proposition B.3.30. *Suppose ω is flat. Then any contractible loop has trivial holonomy.*

Proof. Let γ be a contractible loop based at $x \in X$ and fix any $p \in \pi^{-1}(x)$. Let $P(p)$ denote the set of points in P that are connected to p by a horizontal path. Since ω is flat, TH is closed under the Lie bracket and thus it is completely integrable (that is, for all $x \in X$ there is an immersed submanifold $Y_x \ni x$ such that $T_pY_x = T_pH$ for all $p \in Y$.) by the Frobenius Integrability Theorem ([6, pp. 36, 37]). In particular, that means $P(p)$ is an immersed submanifold of P. By a dimension count we observe that the restricted map $\pi|_{P(a)}$ has an isomorphic pushforward; and is thus locally a diffeomorphism by the Inverse Function Theorem.

Now it suffices to show that $\pi|_{P(a)}$ is a covering map, since that would mean γ lifts to a loop in $P(a)$ by the Covering Homotopy Theorem [37, Theorem 10.5]. To see this, let $V \subseteq P(a)$ be an open subset such that $\pi|_V$ is a diffeomorphism, and let $U \subseteq X$ denote the image of V. Now it is easy to see that for any $q \in V$, if $g \in G$ satisfies $qg \in P(p)$ then $Vg \subseteq P(a)$. It thus follows that $Vg \cap P(a)$ is either empty or equal to Ug for all $g \in G$ hence $\pi|_{P(a)}^{-1}(U) = \bigsqcup_{g \in G}(Vg \cap P(a))$ as desired. $\qquad\square$

This implies that if our connection is flat, then holonomy of a loop is dependent only on the homotopy class of the loop. In particular, that means for any $x \in X$ and $p \in \pi^{-1}(x)$ there is a well-defined map $\rho_{\omega,p} : \pi_1(X, x) \to G$ given by

$$
[\gamma] \mapsto \mathrm{Hol}_p(\omega, \gamma)^{-1}
$$

and in fact this is a group homomorphism by property (3) of Lemma B.3.29. This map is known as the *holonomy representation*. By property (1) of Lemma B.3.29, this representation is defined up to conjugation independently of our choice of p.

B.3.3   The Riemann-Hilbert Correspondence

Theorem B.3.31 (Riemann-Hilbert). *Let G be a Lie group and X a connected manifold. Fix some $x \in X$. Then there is a bijection between the set of pairs (P, ω) where P is some principal G-bundle and ω is a flat connection on P, up to gauge equivalence, and G-representations of $\pi_1(X, x)$ up to conjugation.*

Proof. We will show the map $(P, \omega) \mapsto \rho_{\omega, p}$, where $p \in \pi^{-1}(x)$ is any element in the fibre of x in P and $\rho_{\omega, p}$ is the holonomy representation, is a bijection between the set of pairs (P, ω) up to gauge equivalence and G-representations of the fundamental group up to conjugation. Note that if we choose some other $pg \in \pi^{-1}(x)$, then $\rho_{\omega, pg} = \rho_{\omega, p}^g$ by Lemma B.3.29, thus this representation is well-defined up to conjugation.

First, let $\rho : \pi_1(X, x) \to G$ be a representation. Now let $\widetilde{X}$ denote the universal cover of X, define $\widetilde{P} := \widetilde{X} \times G$ and let $\widetilde{\omega}$ denote the form associated to the trivial connection on $\widetilde{P}$. Now there is a natural isomorphism $\mathrm{Deck}(\widetilde{X}) \cong \pi_1(X, x)$, where $\mathrm{Deck}(\widetilde{X})$ is the group of deck transformations of $\widetilde{X} \to X$ ([37, Corollary 10.29]). Since $\mathrm{Deck}\,\widetilde{X}$ acts transitively on the fibres of $\widetilde{X}$ ([37, Theorem 10.18]), the quotient of this action is X. Now we define an action of $\pi_1(X, x)$ on $\widetilde{P}$ as follows:

$$[\gamma] \cdot (\widetilde{x}, g) := ([\gamma] \cdot \widetilde{x}, g\rho([\gamma])^{-1})$$

We can also identify the tangent spaces $T_{(\widetilde{x}, g)}\widetilde{P}$ and $T_{[\gamma] \cdot (\widetilde{x}, g)}\widetilde{P}$ by the pushforward induced by this action. Now we define P to be the quotient of $\widetilde{P}$ by this action; we can easily show that this is a principal G-bundle. Observe that $\widetilde{\omega}$ induces a connection on P, which we will denote ω. It is easily checked that ω is also flat.

