Analytic Torsion and the Cheeger-Müller Theorem

Elizabeth Jagersma

Department of Mathematics
The University of Auckland

Supervisor: Pedram Hekmati

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Abstract

Reidemeister torsion (or R-torsion) was originally introduced by K. Reidemeister in 1935, who used it to classify 3-dimensional lens spaces. R-torsion is a homeomorphism invariant which may be defined using core concepts in algebraic topology and linear algebra.

Later, in 1971, D. Ray and I. Singer defined an analytic analogue of R-torsion, which involved using the zeta function to define a regularized determinant of the Laplacian on the space of differential forms. After proving that their analytic torsion (which has come to be known as Ray-Singer torsion, or RS-torsion) satisfies many of the same properties of R-torsion, Ray and Singer conjectured that RS-torsion and R-torsion are equal for closed Riemannian manifolds, and provided computational evidence. This conjecture was proven independently in celebrated papers by W. Müller and J. Cheeger.

In 1994, J. M. Bismut and W. Zhang gave an analytic proof of a generalization of the Cheeger-Müller theorem. Their approach utilizes the Witten deformation of the Laplacian to factorize the Ray-Singer torsion into large and small components, which then may be analyzed separately. In 2003, M. Braverman gave another proof which uses Bismut and Zhang's analysis of the small component of the RS-torsion, but introduces a clever comparison analysis of the large component of the RS-torsion.

In this thesis we present Braverman's analytic approach. However, we also provide original proofs for some of the results which are used.

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Contents

Chapter 1

Introduction

Torsion is a powerful and intensely studied invariant associated to a cochain complex endowed with a distinguished basis, or equivalently, an inner product. Reidemeister first introduced torsion in 1935 to classify the *Lens spaces*, $L(p,q)$, where p and q are coprime integers. The 3-dimensional lens spaces are obtained as a quotient of S^3 by an action of $\mathbb{Z}_p$. In the same year Wolfgang Franz, who was a student of Reidemeister, extended R-torsion to arbitrary dimensions and used it to obtain the combinatorial classification of higher dimensional lens spaces.

Remarkably, for fixed p and varying q, the spaces $L(p,q)$ could not be classified up to homotopy or homeomorphism using homology or homotopy groups. Eventually, the lens spaces were classified up to homeomorphism using *Reidemeister torsion*, or R-torsion, which is the torsion associated to the simplicial complex. Thus, Reidemeister torsion is a secondary invariant; it can distinguish homeomorphism classes *within* homotopy classes.

In 1971, Ray and Singer defined an analytic analogue of R-torsion for smooth manifolds, called Ray-Singer analytic torsion or RS-torsion [RS71]. RS-torsion is the torsion of the de Rham complex. As a real vector space, the de Rham complex is infinite dimensional, so Ray and Singer utilized the zeta function to define the regularized determinant of the Laplacian, which was then used to define RS-torsion.

In their paper, Ray and Singer computed the RS-torsion for S^1 and found that it was equal to the R-torsion for S^1. Additionally, they proved that RS-torsion was independent of the metric and satisfied many of the same properties as R-torsion. Thus, they conjectured that R-torsion and RS-torsion are equal for closed Riemannian manifolds. This conjecture was proven independently in the celebrated papers of Cheeger [Che79] and Müller [Mül78], and is now known as the Cheeger-Müller theorem. Cheeger's approach heavily utilized surgery techniques, while Müller used the fact that torsion is invariant under cellular subdivision.

Since its introduction, RS-torsion has been widely studied. Notably, it features as the partition function in Schwarz's BF-theory [Sch78].

Later in 1992, Bismut and Zhang generalized the Cheeger-Müller theorem with a direct analytic proof [BZ92]. Their proof is a generalization in the following sense. Originally, Ray and Singer only defined analytic torsion when the closed Riemannian manifold (M, g^{TM}) is equipped with a flat vector bundle associated to an *orthogonal* representation of the fundamental group $\pi_1(M)$. In this case, the Ray-Singer torsion is independent of the metric [RS71]. Bismut and Zhang allowed a flat vector bundle F associated to any representation, but required that F is equipped with a metric which induces a flat metric on the determinant line $\det F = \bigwedge^{\mathrm{rank}(F)} F$. In this case, the Ray-Singer torsion remains a topological invariant when M is odd dimensional, however when M is even dimensional the Ray-Singer torsion varies with the metric. Thus they proved an anomaly formula, which tracks the variation of the Ray-Singer torsion with the metric, and using that formula they related the *Milnor torsion* to the Ray-Singer torsion.

The Milnor torsion is known to be equal to the Reidemeister torsion on CW-complexes [Mil66], and is much more convenient to use in the analytic approach.

Bismut and Zhang's 1992 proof utilizes the Witten deformation of the Laplacian to factorize the Ray-Singer torsion into two components. In the large t limit, the spectrum of the Witten Laplacian is split into two components by the interval $(e^{-tC_1}, C_2 t)$, for some positive constants $C_1, C_2 > 0$. Thus we may consider small and large eigenvalues of the Witten Laplacian, and compute the Ray-Singer torsion separately on subspaces spanned by small and large eigenforms. We will show that the product of the small and large torsion recovers the whole Ray-Singer torsion, so it sufficies to compute the asymptotics of these components independently.

Bismut and Zhang's original analysis of the small torsion was later simplified in 1994 [BZ94]. Although [BZ94] proves the Cheeger-Müller theorem in the equivariant setting, we can still use their results by assuming the group action is trivial.

In 2003, Braverman provided a simplified analysis of the large torsion when M is odd dimensional [Bra03]. By combining Bismut-Zhang's and Braverman's results, we obtain a simplified analytic proof of the Cheeger-Müller theorem for odd dimensional closed Riemannian manifolds.

Additionally, we provide original proofs for some of the results which are used. In particular the proofs for Theorem 4.2.1, Proposition 6.1.3, Proposition 7.2.2, Lemma 7.2.6, Corollay 6.0.2, and the construction of certain Morse functions on $M \times S^1 \times S^1$ and $M \times S^2$ are original, among others.

The rest of this thesis is structured as follows. In the remainder of Chapter 1 we give a brief sketch of the proof.

In Chapter 2 we define the torsion for a cochain complex of finite dimensional vector spaces and the torsion metric, which is a more natural conceptualization of torsion.

In Chapter 3 we define the Reidemeister and Milnor metrics.

In Chapter 4 we define the Ray-Singer torsion and Ray-Singer metric and prove a product formula.

In Chapter 5 we give a proof of the spectral gap theorem for the Witten Laplacian.

In Chapter 6 we present Bismut and Zhang's analysis of the asymptotics of the small torsion.

In Chapter 7 we present Braverman's analysis of the asymptotics of the large torsion, and then prove the Cheeger-Müller theorem.

1.1 Outline of the Proof

Let M be a closed n-dimensional Riemannian manifold, and let $F \to M$ be a flat real vector bundle over M. Suppose that F is equipped with a Euclidean metric g^F so that the induced metric on the determinant line bundle $\det F := \bigwedge^{\operatorname{rank} F} F$ is flat. Using this data, we may define the *Ray-Singer metric* or *RS metric* on the determinant line $\det H^\bullet(M, F)$. We will denote the RS metric by $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$. If n is odd, or if the vector bundle F is associated to an orthogonal representation, then $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$ is a topological invariant of M, so it does not depend on the choices of metrics g^{TM} or g^F. Additionally, when n is even and g^F is not assumed to be flat, the variation of $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$ with respect to different choices of Riemannian metric is known [BZ92]. More details on the construction of $\|\cdot\|^{RS}_{\det H^\bullet(M,F)}$ are given in Chapter 4.

Suppose that $f : M \to \mathbb{R}$ is a Morse function which satisfies the *Thom-Smale transversality conditions*. Using this data, we may define the *Milnor metric* on the determinant line, $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$. Milnor [Mil66] proved that $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ is actually a topological invariant, so it does not depend on the choice of Morse function f. More details on the construction of $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ are given in Chapter 3.

If n is odd, or if g^F is flat, then the Cheeger-Müller Theorem states that

Theorem 1.1.1.

$$\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)} = \|\cdot\|^{RS}_{\det H^\bullet(M,F)}. \tag{1.1}$$

A generalized version of the Cheeger-Müller Theorem, where n is not necessarily odd and g^F

is not necessarily flat has been proven by Bismut and Zhang [BZ92]. In this thesis, however, we will restrict ourselves to the case where n is odd so we may use the elegant results of Braverman [Bra03].

The following is a sketch of the proof of Theorem 1.1.1. Consider the de Rham complex $\Omega^\bullet(M,F)$ with coefficients in the vector bundle F. Using the metrics g^{TM} and g^F, we may equip $\Omega^\bullet(M,F)$ with an inner product, and consider the Hilbert space $\overline{\Omega^\bullet(M,F)}$, the completion of $\Omega^\bullet(M,F)$. Then we may define a formal adjoint of the differential $d^F : \Omega^\bullet(M,F) \to \Omega^{\bullet+1}(M,F)$, which we will denote by δ^F, and the Laplacian $\Delta := d^F\delta^F + \delta^F d^F$.

By Hodge Theory, we have the canonical decomposition $\Omega^i(M,F) = \ker\Delta^i \oplus \operatorname{im} d^F \oplus \operatorname{im}\delta^F$, where Δ^i is the restriction of Δ to $\Omega^i(M,F)$. Consequently, $\ker\Delta^i$ is canonically isomorphic to the i-th de Rham cohomology group $H^i_{dR}(M,F)$. As a subspace of $\Omega^i(M,F)$, $\ker\Delta^i$ naturally inherits an inner product from the inner product on $\Omega^\bullet(M,F)$, which is induced from the metrics g^{TM} and g^F. This inner product yields a metric on $H^i_{dR}(M,F) \cong H^i(M,F)$, which defines a metric on the determinant line $\det H^\bullet(M,F)$ called the *Hodge metric* and denoted by $|\cdot|^{Hodge}_{\det H^\bullet(M,F)}$.

It turns out that the Milnor metric is related to the Hodge metric by the *Milnor torsion*, $\mathcal{M}(M,f)$. In particular, $\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)} = |\cdot|^{Hodge}_{\det H^\bullet(M,F)} \cdot \mathcal{M}(M,f)$. Also, the RS metric is by definition related to the Hodge metric via the Ray-Singer torsion τ_{RS} in the same way.

Now we will consider the *Witten deformation* of the differential, $d^F_t := e^{-tf}d^F e^{tf}$, which is a family of differentials on $\Omega^\bullet(M,F)$ parametrized by $t \in \mathbb{R}$. Let $\delta^F_t := d^{F*}_t$. Then the *Witten Laplacian* is defined as $\Delta_{f,t} := d^F_t\delta^F_t + \delta^F_t d^F_t$. We may define the Witten deformed Ray-Singer torsion $\tau_{RS}(f,t)$ by using the Witten Laplacian rather than the usual Laplacian. However, crucially, since the dimension of M is odd, τ_{RS} is a topological invariant, so $\tau_{RS} = \tau_{RS}(f,t)$. This is useful, because the Witten Laplacian has a *spectral gap property*.

Theorem 1.1.2. *There exist positive constants C_1, C_2 and $t_0 > 1/C_2$ so that for $|t| \geq t_0$,* $\operatorname{spec}(\Delta_{f,t}) \subset [0, e^{-|t|C_1}) \cup (C_2|t|, \infty]$.

We may split the spectrum of $\Delta_{f,t}$ into small eigenvalues, which belong to $[0, e^{-|t|C_1}]$, and large eigenvalues, which belong to $[C_2|t|, \infty)$. Thus we may define small and large torsions, $\tau_{RS,sm}(f,t)$ and $\tau_{RS,la}(f,t)$. The Ray-Singer torsion factorizes as $\tau_{RS}(f,t) = \tau_{RS,sm}(f,t) \cdot \tau_{RS,la}(f,t)$, so we may analyze the small and large torsions separately.

Let $\chi(F)$ denote the Euler characteristic, let $\operatorname{Tr}^{\operatorname{Cr}(f)}_s[f] = \sum_{x\in\operatorname{Cr}(f)}(-1)^{\operatorname{ind}(x)}f(x)$, and let $\tilde{\chi}'(F) = \operatorname{rank}(F)\sum_{x\in\operatorname{Cr}(f)}(-1)^{\operatorname{ind}(x)}\operatorname{ind}(x)$. For the analysis of the small torsion, we will follow the results of Bismut and Zhang [BZ94] to prove the following theorem,

Theorem 1.1.3. *Let M be a closed manifold Riemannian of dimension n, and let $F \to M$ be* *a vector bundle associated to a representation $\pi_1(M) \to \mathrm{GL}(n, \mathbb{R})$ equipped with a metric g^F whose induced metric on the line bundle $\det F$ is flat. Let (g^{TM}, f) be a generalized triangulation for M. Then the following identity holds.*

$$\begin{aligned}\lim_{t\to+\infty}\Bigg[\log(\tau_{RS,sm}(f,t)^2) + \log\left(\frac{|\cdot|^{Hodge}_{\det H^\bullet(M,F)}}{|\cdot|^{Hodge}_{\det H^\bullet(M,F)}}\right)^2 \\ + \log\left(\frac{t}{\pi}\right)\left(\frac{n}{2}\chi(F) - \tilde{\chi}'(F)\right) + 2t\,\mathrm{rank}(F)\,\mathrm{Tr}_s^{\mathrm{Cr}(f)}[f]\Bigg] \\ = \log\left(\frac{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}}{|\cdot|^{Hodge}_{\det H^\bullet(M,F)}}\right)^2.\end{aligned} \tag{1.2}$$

We will then use this theorem and the factorization of the Ray-Singer torsion to prove the following.

Theorem 1.1.4. *As $t \to \infty$,*

$$\log(\tau_{RS,la}(f,t)) = \log\left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}}\right) + t\,\mathrm{rank}(F)\,\mathrm{Tr}_s^{\mathrm{Cr}(f)}[f] - \frac{1}{2}\tilde{\chi}'(F)\log\left(\frac{t}{\pi}\right) + O(1), \tag{1.3}$$

Following Braverman's approach, we consider another n-dimensional Riemannian manifold $\tilde{M}$ with a vector bundle $\tilde{F} \to \tilde{M}$ so that $\mathrm{rank}(F) = \mathrm{rank}(\tilde{F})$. Additionally, we assume that $\tilde{F}$ is equipped with a Euclidean metric $g^{\tilde{F}}$ so that the induced metric on $\det \tilde{F}$ is flat. Finally, we assume that there exists a Morse function $\tilde{f} : \tilde{M} \to \mathbb{R}$ so that f and $\tilde{f}$ have the same critical point structure. This means that we may choose open neighborhoods around the set of critical points $B \subset U \subset M$ and $\tilde{B} \subset \tilde{U} \subset \tilde{M}$ so that there exists an isometry $\phi : U \to U$ such that $f = \tilde{f} \circ \phi$. This means that $\mathrm{Tr}_s^{\mathrm{Cr}(f)}[f] = \mathrm{Tr}_s^{\mathrm{Cr}(\tilde{f})}[\tilde{f}]$, and $\tilde{\chi}'(F) = \tilde{\chi}'(\tilde{F})$.

We examine the asymptotics of $\log(\tau_{RS,la}(f,t)) - \log(\tau_{RS,la}(\tilde{f},t))$. In particular, we show that $\log(\tau_{RS,la}(f,t)) - \log(\tau_{RS,la}(\tilde{f},t))$ has a *nice asymptotic expansion*, which means that as $t \to \pm\infty$,

$$\log(\tau_{RS,la}(f,t)) - \log(\tau_{RS,la}(\tilde{f},t)) = \sum_{j=0}^{n} a_j(t/|t|)t^j + \sum_{k=0}^{n} b_j(t/|t|)t^j \log|t| + O(1). \tag{1.4}$$

where the constant term a_0 satisfies $a_0(1) + a_0(-1) = 0$.

Now consider $M_1 = M \times S^2$ and $M_2 = M \times S^1 \times S^1$. By a result of Milnor, there exist Morse functions $f_1 : M_1 \to \mathbb{R}$ and $f_2 : M_2 \to \mathbb{R}$ with the same critical point structure. Let $F_1 \to M_1$ and $F_2 \to M_2$ be vector bundles obtained by lifting F to the product manifolds M_1 and M_2 respectively. Then since f_1 and f_2 have the same critical point structure, $\log \tau_{RS,la}(f_1, t) - \log \tau_{RS,la}(f_2, t)$ has a nice asmptotic expansion. Using this fact and Theorem 1.1.4, we prove that

$$\log \left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M_1,F_1)}}{\|\cdot\|^{\mathcal{M},f_1}_{\det H^\bullet(M_1,F)}} \right) = \log \left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M_2,F_2)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M_2,F_2)}} \right). \tag{1.5}$$

Using a well known product formula for the Ray-Singer torsion and Theorem 1.1.4, we prove that

$$\log \left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}} \right) = \log \left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M_1,F_1)}}{\|\cdot\|^{\mathcal{M},f_1}_{\det H^\bullet(M_1,F)}} \right). \tag{1.6}$$

and

$$\log \left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M_2,F_2)}}{\|\cdot\|^{\mathcal{M},f_2}_{\det H^\bullet(M_2,F_2)}} \right) = 0. \tag{1.7}$$

It follows that

$$\begin{aligned} \log \left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}} \right) &= \log \left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M_1,F_1)}}{\|\cdot\|^{\mathcal{M},f_1}_{\det H^\bullet(M_1,F)}} \right) \\ &= \log \left(\frac{\|\cdot\|^{RS}_{\det H^\bullet(M_2,F_2)}}{\|\cdot\|^{\mathcal{M},f_2}_{\det H^\bullet(M_2,F_2)}} \right) = 0. \end{aligned} \tag{1.8}$$

Thus,

$$\|\cdot\|^{RS}_{\det H^\bullet(M,F)} = \|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}. \tag{1.9}$$

Chapter 2

The Knudsen-Mumford Map

In this chapter, we will describe the *Knudsen-Mumford map*, which originally appeared in [KM76]. All vector spaces in this thesis are over $\mathbb{R}$.

2.1 The Determinant Line of a Chain Complex

If λ is a one-dimensional vector space, let λ^{-1} denote the dual space of λ. If $v \in \lambda$ we will sometimes denote by $v^{-1} \in \lambda^{-1}$ the dual element to v.

Given a finite dimensional vector space V, define the *determinant line of* V to be $\det V := \bigwedge^{\dim V} V$. The motivation for this is that any given endomorphism $T : V \to V$, the induced map on the determinant line $\det T : \det V \to \det V$ acts by scaling $\det V$ by the usual determinant of T.

Consider the following cochain complex of vector spaces

$$0 \to V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} V^n \to 0. \tag{2.1}$$

Denote $V^\bullet = \bigoplus_{i=0}^n V^i$ and $d^\bullet = d = \bigoplus_{i=0}^n d^i$ so that d is a homomorphism mapping $V^\bullet$ to $V^\bullet$. In the case of a cochain complex $V^\bullet$, we will instead define the *determinant line of the cochain complex* $V^\bullet$ to be

$$\det V^\bullet = \bigotimes_{i=0}^{n} (\det V^i)^{(-1)^i}. \tag{2.2}$$

Denote elements of $\det V^\bullet$ by $v_\bullet = v_0 \otimes v_1^{-1} \otimes \cdots \otimes v_n^{(-1)^n}$.

In the case of a cochain complex $V^\bullet$, we will always define $\det V^\bullet$ using the alternating product (2.2) rather than considering the top degree forms on the direct sum $\oplus_{i=0}^n V^i$.

Consider the *cohomology* of $V^\bullet$. For each i, let $Z^i = \ker d^i \subseteq V^i$ and $B^i = \operatorname{im} d^{i-1} \subset V^i$. Define the i-th cohomology group of $V^\bullet$ to be $H^i(V^\bullet, d) = H^i(V^\bullet) = Z^i/B^i$.

We can also consider the cohomology of $V^\bullet$ to be a cochain complex, whose differential is the zero map:

$$0 \to H^0(V^\bullet) \to H^1(V^\bullet) \to \cdots \to H^n(V^\bullet) \to 0. \tag{2.3}$$

As before, define $H^\bullet(V^\bullet) = \bigoplus_{i=0}^n H^i(V^\bullet)$ and

$$\det H^\bullet(V^\bullet) = \bigotimes_{i=0}^{n} (\det H^i(V^\bullet))^{(-1)^i}. \tag{2.4}$$

Given two finite dimensional vector spaces V, W, there is a canonical isomorphism $\mu_{V,W}$: $\det V \otimes \det W \to \det(V \oplus W)$ given by

$$\mu_{V,W} : \left(\wedge_{i=1}^{\dim V} v_i \right) \otimes \left(\wedge_{i=1}^{\dim W} w_i \right) \to \left(\wedge_{i=1}^{\dim V} v_i \right) \wedge \left(\wedge_{i=1}^{\dim W} w_i \right). \tag{2.5}$$

One may easily check that this construction is associative, so $\mu_{U,V,W} : \det U \otimes \det V \otimes \det W \to \det(U \oplus V \oplus W)$ is well defined.

2.2 The Knudsen-Mumford Map

To define the Knudsen-Mumford map, we require the introduction of a distinguished basis v_i for each V^i and a distinguished basis h_i for each $H^i(V^\bullet)$. We will identify the bases v_i and h_i with a volume element in $\det V^i$ and $\det H^i(V^\bullet)$ respectively in the following way. If $v_i = (v_{i,1}, ..., v_{i,\dim V^i})$, then consider v_i as an element of $\det V^i$ as follows. Let

$$v_i = v_{i,1} \wedge \cdots \wedge v_{i,\dim V^i} \in \det V^i. \tag{2.6}$$

We may also consider h_i as an element of $H^i(V^\bullet)$ in the same way.

Note that

$$0 \to Z^i \hookrightarrow V^i \xrightarrow{d^i} B^{i+1} \to 0 \tag{2.7}$$

is a short exact sequence. Thus there is a non-canonical splitting $V^i \cong Z^i \oplus B^{i+1}$. Similarly,

$$0 \to B^i \hookrightarrow Z^i \to H^i(V^\bullet) \to 0 \tag{2.8}$$

is a short exact sequence, so there is a non-canonical splitting $Z^i \oplus H^i(V^\bullet) \oplus B^i$. Hence $V^i \cong H^i(V^\bullet) \oplus B^{i+1} \oplus B^i$ non-canonically. This isomorphism is determined by a choice of basis for each B^i. We can remove this dependence by passing to the determinant line. For now, let b_i be a basis for each B^i.

The isomorphism $V^i \cong H^i(V^\bullet) \oplus B^{i+1} \oplus B^i$ determines an isomorphism $\det V^i \cong \det H^i(V^\bullet) \otimes \det B^{i+1} \otimes \det B^i$. Then, for any $u_i \in \det V^i$, there exists some $a_i \in \mathbb{R}$ so that

$$a_i \mu_{\det H^i(V^\bullet), B^i, B^{i+1}}(h_i \otimes b_i \otimes b_{i+1}) = u_i. \tag{2.9}$$

Let $u_\bullet = \otimes_{i=0}^n u_i^{(-1)^i} \in \det V^\bullet$. Also, let

$$ah_\bullet = (a_0 h_0) \otimes (a_1 h_1)^{-1} \otimes \cdots \otimes (a_n h_n)^{(-1)^n}. \tag{2.10}$$

Define the Knudsen-Mumford map $\kappa : \det V^\bullet \to \det H^\bullet(V^\bullet)$ by

$$u_\bullet \mapsto ah_\bullet. \tag{2.11}$$

We immediately see that κ does not depend on the choices of b_i, since if we were to choose some other basis $b_i' = B_i b_i$, where $B_i \in \mathrm{Aut}(B^i)$, then

$$\det B_i a_i \det B_{i+1} \mu_{\det H^i(V^\bullet), B^i, B^{i+1}}(h_i \otimes b_i' \otimes b_{i+1}') = a_i'(h_i \otimes b_i' \otimes b_{i+1}') = u_i \tag{2.12}$$

where in the above equation $\det B_i$ and $\det B_{i+1}$ denote the usual determinant of an endomorphism. But then

$$a'h_\bullet = \det B_0 \det B_1 h_0 \otimes (\det B_1 \det B_2 h_1)^{-1} \otimes \cdots \otimes (\det B_n \det B_{n+1} h_n)^{(-1)^n} \tag{2.13}$$

Since $B^0 = B^{n+1} = 0$, B_0 and B_{n+1} must be the identity, so $\det B_0 = \det B_{n+1} = 1$. Additionally, each other instance of $\det B_i$ is cancelled by $(\det B_i)^{-1}$ in the preceeding or subsequent term, so there is no net effect of the changes of basis for each B^i. Therefore

$$a'h_\bullet = ah_\bullet = \kappa(u_\bullet). \tag{2.14}$$

Since both $\det H^\bullet(V^\bullet)$ and $\det V^\bullet$ are one dimensional, to check that κ is an isomorphism it is sufficient to check that it is not the zero map. However, this is clear, since each isomorphism $V^i \cong H^i(V^\bullet) \oplus B^{i+1} \oplus B^i$ is not the zero map, unless $V^i = 0$ for each i.

In the special case where the cochain complex $V^\bullet$ is acyclic, $H^\bullet(V^\bullet) \cong \mathbb{R}$. In this case, we define the *torsion* of the cochain complex $V^\bullet$ by $\tau = \kappa(v_\bullet) \in \mathbb{R}$. This definition agrees with the classical definition of torsion given in, for example, [Mil66].

2.3 The Determinant of the Laplacian

In this section, we will derive a concrete formula for the Knudsen-Mumford map using a Hodge-theoretic approach.

Recall that choosing a distinguished basis v_i for each V^i fixes a canonical inner product $\langle \cdot, \cdot \rangle_{V^i}$ for each V^i, which we obtain by declaring the basis elements to be orthonormal. Then with respect to these inner products, we may define the adjoint $\delta^i = d^{i*} : V^{i+1} \to V^i$ of the differential d^i. Define the *Hodge Laplacian*,

$$\Delta^i = \delta^i d^i + d^{i-1}\delta^{i-1}. \tag{2.15}$$

Denote $\Delta = \oplus_{i=0}^n \Delta^i : V^\bullet \to V^\bullet$. The Laplacian is a non-negative self-adjoint operator. The following theorem is well-known, and a proof can be found in [Ros97].

Theorem 2.3.1 (Hodge Decomposition Theorem)**.** *For each V^i there is an orthogonal decomposition*

$$V^i = \ker \Delta^i \oplus \operatorname{im} d^{i-1} \oplus \operatorname{im} \delta^i. \tag{2.16}$$

Recall that $\operatorname{im} \delta^i = (\ker d^i)^\perp$. Since the above decomposition is orthogonal, $\ker d^i = \ker \Delta^i \oplus \operatorname{im} d^{i-1}$, therefore we get

$$H^i(V^\bullet) = \ker d^i / \operatorname{im} d^{i-1} = (\ker \Delta^i \oplus \operatorname{im} d^{i-1}) / \operatorname{im} d^{i-1} \cong \ker \Delta^i. \tag{2.17}$$

Recall the distinguished volume elements v_i for each $\det V^i$ and h_i for each $\det H^i(V^\bullet)$.

Proposition 2.3.2. *The Knudsen-Mumford map is given by*

$$\kappa(v_\bullet) = \prod_{k=0}^n \left(\det{}'(\delta^k d^k)^{\frac{1}{2}(-1)^k} \right) h_\bullet \in \det H^\bullet(V^\bullet), \tag{2.18}$$

where $\det{}'(\delta^k d^k)$ *denotes the product of the non-zero eigenvalues of the self-adjoint map* $\delta^k d^k$*.*

Proof. It suffices to show that $a_k = \det{}'(\delta^k d^k)^{\frac{1}{2}}$ for each k. Let $\Delta^k_{\operatorname{im} d^{k-1}}$ denote the restriction of Δ^k to $\operatorname{im} d^{k-1}$. Then $\Delta^k_{\operatorname{im} d^{k-1}} = d^{k-1}\delta^{k-1}$. Choose an orthonormal basis of eigenvectors $\{b_{k,i}\}$, i.e. $\Delta^k_{\operatorname{im} d^{k-1}} b_{k,i} = \lambda_{k,i} b_{k,i}$. Now set $\tilde{b}_{k-1,i} = \lambda_{k,i}^{-1} \delta^{k-1} b_{k,i}$. Then $d^{k-1}\tilde{b}_{k-1,i} = b_{k,i}$.

Observe that $b_{k,i}$ is a basis for B^i, and also since $d^k \tilde{b}_{k,i} = b_{k+1,i}$, $\operatorname{span}\{\tilde{b}_{k,i}\}$ is isomorphic to B_{i+1}. Then

$$V^k \cong H^k(V^\bullet) \oplus B^k \oplus B^{k+1} \cong \ker \Delta^k \oplus \operatorname{span}\{b_{k,i}\} \oplus \operatorname{span}\{\tilde{b}_{k,i}\}. \tag{2.19}$$

With respect to the inner product determined by v_k, $\|b_{k,i}\| = 1$, and

$$\begin{aligned}
\|\tilde{b}_{k-1,i}\|^2 &= \langle \tilde{b}_{k-1,i}, \tilde{b}_{k-1,i} \rangle \\
&= \langle \lambda_{k,i}^{-1} \delta^{k-1} b_{k,i}, \lambda_{k,i}^{-1} \delta^{k-1} b_{k,i} \rangle \\
&= \langle \lambda_{k,i}^{-1} b_{k,i}, \lambda_{k,i}^{-1} d^{k-1} \delta^{k-1} b_{k,i} \rangle \\
&= \lambda_{k,i}^{-1} b_{k,i} \langle b_{k,i}, b_{k,i} \rangle \\
&= \lambda_{k,i}^{-1}.
\end{aligned} \tag{2.20}$$

Thus $\|\tilde{b}_{k-1,i}\| = \lambda_{k,i}^{-\frac{1}{2}}$. We find that

$$a_k = \prod_i \|\tilde{b}_{k,i}\|^{-1} = \prod_i \lambda_{k+1,i}^{\frac{1}{2}}. \tag{2.21}$$

However, each $\lambda_{k+1,i}$ is a non-zero eigenvalue of $\delta^k d^k$. To see this, note that

$$\begin{aligned}
\delta^k d^k (\tilde{b}_{k,i}) &= \delta^k d^k (\lambda_{k+1,i}^{-1} \delta^k b_{k+1,i}) \\
&= \delta^k \lambda_{k+1,i}^{-1} d^k \delta^k b_{k+1,i} \\
&= \delta^k b_{k+1,i} \\
&= \lambda_{k+1,i} \tilde{b}_{k,i}.
\end{aligned} \tag{2.22}$$

Thus $a_k = \det\text{'}(\delta^k d^k)^{\frac{1}{2}}$. □

Proposition 2.3.3. *The Knudsen-Mumford map is given by*

$$\kappa(v_\bullet) = \Big(\prod_{k=0}^{n} (\det\text{'}\, \Delta^k)^{\frac{1}{2} k (-1)^{k+1}} \Big) h_\bullet. \tag{2.23}$$

Proof. If λ is an eigenvector of $\delta^k d^k$ with eigenvector v, then $(d^k \delta^k) d^k v = \lambda d^k v$. Thus λ is also an eigenvector of $d^k \delta^k$ with eigenvector $d^k v$. It follows that $\operatorname{spec}(d^k \delta^k) = \operatorname{spec}(\delta^k d^k)$. Also, since the domains of $\delta^k d^k$ and $d^{k-1} \delta^{k-1}$ are orthogonal,

$$\operatorname{spec}(\Delta^k) = \operatorname{spec}(\delta^k d^k) \cup \operatorname{spec}(d^{k-1} \delta^{k-1}). \tag{2.24}$$

Therefore,

$$\det\text{'}(\Delta^k) = \det\text{'}(\delta^k d^k) \det\text{'}(\delta^{k-1} d^{k-1}) \tag{2.25}$$

and we obtain

$$\begin{aligned}\Big(\prod_{k=0}^{n}(\det{}'\,\Delta^k)^{\frac{1}{2}k(-1)^{k+1}}\Big) &= \Big(\prod_{k=0}^{n}(\det{}'(\delta^k d^k)\det{}'(\delta^{k-1}d^{k-1}))^{-\frac{1}{2}k(-1)^k}\Big)\\ &= \Big(\prod_{k=0}^{n}(\det{}'(\delta^k d^k))^{-\frac{1}{2}(k(-1)^k+(k+1)(-1)^{k+1})}\Big)\\ &= \Big(\prod_{k=0}^{n}(\det{}'(\delta^k d^k))^{\frac{1}{2}(-1)^k}\Big).\end{aligned} \tag{2.26}$$

□

2.4 Torsion as a Metric on the Determinant Line

Suppose that each $\det V^i$ is equipped with an inner product $\langle\cdot,\cdot\rangle_{\det V^i}$. Since each $\det V^i$ is a one-dimensional real vector space, defining an inner product is equivalent to defining a norm $\|\cdot\|_{\det V^i}$. Equip the line $\det V^\bullet$ with the norm

$$\|\cdot\|_{\det V^\bullet} = \bigotimes_{k=0}^{n}\|\cdot\|_{(\det V^k)^{(-1)^k}}. \tag{2.27}$$

Let $\|\cdot\|_{\det H^\bullet(V^\bullet)}$ be the norm on the line $H^\bullet(V^\bullet)$ which is induced by the norm $\|\cdot\|_{\det V^\bullet}$ via the Knudsen-Mumford map. We call $\|\cdot\|_{\det H^\bullet(V^\bullet)}$ the *torsion metric*.

For each i, let $\langle\cdot,\cdot\rangle_{V^i}$ be an inner product on the space V^i which induces the norm $\|\cdot\|_{\det V^i}$ on $\det V^i$. Let $\langle\cdot,\cdot\rangle_{V^\bullet} = \bigoplus_{i=0}^{n}\langle\cdot,\cdot\rangle_{V^i}$, which is the orthogonal sum of the inner products on each V^i.

Using these inner products we may define the Hodge Laplacian Δ, with $\ker\Delta^i \cong H^i(V^\bullet)$ canonically. Since each $\ker\Delta^i$ is a vector subspace of V^i, it inherits the inner product $\langle\cdot,\cdot\rangle_{V^i}$, which we may use to equip $H^i(V^\bullet)$ with an inner product via the canonical isomorphism $\ker\Delta^i \cong H^i(V^\bullet)$. We will denote this inner product by $\langle\cdot,\cdot\rangle_{H^i(V^\bullet)}$. The inner products $\langle\cdot,\cdot\rangle_{H^i(V^\bullet)}$ determines a norm on $\det H^i(V^\bullet)$, which we will denote by $|\cdot|^{Hodge}_{\det H^i(V^\bullet)}$. Let

$$|\cdot|^{Hodge}_{\det H^\bullet(V^\bullet)} = \bigotimes_{k=0}^{n}|\cdot|^{Hodge}_{(\det H^k(V^\bullet)^{(-1)^k}}. \tag{2.28}$$

By choosing orthonormal bases v_i for V^i and h_i for $H^i(V^\bullet)$ and utilizing Proposition 2.3.3, we see that

$$\|\cdot\|_{\det H^\bullet(V^\bullet)} = \Big(\prod_{k=0}^{n}(\det{}'\,\Delta^k)^{\frac{1}{2}k(-1)^{k+1}}\Big)|\cdot|^{Hodge}_{\det H^\bullet(V^\bullet)}. \tag{2.29}$$

Remark 2.4.1. It should be noted that $\left(\prod_{k=0}^{n}(\det{}' \Delta^k)^{\frac{1}{2}k(-1)^{k+1}}\right)$ and $|\cdot|_{\det H^\bullet(V^\bullet)}^{Hodge}$ both depend on the inner products $\langle\cdot,\cdot\rangle_{V^i}$, however $\|\cdot\|_{H^\bullet(V^\bullet)}$ does not.

Chapter 3

The Milnor Metric

3.1 The Reidemeister Metric

Given a closed Riemannian manifold (M, g^{TM}), we may consider a number of cochain complexes associated to M which capture topological data. In the case where the cochain complex is finite dimensional, we may use the Knudsen-Mumford map to obtain a torsion metric on the cohomology determinant line. For simplicial cohomology, the associated torsion metric is called the *Reidemeister metric*.

Let K be a triangulation of M, and choose some lift of K, $\tilde{K}$ to the universal cover $\tilde{M}$. Then the fundamental group $\pi_1(M)$ acts on $\tilde{M}$, and thus $\tilde{K}$, by deck transformations. We may extend this action to an action of the group ring $\mathbb{Z}(\pi_1(M))$ on the groups of simplicial cochains with real coefficients $C^i(\tilde{K}, \mathbb{R})$. In this way, $C^i(\tilde{K}, \mathbb{R})$ may naturally be considered as a right $\mathbb{Z}(\pi_1(M))$-module.

Given an orthogonal representation of the fundamental group $\rho : \pi_1(M) \to O(n)$, define the *ρ-twisted simplicial cochain complex* as

$$C^i(\tilde{K}, \rho) = C^i(\tilde{K}, \mathbb{R}) \otimes_\rho \mathbb{R}^n. \tag{3.1}$$

In particular, for $\gamma \in \pi_1(M)$, $c\gamma \otimes v = c \otimes \rho(\gamma)v$. $\mathbb{R}$ also has a natural left action on $C^i(\tilde{K}, \mathbb{R})$ and a natural right action on $\mathbb{R}^n$, so we may consider each $C^i(\tilde{K}, \rho)$ as a real vector space, which is finite dimensional since M is closed.

Let $\{\sigma^i_j\}$ be the simplicial i-cochains and let $\{e_k\}$ be the usual basis for $\mathbb{R}^n$. Define an inner product on $C^i(\tilde{K}, \rho)$ by declaring elements of the form $\sigma^i_j \otimes e_k$ to be orthonormal.

Definition 3.1.1. The *Reidemeister metric* $\|\cdot\|^R_{\det H^\bullet(\tilde{K},\rho)}$ on $\det H^\bullet(\tilde{K},\rho)$ is the metric obtained by applying the construction in Section 2.4 to the ρ-twisted simplicial cochain complex.

Historically, Reidemeister considered the case when $C^\bullet(\tilde{K},\rho)$ was acyclic. In this case, we associate the Reidemeister metric to a real number, which we call the *Reidemeister torsion*.

Definition 3.1.2. If the complex $C^\bullet(\tilde{K},\rho)$ is acyclic, the *Reidemeister torsion* associated to the representation ρ is defined to be

$$\tau_R(M,\rho) = \sum_{k=0}^{n} (\det \Delta^k)^{\frac{1}{2}k(-1)^{k+1}}, \tag{3.2}$$

where Δ^k is the restriction of the Hodge Laplacian Δ to the cochain group $C^k(\tilde{K},\rho)$.