Now we may compute the holonomy of (P, ω). Suppose $[\gamma] \in \pi_1(X, x)$, and let $(\widetilde{x}, g) \in P$ be a representative of some point in the fibre of x. Then

$$P_\gamma(\widetilde{x}, g) = [\gamma] \cdot (\widetilde{x}, g) = ([\gamma] \cdot \widetilde{x}, g) = (\widetilde{x}, g(\rho([\gamma])^{-1})) = (\widetilde{x}, g) \cdot \rho([\gamma])^{-1}$$

hence $\mathrm{Hol}_{(\widetilde{x}, g)}(\omega, [\gamma]) = \rho([\gamma])^{-1}$ as required. This proves surjectivity.

Now suppose that (P, ω) and (P', ω') have the same holonomy representation, that is for $p \in P$ and $p' \in P'$ we have $\rho_{\omega, p} = \rho_{\omega, p'}$ (observe that by Lemma B.3.29 we can assume that the representations are in fact equal and not just conjugate equivalent). We will construct a gauge equivalence $\Phi : P \to P'$ that preserves horizontal vectors as follows: for any path δ from x to y in X, we define

$$\Phi(P_\delta^\omega(p)) := P_\delta^{\omega'}(p')$$

To see that this is well-defined, suppose δ' is some other path from x to y such that $P_\delta^\omega(p) = P_{\delta'}^\omega(p)$. Then $\mathrm{Hol}_p(\omega, \delta^{-1} \cdot \delta') = 1 = \mathrm{Hol}_{p'}(\omega', \delta^{-1} \cdot \delta')$, hence

$$P_\delta^{\omega'}(p') = P_{\delta'}^{\omega'}(p') = (\delta^{-1} \cdot \delta')_{p'}^\sharp(1/2) \tag{B.4}$$

This determines Φ on all points q with a horizontal path to p. Now we require for all $g \in G$, that $\Phi(qg) = \Phi(q)g$. To see that this is well-defined, suppose that $P_{\delta_1}^\omega(p)g_1 = P_{\delta_2}^\omega(p)g_2$. By multiplying

by g_2^{-1} we may assume $g_2 = 1$. Writing $g := g_1$ we observe

$$g = \mathrm{Hol}_{P_{\delta_2}^{\omega}(p)}(\omega, \delta_1 \cdot \delta_2^{-1})$$
$$= \mathrm{Hol}_p(\omega, \delta_2^{-1} \cdot \delta_1)$$
$$= \mathrm{Hol}_{p'}(\omega', \delta_2^{-1} \cdot \delta_1)$$
$$= \mathrm{Hol}_{P_{\delta_1}^{\omega'}(p')}(\omega', \delta_1 \cdot \delta_2^{-1})$$

and hence $P_{\delta_1}^{\omega'}(p')g = P_{\delta_2}^{\omega'}(p')$ as desired. Since X is path connected this now defines Φ on the entirety of P. Also for this reason, this map is surjective. To see that Φ is injective, suppose that $\Phi(p_1) = \Phi(p_2)$. By multiplying by elements of G, we may assume that $p_1 = P_{\delta}^{\omega}(p)$ for some path δ. Then $\Phi(p_2) = P_{\delta}^{\omega'}(p')$ and hence $p_2 = P_{\delta}^{\omega}(p) = p_1$. $\qquad\square$

Example B.3.32. Being the free group on one element, there is a canonical isomorphism $\mathrm{Hom}_{\mathsf{Gps}}(\mathbb{Z}, G) \cong G$ for any group G. In the case $G = \mathbb{R}^*$, the Riemann-Hilbert correspondence gives a perverse way to see this: firstly, let $U_1 = S^1 \smallsetminus \{1\}$ and $U_2 = S^1 \smallsetminus \{-1\}$. Let $u \in \mathbb{R}^*$, and let $E \to S^1$ be the (real) line bundle defined by gluing $U_1 \times \mathbb{R}$ and $U_2 \times \mathbb{R}$ together with locally constant transition function

$$g_{1,2}(z) := \begin{cases} 1 & \text{if } \Im(z) > 0 \\ u & \text{if } \Im(z) < 0 \end{cases}$$

on the intersection of U_1 and U_2 (which is just the intersection of S^1 and the union of the two half-planes). Observe that if $u < 0$ we get a Möbius strip, but if $u > 0$ we get a cylinder. Now define s to be the section $s(z) := (z, 1) \in U_1 \times \mathbb{R}$, and define ω to be the connection form which is zero on s. Now to compute the holonomy, we define the path $\gamma : [0, 1) \to E$ as

$$\gamma(t) := \begin{cases} (1, 1) \in U_2 \times \mathbb{R} & \text{if } t = 0 \\ s(\exp(2i\pi t)) & \text{otherwise} \end{cases},$$

which has winding number 1, and thus is a generator of $\pi_1(S^1)$, and note that

$$\lim_{t \to 1} \gamma(t) = (1, u) \in U_2 \times \mathbb{R};$$

and in particular the holonomy is just u. This defines the map $\mathbb{Z} \cong \pi_1(S^1) \to \mathbb{R}^*$ sending 1 to u.

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