Remark 3.1.3. $\tau_R(M,\rho)$ actually may be defined for arbitrary simplicial complexes using the above process. However, we are only interested in the case where M is a manifold, since Ray-Singer torsion may only be defined for manifolds.

Initially it was not known that $\tau_R(M,\rho)$ is a topological invariant. First, in 1949 Whitehead [Whi49] proved that $\tau_R(M,\rho)$ is invariant under simplicial subdivisions of the triangulation K. Later, in 1969, Kirby and Siebenmann [KS69] proved that $\tau_R(M,\rho)$ is a topological invariant for manifolds. Finally, in 1974, Chapman [Cha74] proved that $\tau_R(M,\rho)$ is a topological invariant for simplicial complexes.

3.2 Homology With Local Coefficients

When considering R-torsion, we utilized the ρ-twisted simplicial complex. When we later move on to defining Milnor and Ray-Singer torsion, it is far more easier to consider *simplicial cohomology with local coefficients*, which is an equivalent construction. We will very briefly define simplicial cohomology with local coefficients here, and then state a theorem which establishes how simplicial cohomology with local coefficients is equivalent to ρ-twisted simplicial cohomology. This section follows [Dav01, Chapter 5]

Throughout this section, assume that M is a closed manifold, however this construction works more generally for CW-complexes.

Definition 3.2.1. A *system of local coefficients on* M is a fiber bundle $F \to M$ whose fibers F_x are discrete abelian groups.

Suppose (σ_i^k) is a triangulation of M with simplicies $\sigma_i^k : \Delta^k \to M$. Let $C_k(M, F)$ consist of the formal sums

$$\sum_i a_i \sigma_i^k \tag{3.3}$$

where $a_i \in F_{\sigma_i^k(e_0)}$ and $e_0 = (1, 0, ..., 0) \in \Delta^k$. We will equip $C_k(M, F)$ with the obvious addition operation, which turns $C_k(M, F)$ into an abelian group.

Now we will describe the differential. Recall the usual face maps $f_m^k : \Delta^{k-1} \to \Delta^k$ defined by $f_m^k(t_0, t_1, ..., t_{k-1}) = (t_0, ..., t_{m-1}, 0, t_m, ..., t_{k-1})$. Also, for a given simplex $\sigma_i^k : \Delta^k \to M$, let $\gamma_{\sigma_i^k} : [0, 1] \to M$ be the continuous path $\sigma_i^k(t, 1-t, 0, ..., 0)$. By lifting this path, we may define an isomorphism of the fibers $\gamma_{\sigma_i^k} : E_{\sigma_i^k(0,1,...,0)} \to E_{\sigma_i^k(1,0,...,0)}$.

With these maps, we define the differential $d_k : C_k(M, F) \to C_{k-1}(M, F)$ to be

$$d_k(a\sigma) = \gamma_\sigma(a)(\sigma \circ f_0^k) + \sum_{i=1}^{k} (-1)^m a(\sigma \circ f_m^k). \tag{3.4}$$

Theorem 3.2.2 ([Dav01, Theorem 5.8]). *The map d satisfies $d^2 = 0$, and is thus a differential. Additionally, the homology groups $H_k(C_\bullet(M, F))$ equals the ρ-twisted simplicial homology $H_k(M, \rho)$ for some representation ρ of the fundamental group $\pi_1(M)$.*

A similar construction applies to cohomology. Let $C^k(M, F)$ be the set of all functions c which map a simplicial simplex σ to an element $c(\sigma) \in E_{\sigma(e_0)}$. Then $C^k(M, F)$ is an abelian group, and a differential $d^k : C^k(M, F) \to C^{k+1}(M, F)$ may be defined by

$$d^k(c)[\sigma] = (-1)^k \Big(\gamma_\sigma^{-1}(c(d_0\sigma)) + \sum_{i=1}^{k+1} (-1)^i c(d_i \sigma) \Big). \tag{3.5}$$

By [Dav01, Theorem 5.9], d is indeed a differential for the cochain complex $C^\bullet(M, F)$. We will denote the cohomology of this complex $H^\bullet(C^\bullet(M, F))$.

In the above constructions we are using homology and cohomology with integral coefficients. We may also consider simplicial chain groups with real coefficients $C_k(M, \mathbb{R}) = C_k(M) \otimes \mathbb{R}$. If we do so, we may allow the fibers of F to be real vector spaces, and define the homology and cohomology with local real coefficients $H_\bullet(M, F)$, and $H^\bullet(M, F)$ respectively.

3.3 The Thom-Smale Complex

In this section we will define the *Milnor metric*, which arises as the torsion metric obtained from the Thom-Smale complex. Milnor [Mil66] proved that the Milnor metric and the Reidemeister metric agree on smooth manifolds. To prove the Cheeger-Müller theorem for odd dimensional manifolds, we will show that the Milnor metric and Ray-Singer metric agree on closed odd dimensional Riemanninan manifolds. The reason the Milnor metric is preferred over the Reidemeister metric is that the Milnor metric is better suited to the analytic approach.

Let (M, g^{TM}) be a closed Riemannian manifold and let F be a real vector bundle equipped with a metric g^F so that g^F induces a flat metric on $\det F$. Let $f : M \to \mathbb{R}$ be a smooth function. Recall that the points $p \in M$ where $df_p = 0$ are called the *critical points of* f. Let p be a critical point of f, and choose local coordinates $(x^1, ..., x^n)$ in a neighborhood of p. Recall that a critical point $p \in M$ is called *non-degenerate* if the Hessian matrix

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p) \right) \tag{3.6}$$

is non-singular. A *Morse function* is a smooth function $f : M \to \mathbb{R}$ whose critical points are all non-degenerate. Also recall that the *index of the critical point* p is the number of negative eigenvalues of the Hessian matrix at p. The index of p is denoted $\mathrm{ind}(p)$.

Let ∇f be the gradient vector field of f with respect to the metric g. Consider the gradient flow equation

$$\frac{dy}{dt} = -\nabla f(y). \tag{3.7}$$

This differential equation determines a set of diffeomorphisms $(\phi_t)_{t\in\mathbb{R}}$ on M. If p is a critical point of f, define the *unstable manifold of* p to be

$$W^u(p) = \{x \in M \mid \lim_{t\to-\infty} \phi_t(x) = p\}. \tag{3.8}$$

Similarly, define the *stable manifold of* p to be

$$W^s(p) = \{x \in M \mid \lim_{t\to\infty} \phi_t(x) = p\}. \tag{3.9}$$

We will make a few assumptions about our Morse function f. We will assume that ∇f satisfies the *Smale transversality conditions.* In particular, we will assume that if p and q are two critical points of f, with $p \neq q$, then $W^u(p)$ and $W^s(q)$ intersect transversally. This means for all $x \in W^u(p) \cap W^s(q)$,

$$T_x W^u(p) + T_x W^s(q) = T_x M. \tag{3.10}$$

By [Sma61], for a given Morse function f, we may always choose a metric g^{TM} so that ∇f satisfies the Smale transversality conditions. Furthermore, we will also assume that f is *self-indexing*, which means that for all critical points $x \in M$, $f(x) = \mathrm{ind}(x)$. We will also utilize the Morse Lemma to guarantee the existence of certain coordinate neighborhoods around each critical point which has particularly nice coordinate functions. All of these assumptions are summed up in the following definition.

Definition 3.3.1. The pair (g^{TM}, f) is called a *generalized triangulation of* M if $f : M \to \mathbb{R}$ is a self-indexing Morse function which satisfies the Smale transversality conditions and in a neighborhood U_x of each critical point $x \in M$ of f, we may choose local coordinates $(y^1, ..., y^n)$ so that g^{TM} is Euclidean with respect to these coordinates, and f takes the form

$$f(y) = \mathrm{ind}(x) - \frac{1}{2}(y_1^2 + ... + y_k^2) + \frac{1}{2}(y_{k+1}^2 + ... + y_n^2). \tag{3.11}$$

We will always assume that g^{TM} and f are chosen so that (g^{TM}, f) is a generalized triangulation.

We proceed to construct the Thom-Smale complex. If $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$, it is known that $W^u(x) \cap W^s(y)$ consists of a finite set of intergral curves, γ, of the vector field $-\nabla f$, with $\gamma_{-\infty} = x$, $\gamma_{+\infty} = y$ and along which $W^u(x)$ and $W^s(y)$ intersect transversely. We will denote this finite set of integral curves $\Gamma(x, y)$.

Fix an orientation on each $W^u(x)$, where $x \in B$, and some $x, y \in B$ with $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$. Let $\gamma \in \Gamma(x, y)$. Note that $T_y W^u(y)$ is orthogonal to $T_y W^s(y)$, and $T_y W^u(y)$ inherits an orientation from $W^u(y)$. Then for $t \in (-\infty, \infty]$, the orthogonal complement $T^{\perp}_{\gamma_t} W^s(y)$ to $T_{\gamma_t} W^s(y)$ also has a natural orientation.

Also, for $t \in (-\infty, \infty)$, the subbundle, $T^{\perp\prime}_{\gamma_t} W^s(x)$, which is defined to be orthogonal to $-\nabla f(\gamma_t)$ in $T_{\gamma_t} W^u(x)$ can be oriented in such a way that b is an oriented base of $T^{\perp\prime}_{\gamma_t} W^u(x)$ if $(-\nabla f(\gamma_t), b)$ is an oriented base of $T_{\gamma_t} W^u(x)$. Since $W^u(x)$ and $W^s(y)$ are transversal along γ, for $t \in (-\infty, \infty)$, $T^{\perp}_{\gamma_t} W^s(y)$ and $T^{\perp\prime}_{\gamma_t} W^u(x)$ can be identified, and their orientations can be compared. Let

$$\nu_\gamma(x, y) = \begin{cases} +1 \text{ if the orientations are the same,} \\ -1 \text{ if the orientations are different.} \end{cases} \tag{3.12}$$

By [Lau92, Proposition 2, Remark 3], the unstable cells $W^u(x)$ form a cellular decomposition of M. Therefore, we may form the cellular chain groups with real coefficients, $C_\bullet(W^u, \mathbb{R})$. For any critical point x, let $[W^u(x)] \subseteq C_\bullet(W^u, \mathbb{R})$ be the subspace which is spanned by $W^u(x) \in C_\bullet(W^u, \mathbb{R})$.

Let F be a flat vector bundle on M, and let F^* be the dual bundle. Define

$$\begin{aligned} C_\bullet(W^u, F^*) &= \bigoplus_{x \in B} [W^u(x)] \otimes_{\mathbb{R}} F_x^*, \\ C_i(W^u, F^*) &= \bigoplus_{\substack{x \in B \\ \mathrm{ind}(x) = i}} [W^u(x)] \otimes_{\mathbb{R}} F_x^*. \end{aligned} \tag{3.13}$$

Note that if $x \in B$, then the bundle F^* is canonically trivialized on $W^u(x)$. If $x, y \in B$ are such that $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$, if $\gamma \in \Gamma(x, y)$, and if $f^* \in F_x^*$, let $\tau_\gamma(f^*) \in F_y^*$ be the parallel transport of f into F_y^* along γ with respect to the flat connection of F^*.

Now we will define a boundary map. If $x \in B$, and if $f^* \in F_x^*$, define

$$\partial(W^u(x) \otimes f^*) = \sum_{\substack{y \in B \\ \mathrm{ind}(y)=\mathrm{ind}(x)-1}} \sum_{\gamma \in \Gamma(x,y)} \nu_\gamma(x, y) W^u(y) \otimes \tau_\gamma(f^*). \tag{3.14}$$

Extend ∂ by linearity so that ∂ maps $C_i(W^u, F^*)$ into $C_{i-1}(W^u, F^*)$. By [Lau92, Proposition 6], $\partial^2 = 0$, so $C_\bullet(W^u, F^*)$ is a chain complex. Since $C_\bullet(W^u, F^*)$ is a cellular complex $H_k(C^\bullet(W^u, F^*))$ is canonically isomorphic to $H_k(M, F)$ for all k by [Hat02, Theorem 2.35].

We may also consider the associated cochain complex $C^\bullet(W^u, F^*)$. If $x \in B$, let $[W^u(x)]^*$ be the line which is dual to $[W^u(x)]$. Let $(C^\bullet(W^u, F), d)$ be the dual complex to $(C_\bullet(W^u, F^*), d)$. Then

$$C^i(W^u, F) = \bigoplus_{\substack{x \in B \\ \mathrm{ind}(x)=i}} [W^u(x)]^* \otimes_{\mathbb{R}} F_x. \tag{3.15}$$

$C^\bullet(W^u, F)$ is a cellular complex, there is also an isomorphism between Thom-Smale cohomology and simplicial cohomology. The following theorem is proven in [Hat02, Theorem 3.5].

Theorem 3.3.2. *There is a canonical isomorphism*

$$H^\bullet(C^\bullet(W^u, F)) \cong H^\bullet(M, F). \tag{3.16}$$

The complex $C^\bullet(W^u, F)$ is called the *Thom-Smale complex.*

3.4 The Milnor Metric

For each $x \in \mathrm{Cr}_k(f)$, the elements $[W^u(x)]^*$ form a natual basis for $C^k(W^u)$, which provides us with an inner product $\langle \cdot, \cdot \rangle_{C^k(W^u)}$. Define a preferred inner product on $C^k(W^u, F)$ by $\langle \cdot, \cdot \rangle_{C^k(W^u, F)} = \langle \cdot, \cdot \rangle_{C^k(W^u)} \otimes g^F$.

Via the Knudsen-Mumford isomorphism, $\det C^\bullet(W^u, F) \cong \det H^\bullet(C^\bullet(W^u, F))$. By the previous section, $H^\bullet(C^\bullet(W^u, F)) \cong H^\bullet(M, F)$, which induces a canonical isomorphism $\det H^\bullet(C^\bullet(W^u, F)) \cong \det H^\bullet(M, F)$. Then there is a canonical isomorphism $\det C^\bullet(W^u, F) \cong \det H^\bullet(M, F)$.

Definition 3.4.1. The *Milnor metric* $\|\cdot\|^{\mathcal{M}}_{\det H^{\bullet}(M,F)}$ on the real line $\det H^{\bullet}(M,F)$ is the metric corresponding to the induced metric on $\det C^{\bullet}(W^u, F)$ by the canonical isomorphism

$$\det H^{\bullet}(M,F) \cong \det C^{\bullet}(W^u, F). \tag{3.17}$$

Let Δ be the Hodge Laplacian on $C^{\bullet}(W^u, F)$ associated to the inner product $\langle\cdot,\cdot\rangle_{C^k(W^u,F)}$. Set

$$\tau_{\mathcal{M}}(M,f) = \Big(\prod_{k=0}^{n} (\det{}' \, \Delta^k)^{\frac{1}{2}k(-1)^{k+1}}\Big). \tag{3.18}$$

We call $\tau_{\mathcal{M}}(M,f)$ the *Milnor torsion*. By (2.29)

$$\|\cdot\|^{\mathcal{M}}_{\det H^{\bullet}(M,F)} = \tau_{\mathcal{M}}(M,f) |\cdot|^{Hodge}_{\det H^{\bullet}(M,F)}. \tag{3.19}$$

The following is proved in [Mil66, Theorem 9.3].

Theorem 3.4.2. *Suppose that K is a triangulation of M. Let a representation ρ be chosen so that the ρ-twisted cohomology is isomorphic to $H^{\bullet}(M,F)$. Then*

$$\tau_R(\tilde{K}, \rho) = \tau_{\mathcal{M}}(M,f). \tag{3.20}$$

Moreover, $\tau_{\mathcal{M}}(M,f)$ does not depend on the choice of generalized triangulation (g^{TM}, f).

By Theorem 3.4.2, the Milnor metric and Reidemeister metric agree, so the Milnor metric is a homeomorphism invariant. In the sequeal, the Milnor torsion will still be denoted $\tau_{\mathcal{M}}(M,f)$ since it will be useful to keep track of the Morse function.

Chapter 4

The Ray-Singer Metric

In this chapter, we will define the Ray-Singer metric. Ideally, the Ray-Singer metric would be defined by applying the Knudsen-Mumford isomorphism to the determinant line obtained from the de Rham complex of M. However we immediately encounter a problem, since the Knudsen-Mumford isomorphism is only defined when the underlying cochain complex consists of finite dimensional vector spaces.

We will circumvent this problem by first defining *Ray-Singer analytic torsion* using the regularized determinant of the Hodge Laplacian. The Ray-Singer metric is obtained by scaling the Hodge metric on the determinant line $\det H^{\bullet}_{dR}(M, F)$ by the Ray-Singer torsion.

4.1 Ray-Singer Analytic Torsion

Let (M, g^{TM}) be a closed n-dimensional Riemannian manifold. Let $F \to M$ be a real flat vector bundle equipped with a smooth metric g^F which induces a flat metric on $\det F$.

Let $\Omega^{\bullet}(M, F) = \bigoplus_i \Omega^i(M, F)$ be the space of smooth differential forms on M with values in F. Each $\Omega^i(M, F)$ is the space of smooth sections of the vector bundle $\Lambda^i T^* M \otimes F$.

Let ∇^F be the flat connection on the flat vector bundle F. Since this connection is flat, it may be extended via the Leibniz rule to a differential $d^i : \Omega^i(M, F) \to \Omega^{i+1}(M, F)$.

The metric defines the Hodge star operator, $* : \Omega^i(M, F) \to \Omega^{n-i}(M, F)$ and provides each

$\Omega^i(M, F)$ with an inner product:

$$\langle f, g\rangle_{\Omega^i(M,F)} = \int_M \langle f \wedge *g\rangle_F, \tag{4.1}$$

where $\langle \cdot \wedge \cdot\rangle_F$ is determined by the usual wedge product of differential forms, and the inner product on F. Denote by $\langle \cdot, \cdot\rangle_{\Omega^\bullet(M,F)}$ the orthogonal sum of these inner products. The L^2-completion of each $\Omega^i(M, F)$ with respect to this inner product is a Hilbert space and define the formal adjoint of d^F, δ^F. Define the Hodge Laplacian:

$$\Delta = \delta^F d^F + d^F \delta^F. \tag{4.2}$$

Δ is formally self-adjoint, since

$$\begin{aligned} \langle \Delta f, g\rangle &= \langle(\delta^F d^F + d^F \delta^F) f, g\rangle \\ &= \langle \delta^F d^F f, g\rangle + \langle d^F \delta^F f, g\rangle \\ &= \langle f, \delta^F d^F g\rangle + \langle f, d^F \delta^F g\rangle \\ &= \langle f, \Delta g\rangle. \end{aligned} \tag{4.3}$$

Furthermore, Δ is compact and elliptic [Ber+02, Page 19]. Thus the spectrum of Δ only consists of countably many eigenvalues with finite multiplicity and exactly one accumulation point at ∞. By the Hodge theorem the de Rham's theorem,

$$H^\bullet(M, F) \cong \ker \Delta. \tag{4.4}$$

As $\ker \Delta$ is a vector subspace of $\Omega^\bullet(M, F)$, it inherits the inner product $\langle \cdot, \cdot\rangle_{\Omega^\bullet(M,F)}$. There is an induced inner product on $H^\bullet(M, F)$ via the above isomorphism, which in turn induces the Hodge metric on $\det H^\bullet(M, F)$.

Let Δ^i denote the restriction of Δ to $\Omega^i(M, F)$. If we need to specify the vector bundle F, we will write $\Delta^{F,i}$ to denote the restriction of Δ^F to $\Omega^i(M, F)$. Let $P_i : \Omega^\bullet(M, F) \to \ker \Delta^i$ denote the orthogonal projection to $\ker \Delta^i$. We wish to define "$\det \Delta^i$." One might set

$$\det \Delta^i = \prod_j \lambda_j, \tag{4.5}$$

where each λ_j is an eigenvalue of Δ^i. Unforunately, we encounter two problems. First, if zero is an eigenvalue of Δ^i, then this product evaluates to zero, which is not useful. Second, even if zero is not an eigenvalue, this product diverges. We may avoid the first problem by only considering non-zero eigenvalues. To get around the second problem, we will utilise *zeta function regularization.*

For $s \in \mathbb{C}$, let $\zeta_i(s) = \sum_{\lambda \in \mathrm{spec}(\Delta^i)\setminus 0} \lambda^{-s}$. ζ_i is known as the *zeta function of the operator* Δ^i. ζ_i is well defined for $\mathrm{Re}(s) \gg 0$, and by [Roe98, Chapter 8] ζ_i has a meromorphic extension which is analytic at 0. Define the *regularized determinant* $\det' \Delta^i$ by the formula

$$\log \det' \Delta^i = \zeta_i'(0). \tag{4.6}$$

To motivate this formula, suppose that V is a finite dimensional real vector space with dimension m, and let $L : V \to V$ be a linear map with strictly positive eigenvalues $\{\lambda_1, ..., \lambda_m\}$. Consider the zeta function of A, $\zeta_L(s) := \sum_{i=1}^m \lambda_i^{-s}$. This is a finite sum, so it is well defined for all $s \in \mathbb{C}$. Differentiating ζ_L we obtain

$$\zeta_L'(s) = \sum_{i=1}^m -\lambda_i^{-s} \log(\lambda_i). \tag{4.7}$$

By setting $s = 0$ we obtain

$$\begin{aligned} \zeta_L'(0) &= \sum_{i=1}^m \log(\lambda_i) \\ &= \log\Big(\prod_{i=1}^m \lambda_i\Big) \\ &= \log \det L. \end{aligned} \tag{4.8}$$

To summarize, in finite dimensions the derivative of the zeta function associated to L at 0 is precisely $\log \det L$. Therefore, in infinite dimensions, it is reasonable to *define* the determinant of a linear operator using the zeta function. Using this definition of the regularized determinant of the Laplacian, we can define the *Ray-Singer torsion* by a familiar formula.

Definition 4.1.1. The *Ray-Singer torsion* $\tau_{RS}(M, F)$ is defined by

$$\log \tau_{RS}(M, F) = \frac{1}{2} \sum_{i=0}^n (-1)^i i \log \det' \Delta^i. \tag{4.9}$$

The *Ray-Singer metric* on the line $\det H^\bullet(M, F)$ is defined by

$$\|\cdot\|^{RS}_{\det H^\bullet(M,F)} = |\cdot|^{Hodge}_{\det H^\bullet(M,F)} \tau_{RS}(M, F). \tag{4.10}$$

One might ask whether the Ray-Singer metric is also a topological invariant. Provided the dimension of M is odd, then $\tau_{RS}(M, F)$ is independent of the metric. Ray and Singer originally proved this for the case where F is the associated bundle to an orthogonal representation of the fundamental group of M, $\rho : \pi_1(M) \to O(m)$, [RS71, Theorem 2.1]. However, their proof immediately generalizes to the case where F is the associated bundle to any representation of M. This is stated formally in the following theorem.

Theorem 4.1.2. *Suppose that n is odd. Then $\tau_{RS}(M, F)$ is independent of the metrics g^{TM} and g^F.*

Of course we still assume that g^F is chosen so that the induced metric on $\det F$ is flat.

When n is even, it is not generally true that $\tau_{RS}(M, F)$ is independent of g^{TM} and g^F. However, if it assumed that the vector bundle F is obtained by an *orthogonal* representation of $\pi_1(M)$, we have the following remarkable theorem, which was proved in [RS71, Theorem 2.3].

Theorem 4.1.3. *Suppose that n is even, and F is the associated bundle to an orthogonal representation of $\pi_1(M)$.*

Then $\tau_{RS}(M, F) = 1$.

An analogous result also holds for Reidemeister torsion [Mil66, Section 10]. This fact, along with the product formula proven in the next section, led Ray and Singer to conjecture that the Ray-Singer torsion and Reidemeister torsion were equal.

4.2 A Product Formula for the Ray-Singer Torsion

Ray and Singer proved a product formula for the case of orthogonal representations, which was generalised by Müller [Mül93]. For $i = 1, 2$, let M_i be a closed oriented Riemannian manifold and let $\rho_i : \pi_1(M_i) \to \mathrm{GL}(m_i, \mathbb{R})$ be representations of the fundamental group with associated flat bundles $F_{\rho_i} \to M_i$. Let h_i be a metric on F_{ρ_i}, and let

$$p_i : M_1 \times M_2 \to M_i \tag{4.11}$$

be the usual projection map. Assume $M_1 \times M_2$ has the product metric on its tangent space. Note the flat bundle $p_1^*(F_{\rho_1}) \otimes p_2^*(F_{\rho_2}) \to M_1 \times M_2$ is the associated bundle to the representation $\rho_1 \otimes \rho_2 : \pi_1(M_1) \times \pi_1(M_2) \to \mathrm{GL}(\mathbb{R}^{m_1} \otimes \mathbb{R}^{m_2}) \cong \mathrm{GL}(m_1 m_2, \mathbb{R})$, where we have extended the homomorphisms ρ_i to act on $\pi_1(M_1) \times \pi_1(M_2)$ as follows,

$$\rho_1(g, h) = \rho_1(g), \tag{4.12}$$

and

$$\rho_2(g, h) = \rho_2(h). \tag{4.13}$$

Denote by $h_1 \times h_2$ the product metric on the bundle $p_1^*(F_{\rho_1}) \otimes p_2^*(F_{\rho_2}) \to M_1 \times M_2$.

The following theorem was originally stated in [Mül93], however the proof was obtained by the author by following the proof of [RS71, Theorem 2.5].

Theorem 4.2.1. *With the notation above we have the equality*

$$\begin{aligned}\log \tau_{RS}(M_1 \times M_2, p_1^*(F_{\rho_1}) \otimes p_2^*(F_{\rho_2})) = \chi(M_2, F_2) \log \tau_{RS}(M_1, F_1) \\ + \chi(M_1, F_1) \log \tau_{RS}(M_2, F_2).\end{aligned} \tag{4.14}$$

Proof. Let

$$F = p_1^*(F_{\rho_1}) \otimes p_2^*(F_{\rho_2}) \to M_1 \times M_2. \tag{4.15}$$

Suppose $\omega_1 \in \Omega^p(M_1, F_1)$ and $\omega_2 \in \Omega^q(M_2, F_2)$ and consider the $r = p + q$ form $\omega_1 \wedge \omega_2$, where we have lifted ω_1 and ω_2 to the spaces $\Omega^p(M_1 \times M_2, F)$ and $\Omega^q(M_1 \times M_2, F)$ respectively. It is known that such r-forms span $\Omega^r(M_1 \times M_2, F)$. Denote by d^F the induced differential on F. Since the differential commutes with pullbacks,

$$d^F(\omega_1 \wedge \omega_2) = (d^{F_1}\omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge (d^{F_2}\omega_2). \tag{4.16}$$

Also, since we have chosen the product metric on $M_1 \times M_2$, we have

$$\delta^F(\omega_1 \wedge \omega_2) = (\delta^{F_1}\omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge (\delta^{F_2}\omega_2). \tag{4.17}$$

By the definition of the Hodge Laplacian, $\Delta^F = d^F \delta^F + \delta^F d^F$,

$$\Delta^F(\omega_1 \wedge \omega_2) = (\Delta^{F_1}\omega_1) \wedge \omega_2 + \omega_1 \wedge (\Delta^{F_2}\omega_2). \tag{4.18}$$

Therefore, if ω_1 is an eigenform of Δ^{F_1} with eigenvalue λ_1, and ω_2 is an eigenform of Δ^{F_2} with eigenvalue λ_2, then $\omega_1 \wedge \omega_2$ is an eigenform of Δ^F with eigenvalue $\lambda_1 + \lambda_2$. Since the forms $\omega_1 \wedge \omega_2$ span $\Omega^r(M, F)$, we may obtain all eigenforms of $\Delta^{F,r}$ in this way.

Let $N_p(\lambda, M_1)$ and $N_q(\mu, M_2)$ denote the multiplicites of the eigenvalues λ and μ of the Laplacians $\Delta^{F_1,p}$, $\Delta^{F_2,q}$, respectively. The zeta function for $\Delta^{F,r}$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large is given by

$$\begin{aligned}\zeta_{F,r}(s) &= -\operatorname{Tr}[(\Delta^{F,r})^{-s}(\mathrm{id} - P^{F,r})] \\ &= \sum_{i=0}^{r} (N_i(0, M_1)\zeta_{F_2,r-i}(s)) + \sum_{j=0}^{r} (N_j(0, M_2)\zeta_{F_1,r-j}(s)) + \\ &\sum_{\lambda,\mu\neq 0} \sum_{p+q=r} (-\lambda - \mu)^{-s} N_p(\lambda, M_1) N_q(\mu, M_2).\end{aligned} \tag{4.19}$$

Now we consider the alternating sum in the definition of $\tau_{RS}(M_1 \times M_2, F)$. Let n_1 be the dimension of M_1 and n_2 be the dimension of M_2. Then

$$\begin{aligned}\sum_{r=0}^{n_1 n_2}(-1)^r r\zeta_{F,r}(s) = &\sum_{r=0}^{n_1 n_2}(-1)^r r\left(\sum_{i=0}^{r} N_i(0,M_1)\zeta_{F_2,r-i}(s)\right)\\ &+\sum_{r=0}^{n_1 n_2}(-1)^r r\left(\sum_{j=0}^{r} N_j(0,M_2)\zeta_{F_1,r-j}(s)\right)\\ &+\sum_{r=0}^{n_1 n_2}(-1)^r r\left(\sum_{\lambda,\mu\neq 0}\sum_{p+q=r}(-\lambda-\mu)^{-s}N_p(\lambda,M_1)N_q(\mu,M_2)\right).\end{aligned} \tag{4.20}$$

Suppose λ is a non-zero eigenvalue of $\Delta^{F_1,p}$. Let $E_p(\lambda, M_1) \subset \Omega^p(M_1, F_1)$ denote the eigenspace for λ. Let $E'_p(\lambda, M_1)$ denote the subspace of closed forms in $E_p(\lambda, M_1)$ and let $N'_p(\lambda, M_1)$ be the dimension of $E'_p(\lambda, M_1)$. Similarly, let $E''_p(\lambda, M_1)$ be the subspace of $E_p(\lambda, M_1)$ consisting of forms satisfying $\delta^{F_1}\omega = 0$. Note that for any $\omega \in E_p(\lambda, M_1)$,

$$\omega = \lambda^{-1}\Delta^{F_1}\omega = \lambda^{-1}(d^{F_1}\delta^{F_1}\omega + \delta^{F_1}d^{F_1}\omega). \tag{4.21}$$

Clearly $d^{F_1}\delta^{F_1}\omega \in E'_p(\lambda, M_1)$ and $\delta^{F_1}d^{F_1}\omega \in E''_p(\lambda, M_1)$. Thus $E_p(\lambda, M_1)$ is the direct sum $E'_p(\lambda, M_1) \oplus E''_p(\lambda, M_2)$. The map $\lambda^{-1/2}d$ also defines an isometry of $E''_p(\lambda, M_1)$ onto $E'_{p+1}(\lambda, M_1)$, with inverse $\lambda^{-1/2}\delta$. Thus if if $N'_p(\lambda, M_1)$ is the dimension of $E'_p(\lambda, M_1)$, for any p, then $N_p(\lambda, M_1) = N'_p(\lambda, M_1) + N'_{p+1}(\lambda, M_1)$ and we get

$$\sum_{p=0}^{n_1}(-1)^p N_p(\lambda, M_1) = 0. \tag{4.22}$$

It follows from the definition of the zeta function that

$$\sum_{p=0}^{n_1}(-1)^p \zeta_{F_1,p}(s) = 0 \tag{4.23}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large. Using a similar argument we may conclude that

$$\sum_{q=0}^{n_2}(-1)^q \zeta_{F_2,q}(s) = 0 \tag{4.24}$$

once again, for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large. Now we will separately consider the terms of $\sum_{r=0}^{n_1 n_2}(-1)^r r\zeta_{F,r}(s)$. Note that

$$\begin{aligned}\sum_{r=0}^{n_1 n_2}(-1)^r r\left(\sum_{i=0}^{r} N_i(0,M_1)\zeta_{F_2,r-i}(s)\right)&\\ &= \sum_{i=0}^{n_1}(-1)^i N_i(0,M_1)\left(\sum_{j=0}^{n_2}(-1)^j (j+i)\zeta_{F_2,j}(s)\right)\\ &= \sum_{i=0}^{n_1}(-1)^i N_i(0,M_1)\left(\sum_{j=0}^{n_2}(-1)^j j\zeta_{F_2,j}(s) + i\sum_{j=0}^{n_2}(-1)^j\zeta_{F_2,j}(s)\right)\\ &= \sum_{i=0}^{n_1}(-1)^i N_i(0,M_1)\left(\sum_{j=0}^{n_2}(-1)^j j\zeta_{F_2,j}(s)\right).\end{aligned} \tag{4.25}$$

By the Hodge theorem, $N_i(0,M_1)$ is the i-th Betti number of M_1, so

$$\sum_{r=0}^{n_1 n_2}(-1)^r r\left(\sum_{i=0}^{r} N_i(0,M_1)\zeta_{F_2,r-i}(s)\right) = \chi(M_1,F_1)\left(\sum_{j=0}^{n_2}(-1)^j j\zeta_{F_2,j}(s)\right). \tag{4.26}$$

By an identical argument, we conclude that

$$\sum_{r=0}^{n_1 n_2}(-1)^r r\left(\sum_{i=0}^{r} N_i(0,M_2)\zeta_{F_1,r-i}(s)\right) = \chi(M_2,F_2)\left(\sum_{j=0}^{n_1}(-1)^j j\zeta_{F_1,j}(s)\right). \tag{4.27}$$

Consider the final term in $\sum_{r=0}^{n_1 n_2}(-1)^r r\zeta_{F,r}(s)$. Note that

$$\begin{aligned}\sum_{\lambda,\mu\neq 0}\sum_{p+q=r}(-\lambda-\mu)^{-s}N_p(\lambda,M_1)N_q(\mu,M_2)&\\ &= \sum_{\lambda,\mu\neq 0}(-\lambda-\mu)^{-s}\left(\sum_{i=0}^{n_1}(-1)^i i N_i(\lambda,M_1)\right)\left(\sum_{j=0}^{n_2}(-1)^j N_j(\mu,M_2)\right)\\ &+ \sum_{\lambda,\mu\neq 0}(-\lambda-\mu)^{-s}\left(\sum_{i=0}^{n_1}(-1)^i N_i(\lambda,M_1)\right)\left(\sum_{j=0}^{n_2}(-1)^j j N_j(\mu,M_2)\right).\end{aligned} \tag{4.28}$$

Both terms on the right hand side vanish, since

$$\sum_{i=0}^{n_1}(-1)^i N_i(\lambda,M_1) = \sum_{j=0}^{n_2}(-1)^j N_j(\mu,M_2) = 0, \tag{4.29}$$

provided $\lambda, \mu \neq 0$. Thus

$$\begin{aligned}\sum_{r=0}^{n_1 n_2}(-1)^r r\zeta_{F,r}(s) = \chi(M_2, F_2)\left(\sum_{j=0}^{n_1}(-1)^j j\zeta_{F_1,j}(s)\right)\\ + \chi(M_1, F_1)\left(\sum_{j=0}^{n_2}(-1)^j j\zeta_{F_2,j}(s)\right).\end{aligned} \tag{4.30}$$

Of course, the zeta functions here are only defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large. However, as discussed before, we may analytically continue them to meromorphic extensions which are analytic at $s = 0$. Thus by taking the derivative with respect to s at $s = 0$ we obtain

$$\begin{aligned}\sum_{r=0}^{n_1 n_2}(-1)^r r\zeta'_{F,r}(0) = \chi(M_2, F_2)\left(\sum_{j=0}^{n_1}(-1)^j j\zeta'_{F_1,j}(0)\right)\\ + \chi(M_1, F_1)\left(\sum_{j=0}^{n_2}(-1)^j j\zeta'_{F_2,j}(0)\right),\end{aligned} \tag{4.31}$$

which implies the claim. □

4.3 A Quasi-Isomorphism

In this section, we present a quasi-isomorphism between the de Rham and Thom-Smale cochain complexes, which was defined in [Lau92].

Let (g^{TM}, f) be a generalized triangulation of M. By [Lau92], for each $x \in \mathrm{Cr}(f)$, $\overline{W^u(x)}$ is a *submanifold with conical simplicialities*, and thus integration over $\overline{W^u(x)}$ is well defined.

Definition 4.3.1. Let $P_\infty : \Omega^\bullet(M, F) \to C^\bullet(W^u, F)$ be defined by

$$P_\infty \omega = \sum_{x \in \mathrm{Cr}(f)} W^u(x)^* \otimes \int_{\overline{W^u(x)}} \alpha. \tag{4.32}$$

The following proposition was proven in [Lau92, Proposition 7].

Proposition 4.3.2. *P_∞ is a quasi-isomorphism, that is, the induced map $P_\infty^* : H^\bullet_{dR}(M, F) \to H^\bullet(C^\bullet(W^u, F))$ is an isomorphism.*

Chapter 5

The Witten Laplacian

5.1 The Witten Deformation

Henceforth M is assumed to be odd dimensional, so we can choose the metrics freely without affecting the Ray-Singer torsion. Since the Milnor metric is independent of the choice of Morse function f, we will assume that (g^{TM}, f) is a generalized triangulation of M.

Let $d_t^F = e^{-tf} d^F e^{tf}$ and $\delta_t^F = e^{tf} \delta^F e^{-tf}$. Then δ_t^F is the formal adjoint of d_t^F with respect to the inner product on $\Omega^\bullet(M, F)$,

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha \wedge *\beta \rangle_F. \tag{5.1}$$

Define $\Delta_{f,t} = d_t^F \delta_t^F + \delta_t^F d_t^F$. The operator $\Delta_{f,t}$ was introduced by Witten [Wit82] and is known as the *Witten Laplacian*. Denote by $\Delta_{f,t}^i$ the restriction of $\Delta_{f,t}$ to $\Omega^i(M, F)$. Let $\tau_{RS}(f, t)$ be the Ray-Singer torsion as before, however we replace the Hodge Laplacian Δ with the Witten Laplacian $\Delta_{f,t}$.

It turns out that $\tau_{RS}(f, t)$ agrees with $\tau_{RS}(M, F)$. To see this, we will show that $\tau_{RS}(f, t)$ is actually the Ray-Singer torsion computed with a particular metric, and then conclude by considering the metric independence of the Ray-Singer torsion.

We will now conformally scale the metric on F. For $t \geq 0$, let g_t^F be the smooth metric on F defined as

$$g_t^F = e^{-2tf} g^F. \tag{5.2}$$

Using g^{TM} and g_t^F, we may define another inner product on $\Omega^\bullet(M, F)$, $\langle \cdot, \cdot \rangle_{\Omega^\bullet(M,F),t}$. Let $\delta_t'^F$ be the adjoint of d^F with respect to the inner product $\langle \cdot, \cdot \rangle_{\Omega^\bullet(M,F),t}$. Then we see that $\delta_t'^F$

satisfies the formula

$$\delta_t'^F = e^{2tf}\delta^F e^{-2tf}. \tag{5.3}$$

Let $\Delta'_{f,t} = d^F\delta_t'^F + \delta_t'^F d^F$ be the Hodge Laplacian associated to the inner product $\langle\cdot,\cdot\rangle_{\Omega^\bullet(M,F),t}$. Then

$$\begin{aligned}\Delta_{f,t} &= d_t^F\delta_t^F + \delta_t^F d_t^F \\ &= e^{-tf}d^F e^{2tf}\delta^F e^{-tf} + e^{tf}\delta^F e^{-2tf}d^F e^{tf} \\ &= e^{-tf}(d^F e^{2tf}\delta^F e^{-2tf} + e^{2tf}\delta^F e^{-2tf}d^F)e^{tf} \\ &= e^{-tf}\Delta'_{f,t}e^{tf}.\end{aligned} \tag{5.4}$$

When computing the determinant, e^{-tf} cancels e^{tf}, so for each q, $\det'\Delta^q_{f,t} = \det'\Delta'^q_{f,t}$. Thus $\tau_{RS}(f,t)$ is equal to $\tau_{RS}(M,F)$. In particular,

$$\|\cdot\|^{RS}_{\det H^\bullet(M,F)} = |\cdot|^{Hodge}_{\det H^\bullet(M,F)}\tau_{RS}(f,t). \tag{5.5}$$

To conclude this section, we will expand the Witten Laplacian to obtain a formula in terms of Δ and t. Let $\omega \in \Omega^\bullet(M)$. Then

$$\begin{aligned}d_t\omega &= e^{-tf}de^{tf}\omega \\ &= e^{-tf}(e^{tf}d\omega + te^{tf}df\wedge\omega) \\ &= (d + tdf\wedge)\omega.\end{aligned} \tag{5.6}$$

Thus for any $\omega_1, \omega_2 \in \Omega^\bullet(M,F)$,

$$\begin{aligned}\langle\delta_t\omega_1,\omega_2\rangle &= \langle\omega_1, d_t\omega_2\rangle \\ &= \langle\omega_1, d\omega_2\rangle + \langle\omega_1, tdf\wedge\omega_2\rangle \\ &= \langle\delta\omega_1,\omega_2\rangle + \langle t\iota_{\nabla f}\omega_1,\omega_2\rangle \\ &= \langle(\delta + t\iota_{\nabla f})\omega_1,\omega_2\rangle,\end{aligned} \tag{5.7}$$

where we have used the fact that $\iota_X = (X\wedge)^*$ for any vector field X. We have $\delta_t = \delta + t\iota_{\nabla f}$ and

$$\begin{aligned}\Delta_t &= d_t\delta_t + \delta_t d_t \\ &= (d + tdf\wedge)(\delta + t\iota_{\nabla f}) + (\delta + t\iota_{\nabla f})(d + tdf\wedge) \\ &= d\delta + tdf\wedge\delta + td\iota_{\nabla f} + t^2 df\wedge\iota_{\nabla f} + \delta d + t\iota_{\nabla f}d + t\delta df\wedge + t^2\iota_{\nabla f}df\wedge \\ &= (d\delta + \delta d) + t^2(df\wedge(\iota_{\nabla f}) + \iota_{\nabla f}(df\wedge)) \\ &+ t((\iota_{\nabla f}d + d\iota_{\nabla f}) + (df\wedge\delta + \delta df\wedge)) \\ &= \Delta + t^2(df\wedge(\iota_{\nabla f}) + \iota_{\nabla f}(df\wedge)) + t(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}) \\ &= \Delta + th + t^2(df\wedge(\iota_{\nabla f}) + (\iota_{\nabla f}df) - df\wedge(\iota_{\nabla f})) \\ &= \Delta + th + t^2\iota_{\nabla f}df \\ &= \Delta + th + t^2\|df\|,\end{aligned} \tag{5.8}$$

where $h = \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}$, and $\mathcal{L}_{\nabla f}$ is the *Lie derivative* with respect to ∇f.

5.2 The Spectrum of the Witten Laplacian

In this section, we will prove the *spectral gap theorem*, which will allow us to factorize the Ray-Singer torsion. Here we follow [Bur+96, Section 5].

For $n \geq 1$, $k \in \mathbb{R}$, let $h_k : \mathbb{R}^n \to \mathbb{R}$ be the smooth function defined by

$$h_k(x) = k - \frac{1}{2}\sum_{i=1}^{k}|x_i|^2 + \frac{1}{2}\sum_{i=k+1}^{n}|x_i|^2. \tag{5.9}$$

Then h_k is a self-indexing Morse function with one critical point of index k at $x = 0$.

Let $\Delta^q : \Omega^q(\mathbb{R}^n) \to \Omega^q(\mathbb{R}^n)$ be the Laplacian on q-forms on $\mathbb{R}^n$ and let $\Delta^q_{h_k,t}$ be the Witten Laplacian associated to h_k. By [BZ92, Proposition 8.2], $\Delta^q_{h_k,t}$ takes the form

$$\Delta^q_{h_k,t} = \Delta^q + t^2|x|^2 - t(n-2k) + 2t(N^+_{q,k} - N^-_{q,k}) \tag{5.10}$$

where $N^+_{q,k}$ is defined by

$$N^+_{q,k}(dx_{i_1} \wedge \cdots \wedge dx_{i_q}) = |\{j \mid k+1 \leq i_j \leq n\}| dx_{i_1} \wedge \cdots \wedge dx_{i_q} \tag{5.11}$$

and $N^-_{q,k} = q\,\mathrm{id} - N^+_{q,k}$.

For $t \in \mathbb{R}$, let $\omega_{q,t} \in \Omega^q(\mathbb{R}^n)$ be the Gaussian q-form defined by

$$\omega_{q,t}(x) = (t/\pi)^{n/4} e^{-t|x|^2/2} dx_1 \wedge \cdots \wedge dx_q. \tag{5.12}$$

For $\eta > 0$, let $\nu_\eta : \mathbb{R} \to [0,1]$ be a smooth function so that $\nu_\eta(x) = 1$ for $x \in (-\infty, \eta/2)$ and $\nu_\eta(y) = 0$ for $y \in (\eta, \infty)$. For some $\epsilon > 0$, which will be chosen later, let $\tilde{\psi}_{q,t} \in \Omega^q(\mathbb{R}^n)$ be defined by

$$\tilde{\psi}_{q,t}(x) = \|\nu_\epsilon(|x|)\omega_{q,t}\|^{-1}_{\Omega^q(\mathbb{R}^n)} \nu_\epsilon(|x|)\omega_{q,t}(x) \tag{5.13}$$

Now suppose that F is a flat vector bundle over $\mathbb{R}^n$ equipped with a Euclidean metric g^F. We may consider $\Delta^q_{h_k,t}$ to be an operator acting on $\Omega^q(\mathbb{R}^n, F)$ by letting it act trivially on F. The following proposition is proved in [Bur+96, Section 5, (HO1), (HO3)].

Proposition 5.2.1. *The operators $\Delta^q_{h_k,t}$ are non-negative, self-adjoint, elliptic operators with the following two properties.*

1. $\mathrm{spec}(\Delta^q_{h_k,t})$ *is discrete and contained in* $2t\mathbb{Z}_{\geq 0}$*.*

2. If $\{v_1, ..., v_m\}$ *is an orthonormal basis of* F*, then* $\omega_{q,t,i} = \omega_{q,t} \otimes v_i$ *forms a basis for* $\ker \Delta^q_{h_q,t}$ *and* $\psi_{q,t,i} = \tilde{\psi}_{q,t} \otimes v_i$ *is an orthonormal basis for a subspace of* $\ker \Delta^q_{h_q,t}$*.*

The following estimates are also needed to prove the spectral gap theorem.

Proposition 5.2.2. *There exists constants* $C_0, C, t_0 > 0$ *which depend on* ϵ *so that for any* $x \in \mathbb{R}^n$

$$|\Delta^q_{h_q,t}\psi_{q,t,i}(x)| \leq C_0 e^{-Ct}, \tag{5.14}$$

for $t \geq 0$*,*

$$\langle \Delta^q_{h_k,t}\psi_{q,t,i}, \psi_{q,t,i}\rangle_{\Omega^q(\mathbb{R}^n,F)} \geq 2t|q-k|, \tag{5.15}$$

and for $\omega \in \Omega^q(\mathbb{R}^n, F)$ *with compact support orthogonal to the subspace generated by* $\psi_{q,t,i}$*,*

$$\langle \Delta^q_{h_q,t}\omega, \omega\rangle_{\Omega^q(\mathbb{R}^n,F)} \geq Ct\|\omega\|^2_{\Omega^q(\mathbb{R}^n,F)}. \tag{5.16}$$

Proof. First, notice that

$$(N^+_{q,k} - N^-_{q,k})(dx_1 \wedge \cdots \wedge d_q) = n_{q,k} dx_1 \wedge \cdots \wedge dx_q \tag{5.17}$$

where $n_{q,k} = -q$ if $k \geq q$ and $n_{q,k} = q - 2k$ if $k < q$. Thus, since $\Delta = -\sum_{i=1}^n \partial^2_{x_i}$

$$(\Delta + t^2|x|^2 - t(n-2k) + 2tn_{q,k})e^{-t|x|^2/2} = 2t|q-k|e^{-t|x|^2/2}. \tag{5.18}$$

Also,

$$\begin{aligned}\Delta(\nu_\epsilon(|x|)e^{-t|x|^2/2}) = e^{-t|x|^2/2}\Delta(\nu_\epsilon(|x|)) - 2\sum_{i=1}^n \partial_{x_i}\nu_\epsilon(|x|)\partial_{x_i}e^{-t|x|^2/2} \\ + \nu_\epsilon(|x|)\Delta(e^{-t|x|^2/2}).\end{aligned} \tag{5.19}$$

To prove the first estimate set $k = q$ and conclude from (5.18), (5.19) that

$$\begin{aligned}&|(\Delta + t^2|x|^2 - t(n-2k) + 2tn_{q,q})(\nu_\epsilon(|x|)e^{-t|x|^2/2})| \\ &\leq \Big| -2\sum_{i=1}^n \partial_{x_i}(\nu_\epsilon(|x|)e^{-t|x|^2/2})\Big| + \Big| -\sum_{i=1}^n \partial^2(\nu_\epsilon(|x|))e^{-t|x|^2/2}\Big| \\ &\leq |\dot{\nu}_\epsilon(|x|)|2t|x|e^{-t|x|^2/2} + |\dot{\nu}(|x|)|\frac{n-1}{|x|}e^{-t|x|^2/2} + |\ddot{\nu}(|x|)|e^{-t|x|^2/2}\end{aligned} \tag{5.20}$$

where $\dot{\nu}_\epsilon(t) = \frac{d}{dt}\nu_\epsilon(t)$ and $\ddot{\nu}_\epsilon(t) = \frac{d^2}{dt^2}\nu_\epsilon(t)$. Since the support of both $\dot{\nu}_\epsilon$ and $\ddot{\nu}_\epsilon$ are contained in $[\epsilon/2, \epsilon]$, we may conclude that

$$\begin{aligned} &|(\Delta + t^2|x|^2 - t(n-2k) + 2tn_{q,q})(\nu_\epsilon(|x|)e^{-t|x|^2/2})| \\ &\quad \le \Big(\|\dot{\nu}_\epsilon\|_{L^\infty}\Big(2\epsilon t e^{-t\epsilon^2/16} + 2\frac{n-1}{\epsilon}e^{-t\epsilon^2/16}\Big) + \|\ddot{\nu}_\epsilon\|_{L^\infty}e^{-t\epsilon^2/16}\Big)e^{-t\epsilon^2/16}. \end{aligned} \tag{5.21}$$

Now, to estimate $\beta(t)$, notice that for $t_0 := (2/\epsilon)^2$, one obtains for $t \ge t_0$,

$$\int_{\mathbb{R}^n} \nu_\epsilon(|x|)^2 e^{-t|x|^2}dx \ge \int_{|x|<\epsilon/2} e^{-t|x|^2}dx \ge C' t^{-n/2}\int_0^1 e^{-s^2}s^{n-1}ds. \tag{5.22}$$

Combining (5.21) and (5.22), we conclude that there exists $C > 0$ so that for $t \ge t_0$,

$$|\Delta^q_{h_q,t}\psi_{q,t,i}(x)| \le C\Big[\|\dot{\nu}_\epsilon\|_{L^\infty}\Big(2\epsilon t e^{-t\epsilon^2/16} + \frac{n-1}{2}\Big) + \|\ddot{\nu}_\epsilon\|_{L^\infty}\Big]t^{-n/2}e^{-t\epsilon^2/16}. \tag{5.23}$$

From this, we obtain the first estimate.

To prove the second estimate, we integrate by parts to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-t|x|^2/2}(\Delta\nu_\epsilon)e^{-t|x|^2/2}\nu_\epsilon dx &= -\sum_{i=1}^n \int_{\mathbb{R}^n} e^{-t|x|^2}\nu_\epsilon \partial^2_{x_i}\nu_\epsilon dx \\ &= \sum_{i=1}^n \int_{\mathbb{R}^n} (\partial_{x_i}\nu_\epsilon)^2 e^{-t|x|^2}dx \\ &+ \sum_{i=1}^n \int_{\mathbb{R}^n} \nu_\epsilon \frac{\partial \nu_\epsilon}{\partial x_i} 2\Big(\frac{\partial}{\partial x_i}e^{-t|x|^2/2}\Big)e^{-t|x|^2/2}dx. \end{aligned} \tag{5.24}$$

By combining (5.19) and (5.24) we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} \Delta(\nu_\epsilon e^{-t|x|^2/2})\nu_\epsilon e^{-t|x|^2/2}dx \\ &\quad = \int_{\mathbb{R}^n}\Big(\sum_{i=1}^n(\partial_{x_i}\nu_\epsilon)^2 e^{-t|x|^2} + \nu_\epsilon^2\Delta(e^{-t|x|^2/2})e^{-t|x|^2/2}\Big)dx. \end{aligned} \tag{5.25}$$

By combining (5.18) and (5.25) we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n}(\Delta + t^2|x|^2 - t(n-2k) + 2tn_{q,k})(\nu_\epsilon e^{-t|x|^2/2})\cdot(\nu_\epsilon e^{-t|x|^2/2})dx \\ &\quad = \int_{\mathbb{R}^n}\Big(2t|q-k|\nu_\epsilon^2 e^{-t|x|^2} + \sum_{i=1}^n(\partial_{x_i}\nu_\epsilon)^2 e^{-t|x|^2}\Big)dx \\ &\quad \ge 2t|q-k|\int_{\mathbb{R}^n}(\nu_\epsilon)^2 e^{-t|x|^2}dx. \end{aligned} \tag{5.26}$$

By taking into account the normalization factor $\beta(t)$, we obtain

$$\begin{aligned}\langle \Delta^q_{h_k,t}\psi_{q,t,i},\psi_{q,t,i}\rangle_{\Omega^q(\mathbb{R}^n,F)} &= \frac{\int_{\mathbb{R}^n}(\Delta+t^2|x|^2-t(n-2k)+2tn_{q,q})(\nu_\epsilon e^{-t|x|^2/2})\cdot(\nu_\epsilon e^{-t|x|^2/2})dx}{\int_{\mathbb{R}^q}(\nu_\epsilon)^2 e^{-t|x|^2}dx}\\ &\geq 2t|q-k|,\end{aligned} \tag{5.27}$$

which proves the second estimate.

To prove the final estimate, it suffices to consider $\omega\in\Omega^q(\mathbb{R}^n,F)$ of the form $\omega=\phi dx_{i_1}\wedge\cdots\wedge dx_{i_q}\otimes v$ with $v\in F$ and $\phi\in C^\infty(\mathbb{R}^n,\mathbb{R})$ with compact support. We need to show that there exists t_0 and C_0 so that for any $\phi\in C^\infty(\mathbb{R}^n,\mathbb{R})$ with compact support satisfying

$$\int_{\mathbb{R}^n}\phi(x)\nu_\epsilon(|x|)e^{-t|x|^2/2}dx=0, \tag{5.28}$$

the following estimate holds

$$\int_{\mathbb{R}^n}(\Delta+t^2|x|^2-tn)\phi(x)^2dx\geq C_0t\int_{\mathbb{R}^n}|\phi(x)|^2dx. \tag{5.29}$$

To prove this, consider the function $\phi_2=\phi-\phi_1$, where

$$\phi_1(x)=\frac{\int_{\mathbb{R}^n}\phi(x)e^{-t|x|^2/2}dx}{\int_{\mathbb{R}^n}e^{-t|x|^2}dx}, \tag{5.30}$$

so ϕ_2 is the orthogonal projection of ϕ onto $e^{-t|x|^2/2}$. Then $(\Delta+t^2|x|^2-tn)\phi_1=0$, and due to the properties of the spectrum of the Witten Laplacian, $\mathrm{spec}(\Delta+t^2|x|^2-tn)\subseteq t\mathbb{Z}_{\geq 0}$. Hence

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}(\Delta+t^2|x|^2-tn)\phi(x)^2dx\Big| &\geq \Big|\int_{\mathbb{R}^n}(\Delta+t^2|x|^2-tn)\phi_2(x)^2dx\Big|\\ &\geq t\int_{\mathbb{R}^n}|\phi_2(x)|^2dx\\ &= t\Big(\int_{\mathbb{R}^n}|\phi(x)|^2dx-\int_{\mathbb{R}^n}|\phi_1(x)|^2dx\Big).\end{aligned} \tag{5.31}$$

It remains to calculate an estimate for $\int_{\mathbb{R}^n}|\phi_1(x)|^2dx=|\int_{\mathbb{R}^n}\phi e^{-t|x|^2/2}dx|^2$. By the Cauchy-Schwarz inequality and the fact that

$$\int_{\mathbb{R}^n}\phi(x)\nu_\epsilon(|x|)e^{-t|x|^2/2}dx=0, \tag{5.32}$$

we conclude that

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\phi(x)e^{-t|x|^2/2}dx\Big|^2 &= \Big|\int_{\mathbb{R}^n}\phi(x)(1-\nu_\epsilon(|x|))e^{-t|x|^2/2}dx\Big|^2 \\ &\leq \int_{\mathbb{R}^n}|\phi(x)|^2dx\int_{\mathbb{R}^n}(1-\nu_\epsilon(|x|))^2e^{-t|x|^2}dx.\end{aligned} \tag{5.33}$$

Also, with $t \geq (2/\epsilon)^2$ and $t \geq t_0$,

$$\int_{\mathbb{R}^n}(1-\nu_\epsilon(|x|))^2e^{-t|x|^2}dx \leq \int_{|x|\leq\frac{\epsilon}{2}} e^{-t_0|x|^2} \leq Ct_0^{-n/2}\int_1^\infty e^{-s^2}s^{n-1}ds. \tag{5.34}$$

Using (5.34), we may choose $t_0 \geq (2/\epsilon)^2$ sufficiently large so that

$$\int_{\mathbb{R}^n}(1-\nu_\epsilon(|x|))^2e^{-t|x|^2} \leq 1/2. \tag{5.35}$$

Then by combining (5.31), (5.33), and (5.35) we see that for $t \geq t_0$,

$$\int_{\mathbb{R}^n}(\Delta + t^2|x|^2 - tn)\phi(x)^2dx \geq \frac{t}{2}\int_{\mathbb{R}^n}|\phi(x)|^2dx \tag{5.36}$$

which concludes the proof. □

Before we continue, we will identify a sufficiently small neighborhood of each critical point $x \in \mathrm{Cr}(f)$ with an open ball in $\mathbb{R}^n$, so we may utilize Theorem 5.2.2 in the setting of a general manifold M with a real vector bundle F.

For all q, choose an $\epsilon > 0$ sufficiently small so that for each $x \in \mathrm{Cr}_q(f)$, $B^M(x, 4\epsilon)$ are pairwise disjoint and $B^M(x, 3\epsilon) \subseteq U_x$, where U_x is the neighborhood described in Definition 3.3.1. For each $z \in U_x$, we will identify the fibers F_z and F_x by parallel transport. Fix once and for all a basis for each fiber F_x. Then we can naturally identify forms $\omega \in \Omega^q(M, F)$ with support in U_x with forms in $\Omega^q(\mathbb{R}^n, F)$. In this way, each $\psi_{q,t,i} \in \Omega^q(\mathbb{R}^n, F)$ can be identified with a differential form $\psi_{x,t,i} \in \Omega^q(M, F)$. The forms $\psi_{x,t,i}$ are orthonormal, and satisfy Theorem 5.2.2

Now we are ready to prove the *spectral gap theorem.*

Theorem 5.2.3. *There exist constants $C_1, C_2, t_0 > 0$ so that for $t \geq t_0$ and $0 \leq q \leq n$,* $\mathrm{spec}(\Delta^q_{f,t}) \subseteq [0, e^{-tC_1}] \cup [C_2t, \infty)$.

Proof. Each $\Delta^q_{f,t}$ is non-negative, so it suffices to show that $\mathrm{spec}(\Delta^q_{f,t}) \cap (e^{-tC_1}, C_2t) = \varnothing$. The proof is completed in two steps.

In the first step, we will prove that there exists some constants $t_0, C_1, C_2 > 0$ so that for each $t \geq t_0$ there exists a pair of orthogonal closed subspaces of the L^2 completion of $\Omega^q(M, F)$, W_1 and W_2 with $W_1 \subset \Omega^q(M, F)$, which satisfy the following properties:

1. $W_1 \cap W_2 = 0$.
2. $W_1 + W_2 = \Omega^q(M, F)$.
3. $\langle \Delta^q_{f,t} \omega, \omega \rangle_{\Omega^q(M,F)} \leq e^{-tC_1} \langle \omega, \omega \rangle_{\Omega^q(M,F)}$ for all $\omega \in W_1$.
4. $\langle \Delta^q_{f,t} \omega, \omega \rangle_{\Omega^q(M,F)} \geq C_2 t \langle \omega, \omega \rangle_{\Omega^q(M,F)}$ for all $\omega \in W_2 \cap \Omega^q(M, F)$.

Then in the second step, we will prove the claim by contradiction using the results in the first step.

We will start by proving the first step. Define W_1 to be the subspace generated by the forms $\psi_{x,t,i} \in \Omega^q(M, F)$, and define W_2 to be the orthogonal complement of W_1 in $\Omega^q(M, F)$. By construction, the first two properties are satisfied.

W_1 consists of q-forms of the form $\sum_{i,x \in B^q} a_{x,t,i} \psi_{x,t,i}$ where $a_{x,t,i} : M \to \mathbb{R}$ is a smooth function. By choosing a sufficiently small neighborhood of each critical point $x \in \mathrm{Cr}_q(f)$, U_x, with appropriate local coordinates, $\Delta^q_{f,t}$ coincides with $\Delta^q_{h_q,t}$ acting on $\Omega^q(\mathbb{R}^n, F)$ when restricted to U_x. So $\Delta^q_{f,t}$ is $C^\infty(M, \mathbb{R})$-linear, and we may use Proposition 5.2.2 to obtain an estimate. By choosing $C, C_1, t_1 > 0$ as in Proposition 5.2.2, we obtain for $t \geq t_1$,

$$\begin{aligned}
\langle \Delta^q_{f,t} \omega, \omega \rangle_{\Omega^q(M,F)} &= \sum_{i,x} \langle a_{x,t,i} \Delta^q_{f,t} \psi_{x,t,i}, a_{x,t,i} \psi_{x,t,i} \rangle_{\Omega^q(M,F)} \\
&\leq \sum_{i,x} \|a_{x,t,i}\|^2_{\Omega^0(M,F)} \|\Delta^q_{f,t} \psi_{x,t,i}\|_{\Omega^q(M,F)} \|\psi_{x,t,i}\|_{\Omega^q(M,F)} \\
&\leq C \sum_{i,x} \|a_{x,t,i}\|^2_{\Omega^0(M,F)} e^{-tC_1} \\
&\leq C \|\omega\|^2_{\Omega^q(M,F)} e^{-tC_1}.
\end{aligned} \tag{5.37}$$

By choosing a sufficiently large $t_0 \geq t_1$, we have for $t \geq t_0$,

$$\langle \Delta^q_{f,t} \omega, \omega \rangle_{\Omega^q(M,F)} \leq \|\omega\|^2_{\Omega^q(M,F)} e^{-tC_1} = \langle \omega, \omega \rangle_{\Omega^q(M,F)} e^{-tC_1}. \tag{5.38}$$

Now we need to prove the fourth property. For any critical point x with a sufficiently small neighborhood U_x, we will denote by $\chi_x : M \to \mathbb{R}$ a smooth bump function with support in U_x defined by $\nu_{2\epsilon}$, and define $\chi = \sum_{x \in B} \chi_x$. For $\omega \in W_2 \cap \Omega^q(M, F)$, define $\omega_1 = \chi \omega$ and $\omega_2 = (1 - \chi)\omega$. Observe that by the construction of the forms $\psi_{x,t,i}$, the support of ω_2

is disjoint with the support of any element in W_1. Therefore, both ω_1 and ω_2 are elements of $W_2 \cap \Omega^q(M,F)$. Since $\Delta^q_{f,t}$ is self adjoint,

$$\begin{aligned}\langle \Delta^q_{f,t}\omega,\omega\rangle_{\Omega^q(M,F)} &= \langle \Delta^q_{f,t}\omega_1,\omega_1\rangle_{\Omega^q(M,F)} \\ &\quad + 2\langle \Delta^q_{f,t}\omega_1,\omega_2\rangle_{\Omega^q(M,F)} + \langle \Delta^q_{f,t}\omega_2,\omega_2\rangle_{\Omega^q(M,F)}.\end{aligned} \tag{5.39}$$

We will show that there exist positive constants t_0, K_1, K_2, K_3, K_4 depending only on M, F and ϵ so that for $\omega \in W_2 \cap \Omega^q(M,F)$ and $t > t_0$ the following estimates hold:

$$\langle \Delta^q_{f,t}\omega_2,\omega_2\rangle_{\Omega^q(M,F)} \geq \langle \Delta^q\omega_2,\omega_2\rangle_{\Omega^q(M,F)} + K_1t^2\|\omega_2\|^2_{\Omega^q(M,F)} - K_2t\|\omega_2\|^2_{\Omega^q(M,F)}, \tag{5.40}$$

$$\langle \Delta^q_{f,t}\omega_1,\omega_1\rangle_{\Omega^q(M,F)} \geq K_3t\|\omega_1\|^2_{\Omega^q(M,F)}, \tag{5.41}$$

$$\langle \Delta^q_{f,t}\omega_1,\omega_1\rangle_{\Omega^q(M,F)} \geq \langle \Delta^q\omega_1,\omega_1\rangle_{\Omega^q(M,F)} - K_2t\|\omega_1\|^2_{\Omega^q(M,F)}, \tag{5.42}$$

and for any $\alpha > 0$,

$$\begin{aligned}\langle \Delta^q_{f,t}\omega_1,\omega_2\rangle_{\Omega^q(M,F)} &\geq -K_4(1+\alpha^{-2})(\|\omega_1\|^2_{\Omega^q(M,F)} + \|\omega_2\|^2_{\Omega^q(M,F)}) \\ &\quad - K_4\alpha^2\langle \Delta^q\omega_2,\omega_2\rangle_{\Omega^q(M,F)} - K_4\alpha^2\langle \Delta^q\omega_1,\omega_1\rangle_{\Omega^q(M,F)}.\end{aligned} \tag{5.43}$$

Recall that

$$\Delta^q_{f,t} = \Delta^q + t(\mathcal{L}_q + \mathcal{L}^*_q) + t^2\|\nabla f\|^2_{\Omega^q(M,F)}. \tag{5.44}$$

To prove (5.40) choose $K_1 = \inf_{z\in M\setminus\bigcup_x U_x}|\nabla f(z)|^2$ and $K_2 = \sum_{x\in M}\|(\mathcal{L}_q + \mathcal{L}^*_q)\|_{\Omega^q(M,F)}$. The estimate then follows from (5.44).

To prove (5.41), we may note that the support of ω_1 is contained in $\bigcup_x U_x$ and ω_1 is orthogonal to each $\psi_{x,t,i}$. Then (5.41) follows from Proposition 5.2.2 by setting $K_3 = C_0$.

(5.42) immediately follows from (5.44).

To prove (5.43), note that

$$|\langle(\mathcal{L}_q + \mathcal{L}^*_q)\omega_1,\omega_2\rangle_{\Omega^q(M,F)} \leq K_2|\langle\omega_1,\omega_2\rangle_{\Omega^q(M,F)}| = K_2|\langle\omega_1,\omega_2\rangle_{\Omega^q(M,F)}|, \tag{5.45}$$

and using the fact that the support of ω_2 is disjoint with the neighborhoods U_x,

$$\langle|\nabla f|^2\omega_1,\omega_2\rangle_{\Omega^q(M,F)} \geq K_1\langle\chi(1-\chi)\omega,\omega\rangle_{\Omega^q(M,F)} \geq 0. \tag{5.46}$$

Then conclude that

$$\begin{aligned}\langle \Delta^q_{f,t}\omega_1,\omega_2\rangle_{\Omega^q(M,F)} &= \langle \Delta^q\omega_1,\omega_2\rangle_{\Omega^q(M,F)} + t\langle(\mathcal{L}_q + \mathcal{L}^*_q)\omega_1,\omega_2\rangle_{\Omega^q(M,F)} \\ &\quad + t^2\langle|\nabla f|^2\omega_1,\omega_2\rangle_{\Omega^q(M,F)}\end{aligned} \tag{5.47}$$

may be used to estimate

$$\langle \Delta^q_{f,t}\omega_1, \omega_2\rangle_{\Omega^q(M,F)} \geq \langle \Delta^q \omega_1, \omega_2\rangle_{\Omega^q(M,F)} + (K_1 t^2 - K_2 t)\langle \omega_1, \omega_2\rangle_{\Omega^q(M,F)}. \tag{5.48}$$

Since $\langle \omega, \omega\rangle_{\Omega^q(M,F)}$ is real and nonnegative, for $t > K_2/K_1$

$$\langle \Delta^q_{f,t}\omega_1, \omega_2\rangle_{\Omega^q(M,F)} \geq \langle \Delta^q \omega_1, \omega_2\rangle_{\Omega^q(M,F)}. \tag{5.49}$$

Hence (5.43) follows from the fact that $2(\|\omega_1\|^2_{\Omega^q(M,F)} + |\omega_2\|^2_{\Omega^q(M,F)}) \geq \|\omega\|^2_{\Omega^q(M,F)}$ and the following proposition. We claim that there exists a positive constant $K_4 > 0$ so that for any $\alpha > 0$,

$$\begin{aligned}\langle \Delta^q \omega_1, \omega_2\rangle_{\Omega^q(M,F)} \geq -K_4(1+\alpha^{-2})\|\omega\|^2_{\Omega^q(M,F)} - K_4\alpha^2\langle \Delta^q\omega_2, \omega_2\rangle_{\Omega^q(M,F)} \\ - K_4\alpha^2\langle \Delta^q \omega_1, \omega_1\rangle_{\Omega^q(M,F)}.\end{aligned} \tag{5.50}$$

To prove (5.50), recall that $\Delta^q = d^{q-1}\delta^{q-1} + \delta^q d^q$, $\omega_1 = \chi\omega$, and $\omega_2 = (1-\chi)\omega$. Also recall that $\delta^{q-1} = -(-1)^{nq+n+1} *_{n-q+1} d^{n-q} *_q$, where $*_k$ denotes the Hodge star operator acting on k-forms. Then

$$\begin{aligned}\langle \Delta^q \omega_1, \omega_2\rangle_{\Omega^q(M,F)} &= \langle d\omega_1, d\omega_2\rangle_{\Omega^q(M,F)} + \langle d*\omega_1, d*\omega_2\rangle_{\Omega^q(M,F)} \\ &\geq A + B - \|d\chi\wedge\omega\|^2_{\Omega^q(M,F)} - \|d\chi \wedge *\omega\|^2_{\Omega^q(M,F)} \\ &+ \langle \chi d\omega, (1-\chi)d\omega\rangle_{\Omega^q(M,F)} + \langle \chi d*\omega, (1-\chi)d*\omega\rangle_{\Omega^q(M,F)},\end{aligned} \tag{5.51}$$

where

$$\begin{aligned}A &:= \langle d\chi\wedge\omega, u(1-\chi)d\omega\rangle_{\Omega^q(M,F)} + \langle d\chi\wedge *\omega, u(1-\chi)d*\omega\rangle_{\Omega^q(M,F)}, \\ B &:= -\langle \chi d\omega, u d\chi\wedge\omega\rangle_{\Omega^q(M,F)} - \langle \chi d*\omega, u d\chi \wedge *\omega\rangle_{\Omega^q(M,F)}\end{aligned} \tag{5.52}$$

and u is the characteristic function of $M \setminus \operatorname{supp}\chi$. Since $\langle \chi d\omega, (1-\chi)d\omega\rangle_{\Omega^q(M,F)}$ and $\langle \chi d * \omega, (1-\chi)d*\omega\rangle_{\Omega^q(M,F)}$ are real and nonnegative,

$$\langle \Delta^q \omega_1, \omega_2\rangle_{\Omega^q(M,F)} \geq A + B - \|d\chi\wedge\omega\|^2_{\Omega^q(M,F)} - \|d\chi\wedge *\omega\|^2_{\Omega^q(M,F)}. \tag{5.53}$$

Consider the constant $K_5 = \sup_{1\leq k\leq n}\|H_k\|_{\Omega^q(M,F)}$, where $H_k : \Omega^k(M,F) \to \Omega^{k+1}(M,F)$ is the left exterior multiplication by $d\chi$. Note that $\|H_k\|_{\Omega^q(M,F)} = \|H_k^*\|_{\Omega^q(M,F)}$, where H_k^* is the adjoint of H_k, and $\|\omega\|_{\Omega^q(M,F)} = \|{*\omega}\|_{\Omega^q(M,F)}$. Then

$$\begin{aligned}|A| &\leq K_5\|\omega\|_{\Omega^q(M,F)}(\|(1-\chi)d\omega\|_{\Omega^q(M,F)} + \|(1-\chi)d*\omega\|_{\Omega^q(M,F)}) \\ &\leq K_5\|\omega\|_{\Omega^q(M,F)}(\|d\omega_2\|_{\Omega^q(M,F)} + \|d\chi\wedge\omega\|_{\Omega^q(M,F)} + \|d*\omega_2\|_{\Omega^q(M,F)} \\ &+ \|d\chi\wedge *\omega\|_{\Omega^q(M,F)}) \\ &\leq K_5\|\omega\|_{\Omega^q(M,F)}(\|d\omega_2\|_{\Omega^q(M,F)} + \|d*\omega_2\|_{\Omega^q(M,F)} + 2K_5\|\omega\|_{\Omega^q(M,F)}) \\ &\leq \sqrt{2}K_5\|\omega\|_{\Omega^q(M,F)}\langle \Delta^q\omega_2, \omega_2\rangle^{1/2}_{\Omega^q(M,F)} + 2K_5^2\|\omega\|^2_{\Omega^q(M,F)}.\end{aligned} \tag{5.54}$$

Thus, for any $\alpha > 0$, using the inequality $bc \leq (b/\alpha)^2 + (\alpha c)^2$ where $b, c \in \mathbb{R}$,

$$|A| \leq (2K_5^2 + K_5\alpha^{-2})\|\omega\|^2_{\Omega^q(M,F)} + \alpha^2\langle\Delta^q\omega_2, \omega_2\rangle_{\Omega^q(M,F)}. \tag{5.55}$$

An almost identical calculation yields

$$|B| \leq (2K_5^2 + K_5\alpha^{-2})\|\omega\|^2_{\Omega^q(M,F)} + \alpha^2\langle\Delta^q\omega_1, \omega_1\rangle_{\Omega^q(M,F)}. \tag{5.56}$$

Noting that $A \geq -|A|$ and $B \geq -|B|$,

$$\begin{aligned}\langle\Delta^q\omega_1, \omega_2\rangle_{\Omega^q(M,F)} &\geq A + B - \|d\chi \wedge \omega\|^2_{\Omega^q(M,F)} - \|d\chi \wedge *\omega\|^2_{\Omega^q(M,F)} \\ &\geq A + B - 2K_5^2\|\omega\|^2_{\Omega^q(M,F)} \\ &\geq -2(2K_5^2 + K_5\alpha^{-2})\|\omega\|^2_{\Omega^q(M,F)} - \alpha^2\langle\Delta^q\omega_1, \omega_1\rangle_{\Omega^q(M,F)} \\ &- \alpha^2\langle\Delta^q\omega_2, \omega_2\rangle_{\Omega^q(M,F)} - 2K_5^2\|\omega\|^2_{\Omega^q(M,F)}.\end{aligned} \tag{5.57}$$

By choosing K_4 sufficiently large we obtain the desired estimate (5.50), and thus (5.43).

To finish the first step, we will prove the fourth property given the estimates (5.40), (5.41), (5.42), and (5.43). For any $0 \leq \gamma \leq 1$, multiply (5.41) by $1 - \gamma$ and multiply (5.42) by γ, and then add them to obtain

$$\langle\Delta^q_{f,t}\omega_1, \omega_1\rangle_{\Omega^q(M,F)} \geq (1-\gamma)\langle\Delta^q\omega_1, \omega_1\rangle_{\Omega^q(M,F)} + t(\gamma K_3 - (1-\gamma)K_2)\|\omega_1\|^2_{\Omega^q(M,F)}. \tag{5.58}$$

Combine (5.58) with (5.43), (5.40), and (5.39) to obtain for $0 < \gamma < 1$, $\alpha > 0$,

$$\begin{aligned}\langle\Delta^q_{f,t}\omega_1, \omega_1\rangle_{\Omega^q(M,F)} \geq\ & (1 - 2K_4\alpha^2)\langle\Delta^q\omega_2, \omega_2\rangle_{\Omega^q(M,F)} + (1 - \gamma - 2K_4\alpha^2)\langle\Delta^q\omega_1, \omega_1\rangle_{\Omega^q(M,F)} \\ &+ (K_1t^2 - K_2t - 2K_4(1 + \alpha^{-2}))\|\omega_2\|^2_{\Omega^q(M,F)} \\ &+ (t(\gamma K_4 - (1-\gamma)K_2)) - 2K_4(1 + \alpha^{-2})\|\omega_1\|^2_{\Omega^q(M,F)}.\end{aligned} \tag{5.59}$$

Proceed by choosing $0 < \gamma < 1$ so that $C_6 := \gamma K_3 - (1 - \gamma)K_2 > 0$. Then choose $\alpha > 0$ sufficiently small so that $1 - \gamma - 2C_4\alpha^2 > 0$. We obtain

$$\begin{aligned}\langle\Delta^q_{f,t}\omega, \omega\rangle_{\Omega^q(M,F)} &\geq (K_1t^2 - K_2t - 4K_4(1 + \alpha^{-2}))\|\omega_2\|^2_{\Omega^q(M,F)} \\ &+ (tK_6 - 4K_4(1 + \alpha^{-2}))\|\omega_1\|^2_{\Omega^q(M,F)}.\end{aligned} \tag{5.60}$$

If necessary, we may alter our constants K_1, K_2, K_3, K_4, so the inequality $2\|\omega_1\|^2_{\Omega^q(M,F)} + 2\|\omega_2\|^2_{\Omega^q(M,F)} \geq \|\omega\|^2_{\Omega^q(M,F)}$ can be used to obtain for some $C_2 > 0$

$$\begin{aligned}\langle\Delta^q_{f,t}\omega, \omega\rangle_{\Omega^q(M,F)} &\geq (K_1t^2 - K_2t - 4K_4(1 + \alpha^{-2}))\|\omega_2\|^2_{\Omega^q(M,F)} \\ &+ (tK_6 - 4K_4(1 + \alpha^{-2}))\|\omega_1\|^2_{\Omega^q(M,F)} \\ &\geq C_2t\langle\omega, \omega\rangle_{\Omega^q(M,F)}.\end{aligned} \tag{5.61}$$

This establishes the fourth property.

Now we will move on to the second step. Assume for the purposes of obtaining a contradiction that there exists some $0 \leq q \leq n, t \geq t_0$ and some $\mu \in \mathbb{R}$ so that $\mu \in \operatorname{spec} \Delta^q_{f,t} \cap (e^{-tC_1}, C_2 t)$. Then there exists a sequence $(u_i) = (u_i)_{i \in \mathbb{Z}_{>0}}$ of unit eigenfunctions $u_i \in \Omega^q(M, F)$ which satisfies

$$\|\Delta^q_{f,t} u_i - \mu u_i\|_{\Omega^q(M,F)} \leq \frac{1}{i}. \tag{5.62}$$

Since $\Omega^q(M, F) \subseteq \Omega^q(M, F) = W_1 + W_2$, for each u_i there exists some $v_i \in W_1$ and $w_i \in W_2$ so that $u_i = v_i + w_i$. Moreover, since each $u_i \in \Omega^q(M, F)$, we may choose the functions w_i so that $w_i \in W_2 \cap \Omega^q(M, F)$. Then, since $\Delta^q_{f,t}$ is self-adjoint,

$$\langle \Delta^q_{f,t} u_i, v_i \rangle_{\Omega^q(M,F)} = \langle \Delta^q_{f,t} v_i, v_i \rangle_{\Omega^q(M,F)} + \langle w_i, \Delta^q_{f,t} v_i \rangle_{\Omega^q(M,F)}, \tag{5.63}$$

$$\langle \Delta^q_{f,t} u_i, w_i \rangle_{\Omega^q(M,F)} = \langle \Delta^q_{f,t} v_i, w_i \rangle_{\Omega^q(M,F)} + \langle w_i, \Delta^q_{f,t} w_i \rangle_{\Omega^q(M,F)}. \tag{5.64}$$

Also,

$$\begin{aligned} \mu \|v_i\|^2_{\Omega^q(M,F)} &= \langle \mu u_i, v_i \rangle_{\Omega^q(M,F)} \\ &= \langle \Delta^q_{f,t} u_i, v_i \rangle_{\Omega^q(M,F)} - \langle \Delta^q_{f,t} u_i - \mu u_i, v_i \rangle_{\Omega^q(M,F)} \end{aligned} \tag{5.65}$$

$$\begin{aligned} \mu \|w_i\|^2_{\Omega^q(M,F)} &= \langle \mu u_i, w_i \rangle_{\Omega^q(M,F)} \\ &= \langle \Delta^q_{f,t} u_i, w_i \rangle_{\Omega^q(M,F)} - \langle \Delta^q_{f,t} u_i - \mu u_i, w_i \rangle_{\Omega^q(M,F)}. \end{aligned} \tag{5.66}$$

Then,

$$\begin{aligned} \mu \|v_i\|^2_{\Omega^q(M,F)} - \langle \Delta^q_{f,t} v_i, v_i \rangle_{\Omega^q(M,F)} + \langle \Delta^q_{f,t} u_i - \mu u_i, v_i \rangle_{\Omega^q(M,F)} \\ = \langle w_i, \Delta^q_{f,t} v_i \rangle_{\Omega^q(M,F)}, \end{aligned} \tag{5.67}$$

and similarly,

$$\begin{aligned} \mu \|w_i\|^2_{\Omega^q(M,F)} - \langle \Delta^q_{f,t} w_i, w_i \rangle_{\Omega^q(M,F)} + \langle \Delta^q_{f,t} u_i - \mu u_i, w_i \rangle_{\Omega^q(M,F)} \\ = \langle \Delta^q_{f,t} v_i, w_i \rangle_{\Omega^q(M,F)}. \end{aligned} \tag{5.68}$$

Note that $\langle w_i, \Delta^q_{f,t} v_i \rangle_{\Omega^q(M,F)} = \langle \Delta^q_{f,t} v_i, w_i \rangle_{\Omega^q(M,F)}$. Therefore,

$$\begin{aligned} \mu \|v_i\|^2_{\Omega^q(M,F)} - \langle \Delta^q_{f,t} v_i, v_i \rangle_{\Omega^q(M,F)} &= -\langle \Delta^q_{f,t} u_i - \mu u_i, v_i - w_i \rangle_{\Omega^q(M,F)} \\ &\quad + \mu \|w_i\|^2_{\Omega^q(M,F)} - \langle \Delta^q_{f,t} w_i, w_i \rangle_{\Omega^q(M,F)}. \end{aligned} \tag{5.69}$$

Using

$$\langle \Delta^q_{f,t} v_i, v_i \rangle_{\Omega^q(M,F)} \leq e^{-tC_1} \langle v_i, v_i \rangle_{\Omega^q(M,F)}, \tag{5.70}$$

$$\langle \Delta^q_{f,t} w_i, w_i \rangle_{\Omega^q(M,F)} \geq C_2 t \langle w_i, w_i \rangle_{\Omega^q(M,F)}, \tag{5.71}$$

$\|w_i\|^2_{\Omega^q(M,F)} = 1 - \|v_i\|^2_{\Omega^q(M,F)} \leq 1$, and

$$\|\Delta^q_{f,t} u_i - \mu u_i\|_{\Omega^q(M,F)} \leq \frac{1}{i}, \tag{5.72}$$

we obtain

$$(\mu - e^{-tC_1})\|v_i\|^2_{\Omega^q(M,F)} \leq \frac{2}{i} + (\mu - C_2 t)(1 - \|v_i\|^2_{\Omega^q(M,F)}). \tag{5.73}$$

Without a loss of generality, we may assume that $\lim_{i\to\infty}\|v_i\|^2_{\Omega^q(M,F)} = x^2$ exists. Then $x^2 \leq 1$ and

$$(\mu - e^{-tC_1})x^2 \leq (\mu - C_2 t)(1 - x^2). \tag{5.74}$$

Note that $(\mu - e^{-tC_1}) \geq 0$, $(\mu - C_2 t)(1 - x^2) \leq 0$, and both $(\mu - e^{-tC_1})$ and $(\mu - C_2 t)(1 - x^2)$ cannot be simultaneously 0, which is a contradiction. $\square$

Using Theorem 5.2.3, we may factorize the Ray-Singer torsion into small and large components.

For t sufficiently large, let $\Omega_{sm,t} = \Omega_{sm,t}(M,F) \subseteq \Omega^\bullet(M,F)$ be the subspace spanned by eigenforms of $\Delta_{f,t}$ with eigenvalues in $\operatorname{spec}(\Delta_{f,t}) \cap [0,1]$. Similarly, let $\Omega_{la,t} = \Omega_{la,t}(M,F) \subseteq \Omega^\bullet(M,F)$ be the subspace spanned by eigenforms of $\Delta_{f,t}$ with eigenvalues in $[1,\infty)$. Both $\Omega_{la,t}$ and $\Omega_{sm,t}$ inherit the inner product $\langle\cdot,\cdot\rangle_{\Omega^\bullet(M,F)}$. We will also denote by $P_{sm,t} : \Omega^\bullet(M,F) \to \Omega_{sm,t}$ the orthogonal projection with respect to $\langle\cdot,\cdot\rangle_{\Omega^\bullet(M,F)}$, and define $P_{la,t} = \operatorname{id} - P_{sm,t}$.

Let $\Delta_{sm,f,t}$ denote the restriction of $\Delta_{f,t}$ to $\Omega_{sm,t}$, and let $\Delta_{la,f,t}$ denote the restriction of $\Delta_{f,t}$ to $\Omega_{la,t}$.

For each q, let $\Omega^q_{sm,t} = \Omega_{sm,t} \cap \Omega^q(M,F)$ and $\Omega^q_{la,t} = \Omega_{la,t} \cap \Omega^q(M,F)$. Finally, let $\Delta^q_{sm,f,t}$ denote the restriction of $\Delta_{sm,f,t}$ to $\Omega^q_{sm,t}$ and let $\Delta^q_{la,f,t}$ denote the restriction of $\Delta_{la,f,t}$ to $\Omega^q_{la,t}$.

Let $\tau_{RS,la}(f,t)$ and $\tau_{RS,sm}(f,t)$ denote the Ray-Singer torsion computed with the determinant of $\Delta_{la,f,t}$ and $\Delta_{sm,f,t}$ respectively. Since for each q, $\operatorname{spec}(\Delta^q_{f,t}) = \operatorname{spec}(\Delta^q_{la,f,t}) \cup \operatorname{spec}(\Delta_{sm,f,t})$, we have

$$\tau_{RS} = \tau_{RS}(f,t) = \tau_{RS,la}(f,t)\tau_{RS,sm}(f,t). \tag{5.75}$$

Thus we may treat $\tau_{RS,sm}(f,t)$ and $\tau_{RS,la}(f,t)$ separately, which we will do in subsequent chapters.

Chapter 6

Asymptotic Expansion of the Small Eigenvalues

The purpose of this chapter is to establish the following theorem.

Theorem 6.0.1. *Suppose that the pair (g^{TM}, f) is a generalized triangulation. Then as $t \to \infty$ we have*

$$\log \frac{|\cdot|^{Hodge}_{\det H^\bullet(M,F)} \tau_{RS,sm}(f,t)}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}} = -t \operatorname{rank}(F) \operatorname{Tr}_s^{\operatorname{Cr}(f)}[f] + \frac{1}{2}\tilde{\chi}'(F) \log\left(\frac{t}{\pi}\right) + O(1), \quad (6.1)$$

where $\operatorname{Tr}_s^{\operatorname{Cr}(f)}[f] = \sum_{x\in \operatorname{Cr}(f)} (-1)^{\operatorname{ind}(x)} f(x)$ *and* $\tilde{\chi}'(F) = \operatorname{rank}(F) \sum_{x \in \operatorname{Cr}(f)} (-1)^{\operatorname{ind}(x)} \operatorname{ind}(x)$.

Note that since we assumed f to be self indexing, $\operatorname{rank}(F) \operatorname{Tr}_s^{\operatorname{Cr}(f)}[f] = \tilde{\chi}'(F)$. From Theorem 6.0.1 we obtain the following corollary.

Corollary 6.0.2. *As $t \to \infty$,*

$$\log \tau_{RS,la}(f,t) = \log \frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}} + t \operatorname{rank}(F) \operatorname{Tr}_s^{\operatorname{Cr}(f)}[f] - \frac{1}{2}\tilde{\chi}'(F) \log\left(\frac{t}{\pi}\right) + O(1). \quad (6.2)$$

Proof. Since $\tau_{RS}(M,F) = \tau_{RS,la}(f,t)\tau_{RS,sm}(f,t)$, $\log \tau_{RS}(M,F) = \log \tau_{RS,sm}(f,t) + \log \tau_{RS,la}(f,t)$, we have that

$$\log \frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}} = \log \frac{|\cdot|^{Hodge}_{\det H^\bullet(M,F)} \tau_{RS,sm}(f,t)}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}} + \log \tau_{RS,la}(f,t). \quad (6.3)$$

Then adding $\log \tau_{RS,la}(f,t)$ to both sides of (6.1) gives

$$\begin{aligned} \log \frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}} &= \log \tau_{RS,la}(f,t) - t\,\mathrm{rank}(F)\,\mathrm{Tr}_s^{\mathrm{Cr}(f)}[f] \\ &\quad + \frac{1}{2}\tilde{\chi}'(F)\log\left(\frac{t}{\pi}\right) + O(1). \end{aligned} \tag{6.4}$$

Rearranging (6.4) gives (6.2). □

Remark 6.0.3. Theorem 6.0.1 was originally proven in [BZ92, Theorem 7.6] using the results of Helffer and Sjöstrand [HS84], [HS85b], [HS85a], [HS85c], and instantons. Later Bismut and Zhang proved Theorem 6.0.1 for the G-equivariant case while avoiding using instantons in [BZ94]. We will take the latter approach while assuming $G = \{1\}$ and adjusting our notation accordingly.

6.1 An Isometry From the Morse Complex to the de Rham Complex

Recall that M is a closed Riemannian manifold manifold of odd dimension and $F \to M$ is a real flat vector bundle. Also, g^{TM} and g^F are metrics on TM and F so that g^F induces a flat metric on $\det F$. Finally, recall that g^{TM} and g^F induce the inner product $\langle\cdot,\cdot\rangle_{\Omega^\bullet(M,F)}$ on $\Omega^\bullet(M,F)$. Additionally, recall that $f : M \to \mathbb{R}$ is a smooth function.

Definition 6.1.1. The *Dirac operator* is defined by

$$D = d^F + \delta^F, \tag{6.5}$$

and the *deformed Dirac operator* is defined by

$$D_{f,t} = d^F_t + \delta^F_t. \tag{6.6}$$

Note that $D^2 = \Delta$ and $D^2_{f,t} = \Delta_{f,t}$.

Let $x \in \mathrm{Cr}_k(f)$. For any $\epsilon > 0$, let $B^M(x,\epsilon)$ be the ball of radius ϵ centered at $x \in M$, where the distance function is determined by g^{TM}. Note that for some $\epsilon > 0$ small enough, $B^M(x,\epsilon) \subseteq U_x$, so we may use the preferred coordinates described in Definition 3.3.1. We will also choose $\epsilon > 0$ small enough so the results of Chapter 5 apply.

Note that $\mathbb{R}^k \cong T_xW^u(x)$ inherits an orientation from the orientation of $W^u(x)$. Let ρ_x be the volume form on $T_xW^u(x)$. Without a loss of generality assume that the coordinate chart $(y^1, ..., y^n)$ is chosen so that

$$\rho_x = dy^1 \wedge \cdots \wedge dy^k. \tag{6.7}$$

Let $\mu : M \to \mathbb{R}$ be a bump function so that $\mu(y) = 1$ for $y \in B^M(x, \epsilon/2)$ and so that μ vanishes outside of $B^M(x, \epsilon)$.

Set

$$\alpha_t = \int_M \mu^2(y) \exp(-t\,\mathrm{dist}(x, y)^2)\omega, \tag{6.8}$$

where ω is the volume form determined by the metric g^{TM} and dist is the distance function determined by the metric g^{TM}. In the case where M is not orientable, we may still perform this integration by utilising densities.

As we will see, α_t is a *normalization factor* which will ensure that the map we define next is an isometry.

Since α_t is the integral of a Gaussian function multiplied by the bump function μ^2, there exists some $c > 0$ so that as $t \to \infty$,

$$\alpha_t = \left(\frac{\pi}{t}\right)^{n/2} + O(e^{-ct}). \tag{6.9}$$

Definition 6.1.2. For $x \in B$, $t > 0$, let J_t be the linear map from $C^\bullet(W^u, F)$ into $\Omega^\bullet(M, F)$ such that if $x \in B$, $h \in F_x$, $y \in B^M(x, \epsilon)$,

$$J_t(W^u(x)^* \otimes h)(y) = \frac{\mu(y)}{(\alpha_t)^{1/2}} \exp\left(\frac{-t\,\mathrm{dist}(x, y)^2}{2}\right)\rho_x \otimes h. \tag{6.10}$$

Proposition 6.1.3. *J_t is an isometry from $C^\bullet(W^u, F)$ into $\Omega^\bullet(M, F)$ which preserves the $\mathbb{Z}$-grading.*

Proof. If $x \in \mathrm{Cr}_q$, $W^u(x)^* \in C^q(W^u, F)$ and $\rho_x \in \Omega^q(M, F)$, by definition $J_t(W^u(x)^* \otimes h) \in \Omega^q(M, F)$. So J_t preserves the $\mathbb{Z}$-grading.

To show that J_t is an isometry, recall that the inner product on $C^\bullet(W^u, F)$ is determined by the elements $W^u(x)^* \in C^q(W^u)$, where $x \in \mathrm{Cr}(f)$, and the metric on F. Suppose that $x_1, x_2 \in \mathrm{Cr}(f)$ so that $x_1 \neq x_2$ and let μ_1, μ_2 be the bump functions associated to $B^M(x_1, \epsilon_1/2)$ and $B^M(x_2, \epsilon_2/2)$ respectively. Then the support of μ_1 and μ_2 are disjoint, since the sets $B^M(x_1, \epsilon_1/2)$, $B^M(x_2, \epsilon_2/2)$ were assumed to be disjoint. Therefore

$$\langle J_t(W^u(x_1)^* \otimes h_1), J_t(W^u(x_2)^* \otimes h_2)\rangle_{\Omega^\bullet(M,F)} = 0 \tag{6.11}$$

for any $h_1, h_2 \in F$.

Now suppose $x \in \mathrm{Cr}(f)$, and $h \in F$ so that h is a unit vector. Then

$$\begin{aligned}
&\|J_t(W^u(x)^* \otimes h)\|^2_{\Omega^\bullet(M,F)} \\
&= \int_M \left(\frac{\mu(y)}{(\alpha_t)^{1/2}} \exp\left(\frac{-t\,\mathrm{dist}(x,y)^2}{2}\right)\rho_x\right) \wedge \star\left(\frac{\mu(y)}{(\alpha_t)^{1/2}} \exp\left(\frac{-t\,\mathrm{dist}(x,y)^2}{2}\right)\rho_x\right) \\
&= \int_M \frac{\mu^2(y)}{\alpha_t} \exp(-t\,\mathrm{dist}(x,y)^2)\omega \\
&= \frac{1}{\alpha_t}\int_M \mu^2(y) \exp(-t\,\mathrm{dist}(x,y)^2)\omega \\
&= \frac{\alpha_t}{\alpha_t} = 1.
\end{aligned} \tag{6.12}$$

So J_t maps the orthonormal basis of $C^\bullet(W^u, F)$ to orthonormal vectors in $\Omega^\bullet(M, F)$. $\square$

Definition 6.1.4. Let $\tilde{e}_t : C^\bullet(W^u, F) \to \Omega_{sm,t}$ be given by

$$\tilde{e}_t = P_{sm,t} J_t. \tag{6.13}$$

We may intuitively consider $\tilde{e}_t$ to be the isometry J_t whose codomain has been restricted to the eigenspace spanned by eigenforms corresponding to small eigenvalues of $\Delta_{f,t}$. We will eventually show that for sufficiently large t, the map $e^{tf}\tilde{e}_t$ is actually an *isometric isomorphism* which preserves the $\mathbb{Z}$-grading. To do so, we need the following theorem.

Theorem 6.1.5. *There exists $c > 0$ such that as $t \to +\infty$, for any $s \in C^\bullet(W^u, F)$,*

$$\|(\tilde{e}_t - J_t)s\|_{\Omega^\bullet(M,F)} = O(e^{-ct})\|s\|_{C^\bullet(W^u,F)} \tag{6.14}$$

uniformly on M.

Proof. It sufficies to consider the case where $s = W^u(x)^* \otimes h$ where $x \in B$, $h \in F_x$. Note $\|s\|_{C^\bullet(W^u,F)} = \|h\|_{F_x}$. Set

$$J_{t,x}h = J_t(W^u(x)^* \otimes h). \tag{6.15}$$

Let S be the oriented circle of center 0 and radius $1/2$ in $\mathbb{C}$. By Proposition 5.2.3, for $t \geq 0$ large enough, for each $\lambda \in \mathrm{spec}(\Delta_{sm,f,t})$ we have that $|\lambda| < \frac{1}{2}$. So we may consider the Riesz projector:

$$P_{sm,t} = \frac{1}{2\pi i}\int_S (\lambda - \Delta_{f,t})^{-1} d\lambda. \tag{6.16}$$

Also, if $\lambda \in \mathbb{C}^*$, then

$$(\lambda - \Delta_{f,t})\frac{J_{t,x}h}{\lambda} - J_{t,x}h = -\frac{\Delta_{f,t}J_{t,x}h}{\lambda}, \tag{6.17}$$

so

$$\frac{J_{t,x}h}{\lambda} - (\lambda - \Delta_{f,t})^{-1}J_{t,x}h = -(\lambda - \Delta_{f,t})^{-1}\frac{\Delta_{f,t}J_{t,x}h}{\lambda}. \tag{6.18}$$

Since $\mu(y) = 1$ for $y \in B^M(x, \epsilon/2)$, by [BZ92, Proposition 8.3]

$$(\Delta_{f,t}J_{t,x}h)(y) = 0 \tag{6.19}$$

for $y \in B^M(x, \epsilon/2)$.

For $p \geq 0$, let $L^{p,2}$ be the p-th Sobolev space of sections of $\Omega^\bullet(M, F)$ over M equipped with the Sobolev norm $\|\cdot\|_{L^{p,2}}$. Using (6.19), for any $p \geq 0$, there is some $c > 0$ such that as $t \to +\infty$,

$$\|\Delta_{f,t}J_{t,x}h\|_{L^{p,2}} = O(e^{-ct})\|f\|_{F_x}. \tag{6.20}$$

Now take $q \geq 1$. Since Δ is elliptic, by elliptic regularity there exists $C_1 > 0$ so that if $s \in \Omega^\bullet(M, F)$, then

$$\|s\|_{L^{2q,2}} \leq C_1(\|\Delta s\|_{L^{2q-2,2}} + \|s\|_{L^{0,2}}). \tag{6.21}$$

By expanding the Witten Laplacian we see that

$$\Delta_{f,t} = \Delta + t[D, \hat{c}(\nabla f)] + t^2|\nabla f|^2. \tag{6.22}$$

The principal symbol of D anticommutes with $\hat{c}(\nabla f)$, so $[D, \hat{c}(\nabla f)]$ is an even operator of order 0. Using (6.22), there exists $C_2 > 0$ such that for $\lambda \in S$, $t \geq 1$, $s \in \Omega^\bullet(M, F)$,

$$\begin{aligned}
\|(\lambda - \Delta_{f,t} + \Delta)s\|_{L^{2q-2,2}} &= \|(\lambda - \Delta - t[D, \hat{c}(\nabla f)] - t^2|\nabla f| + \Delta)s\|_{L^{2q-2,2}} \\
&= \|(\lambda - t[D, \hat{c}(\nabla f)] - t^2|\nabla f|)s\|_{L^{2q-2,2}} \\
&= \left\| \left(\frac{1}{t^2}\lambda - \frac{1}{t}[D, \hat{c}(\nabla f)] - |\nabla f|\right) t^2 s \right\|_{L^{2q-2,2}} \\
&\leq C_2 t^2 \|s\|_{L^{2q-2,2}},
\end{aligned} \tag{6.23}$$

where the last inequality holds by the fact that λ, $[D, \hat{c}(\nabla f)]$, and $|\nabla f|$ all act by multiplication of some constant, and are thus all bounded by some constant $C_2 > 0$.

Using (6.21) and (6.23), we have that for $\lambda \in S$, $t \geq 1$, $s \in \Omega^\bullet(M, F)$,

$$\begin{aligned}
\|s\|_{L^{2q,2}} &\leq C_1(\|\Delta s\|_{L^{2q-2,2}} + \|s\|_{L^{0,2}}) \\
&\leq C_1(\|(\lambda - \Delta_{f,t} + \Delta)s\|_{L^{2q-2,2}} + \|(\lambda - \Delta_{f,t})s\|_{L^{2q-2,2}} + t^2\|s\|_{L^{2q-2,2}}) \\
&\leq C_1(\|(\lambda - \Delta_{f,t})s\|_{L^{2q-2,2}} + C_2 t^2\|s\|_{L^{2q-2,2}} + t^2\|s\|_{L^{2q-2,2}}) \\
&\leq C_1((C_2 + 1)\|(\lambda - \Delta_{f,t})s\|_{L^{2q-2,2}} + (C_2 + 1)t^2\|s\|_{L^{2q-2,2}}) \\
&\leq C_3(\|(\lambda - \Delta_{f,t})s\|_{L^{2q-2,2}} + t^2\|s\|_{L^{2q-2,2}}),
\end{aligned} \tag{6.24}$$

where in the final inequality we have set $C_3 = C_1(C_2 + 1) > 0$.

Using (6.24), there exists $C_4 > 0$ such that for $\lambda \in S$, $t \geq 1$, $s \in \Omega^\bullet(M, F)$,

$$\begin{aligned} \|s\|_{L^{2q,2}} &\leq C_3(\|(\lambda - \Delta_{f,t})s\|_{L^{2q-2,2}} + t^2\|s\|_{L^{2q-2,2}}) \\ &\leq C_3(\|(\lambda - \Delta_{f,t})s\|_{L^{2q-2,2}} + t^2 A_1(\|(\lambda - \Delta_{f,t})s\|_{L^{2q-4,2}} + t^2\|s\|_{L^{2q-4,2}})) \end{aligned} \tag{6.25}$$

for some constant $A_1 > 0$. Proceeding inductively we obtain

$$\begin{aligned} \|s\|_{L^{2q,2}} &\leq C_3\|(\lambda - \Delta_{f,t})s\|_{L^{2q-2,2}} + C_3 A_1 t^2\|(\lambda - \Delta_{f,t})s\|_{L^{2q-4,2}} + \cdots \\ &\quad + C_3 A_1 \ldots A_{q-1} t^{2q-2}\|(\lambda - \Delta_{f,t})s\|_{L^{0,2}} + C_3 A_1 \ldots A_{q-1} t^{2q}\|s\|_{L^{0,2}} \\ &\leq C_4 t^{2q}(\|(\lambda - \Delta_{f,t})s\|_{L^{2q-2,2}} + \|s\|_{L^{0,2}}), \end{aligned} \tag{6.26}$$

where $C_4 > 0$.

By Theorem 5.2.3, for $t \geq 1$ large enough, if $\lambda \in S$, then $\lambda \notin \operatorname{spec}(\Delta_{f,t})$. In particular, there exists some $C_5 > 0$ such that for $t \geq 1$ large enough, $\lambda \in S$, and $s \in \Omega^\bullet(M, F)$,

$$\|(\lambda - \Delta_{f,t})^{-1}s\|_{L^{0,2}} \leq C_5\|s\|_{L^{0,2}}. \tag{6.27}$$

By (6.26), (6.27), for $\lambda \in S$, $t \geq 1$, $s \in \Omega^\bullet(M, F)$,

$$\begin{aligned} \|(\lambda - \Delta_{f,t})^{-1}s\|_{L^{2q,2}} &\leq C_4 t^{2q}(\|s\|_{L^{2q-2,2}} + \|(\lambda - \Delta_{f,t})^{-1}s\|_{L^{0,2}}) \\ &\leq C_4 t^{2q}(\|s\|_{L^{2q-2,2}} + C_5\|s\|_{L^{0,2}}) \\ &\leq C_6 t^{2q}\|s\|_{L^{2q-2,2}}, \end{aligned} \tag{6.28}$$

where $C_6 > 0$.

Using (6.20) and (6.28), for $t \geq 1$ large enough,

$$\begin{aligned} \|(\lambda - \Delta_{f,t})^{-1}\Delta_{f,t}J_{t,x}h\|_{L^{2q,2}} &\leq C_6 t^{2q}\|\Delta_{f,t}J_{t,x}h\|_{L^{2q-2,2}} \\ &= C_6 T^{2q} O(e^{-ct})\|h\|_{F_x} \\ &= O(e^{-ct})\|h\|_{F_x} \end{aligned} \tag{6.29}$$

uniformly in $\lambda \in S$. By choosing $q \geq n/2$, and considering the Sobolev embedding theorem for q-forms, we obtain from (6.29)

$$\|(\lambda - \Delta_{f,t})^{-1}\Delta_{f,t}J_{t,x}h\|_{\Omega^\bullet(M,F)} \leq O(e^{-ct})\|h\|_{F_x} \tag{6.30}$$

uniformly on M. Now, note that

$$\begin{aligned} (\tilde{e}_t - J_t)s &= (P_{sm,t}J_t - J_t)s \\ &= (P_{sm,t} - 1)J_t s \\ &= \frac{1}{2\pi i}\Big(\int_S (\lambda - \Delta_{f,t})^{-1}d\lambda - \int_S \lambda^{-1}d\lambda\Big)J_t s \\ &= \frac{1}{2\pi i}\Big(\int_S \frac{\lambda(\lambda - \Delta_{f,t})^{-1} - 1}{\lambda}d\lambda\Big)J_t s \end{aligned} \tag{6.31}$$

Since $J_t s$ has no dependence on λ,

$$
\begin{aligned}
(\tilde{e}_t - J_t)s &= \frac{1}{2\pi i}\int_S \frac{1}{\lambda}(\lambda(\lambda - \Delta_{f,t})^{-1}J_{t,x}h - J_{t,x}h)d\lambda \\
&= \frac{1}{2\pi i}\int_S \frac{1}{\lambda}(\lambda - \Delta_{f,t})^{-1}\Delta_{f,t}J_{t,x}hd\lambda \\
&\leq \frac{1}{2\pi i}\int_S \frac{1}{\lambda}\|(\lambda - \Delta_{f,t})^{-1}\Delta_{f,t}J_{t,x}h\|_{\Omega^\bullet(M,F)}d\lambda \\
&\leq \frac{1}{2\pi i}\int_S \frac{O(e^{-ct})\|h\|_{F_x}}{\lambda}d\lambda \\
&= \frac{O(e^{-ct})\|h\|_{F_x}}{2\pi i}\int_S \frac{1}{\lambda}d\lambda \\
&= O(e^{-ct})\|h\|_{F_x}
\end{aligned}
\tag{6.32}
$$

□

Definition 6.1.6. For $t \geq 0$, let e_t be the linear map from $C^\bullet(W^u, F)$ into $\Omega^i_{sm,t}$,

$$e_t = e^{tf}\tilde{e}_t. \tag{6.33}$$

Let $\langle\cdot,\cdot\rangle_{\Omega_{sm,t}}$ denote the inner product induced by $\langle\cdot,\cdot\rangle_{\Omega^\bullet(M,F),t}$ on $\Omega_{sm,t}$. Finally, let e_t^* be the adjoint of e_t with respect to this inner product.

Theorem 6.1.7. *There exists $c > 0$ such that as $t \to \infty$*

$$e_t^* e_t = \operatorname{id} + O(e^{-ct}). \tag{6.34}$$

In particular, for $t \geq 0$ large enough, $e_t : C^\bullet(W^u, F) \to \Omega_{sm,t}$ is an isometric isomorphism of $\mathbb{Z}$-graded vector spaces.

Proof. First we will prove that $\tilde{e}_t$ is an isomorphism. Let $X_t = \tilde{e}_t(C^\bullet(M,F)) \subset \Omega_{sm,t}$. We will show that if $u \in \Omega_{sm,t}$ is orthogonal to X_t with respect to $\langle\cdot,\cdot\rangle_{\Omega_{sm,t}}$, then $u = 0$. Recall the differential forms $\psi_{x,t,i}$ were defined in Chapter 5. Since $P_{sm,t}$ is an orthogonal projection, it is self-adjoint so

$$\langle\psi_{x,t,i}, u\rangle_{\Omega^\bullet(M,F)} = \langle\psi_{x,t,i}, P_{sm,t}u\rangle_{\Omega^\bullet(M,F)} = \langle P_{sm,t}\psi_{x,t,i}, u\rangle_{\Omega^\bullet(M,F)} = 0, \tag{6.35}$$

since u was assumed to be orthogonal to X_t, and each $P_{sm,t}\psi_{x,t,i} \in X_t$. Thus, by Theorem 5.2.2,

$$\langle\Delta_{f,t}u, u\rangle_{\Omega^\bullet(M,F)} \geq Ct\|u\|^2_{\Omega^\bullet(M,F)}. \tag{6.36}$$

However, $u \in \Omega_{sm,t}$, so it is the sum of eigenvectors with eigenvalues less than 1. Thus

$$\langle \Delta_{f,t} u, u \rangle_{\Omega^\bullet(M,F)} \leq \|u\|^2_{\Omega^\bullet(M,F)}. \tag{6.37}$$

From the above inequalities we require $\|u\|^2 = 0$, so $u = 0$ as desired. $X_t = \Omega_{sm,t}$, so $\tilde{e}_t$ is an isomorphism. Note that

$$\langle e_t s, e_t s' \rangle_{\Omega_{sm,t},t} = \langle \tilde{e}_t s, \tilde{e}_t s' \rangle_{\Omega^\bullet(M,F)}. \tag{6.38}$$

Using this, we may conclude that since $\tilde{e}_t$ is an isomorphism, e_t must be injective. To conclude that e_t is an isomorphism recall that for sufficiently large t, note that $\mathrm{rank}(\Omega^q_{sm,t}) = \mathrm{rank}(C^q(W^u, F))$.

Also, from Theorem 6.1.5, it follows that

$$e_t^* e_t = 1 + O(e^{-ct}) \tag{6.39}$$

for some $c > 0$. □

6.2 The Asymptotics of $\mathbf{P_{\infty,t} e_t}$

Recall from Chapter 4 the linear map

$$P_\infty : \alpha \in \Omega^\bullet(M,F) \mapsto \sum_{x \in B} W^u(x)^* \otimes \int_{\overline{W}^u(x)} \alpha \in C^\bullet(W^u, F) \tag{6.40}$$

is a quasi-isomorphism of complexes, which induces the identifications

$$H^\bullet(M,F) \cong H^\bullet_{dR}(M,F) \cong H^\bullet(C^\bullet(W^u,F)). \tag{6.41}$$

Note that $(\Omega_{sm,t}, d^F)$ is a subcomplex of $(\Omega^\bullet(M,F), d^F)$ which contains every harmonic form in $\Omega^\bullet(M,F)$. Therefore,

$$H^\bullet(\Omega_{sm,t}) \cong \ker \Delta_{sm,f,t} = \ker \Delta \cong H^\bullet_{dR}(M,F). \tag{6.42}$$

The restriction of P_∞ to $\Omega_{sm,t}$,

$$P_{\infty,t} : \alpha \in \Omega_{sm,t} \mapsto P_\infty \alpha \in C^\bullet(W^u, F) \tag{6.43}$$

is also a quasi-isomorphism of complexes, which induces the canonical identificiation

$$H^\bullet(\Omega_{sm,t}) \cong H^\bullet(C^\bullet(W^u,F)) \cong H^\bullet(M,F). \tag{6.44}$$

Definition 6.2.1. Let $\mathcal{F} \in \mathrm{End}(C^\bullet(W^u, F))$ which, for $x \in B$ acts on $[W^u(x)]^* \otimes F_x$ by multiplication by $f(x)$.

Denote by $N \in \mathrm{End}(C^\bullet(W^u, F))$ the operator acting on $C^i(W^u, F)$ by multiplication by i.

From now on, for $c > 0$, $O(e^{-ct})$ denotes an element of $\mathrm{End}(C^\bullet(W^u, F))$ which preserves the $\mathbb{Z}$-grading and is $O(e^{-ct})$ as $t \to \infty$.

Theorem 6.2.2. *There exists $c > 0$ such that as $t \to +\infty$*

$$P_{\infty,t}e_t = e^{t\mathcal{F}}\left(\frac{\pi}{T}\right)^{N/2-n/4}(1 + O(e^{-ct})). \tag{6.45}$$

In particular for $t \geq 0$ large enough, $P_{\infty,t}e_t \in \mathrm{End}(C^\bullet(W^u, F))$ is one-to-one.

Proof. Let $x \in B$ and let $h \in F_x$. Define

$$s = W^u(x)^* \otimes h. \tag{6.46}$$

Recall that by definition

$$P_\infty : \alpha \mapsto \sum_{x\in B} W^u(x)^* \otimes \int_{\overline{W}^u(x)} \alpha. \tag{6.47}$$

Additionally, $P_{\infty,t}$ is the restriction of P_∞ to $\Omega_{sm,t}(M, F)$. Thus

$$\begin{aligned} P_{\infty,t}e_t s &= \sum_{y\in B^{\mathrm{ind}(x)}} W^u(y)^* \otimes \int_{\overline{W}^u(y)} e^{tf}\tilde{e}_t s \\ &= \sum_{y\in B^{\mathrm{ind}(x)}} e^{tf(y)}W^u(y)^* \otimes \int_{\overline{W}^u(y)} e^{t(f-f(y))}\tilde{e}_t s. \end{aligned} \tag{6.48}$$

Note that if $y \in B$, $f - f(y) \leq 0$ on $\overline{W}^u(y)$ by definition.

Recall from Chapter 3 that each $\overline{W}^u(y)$ is a compact manifold with conical simplicialities. Using Theorem 6.1.5, as $t \to \infty$

$$\begin{aligned} \int_{\overline{W}^u(y)} e^{t(f-f(y))}\tilde{e}_t s &= \int_{\overline{W}^u(y)} e^{t(f-f(y))}(\tilde{e}_t s - J_t s + J_t s) \\ &= \int_{\overline{W}^u(y)} e^{t(f-f(y))}J_t s + \int_{\overline{W}^u(y)} e^{t(f-f(y))}(\tilde{e}_t - J_t)s \\ &= \int_{\overline{W}^u(y)} e^{t(f-f(y))}J_t s + O(e^{-ct})h. \end{aligned} \tag{6.49}$$

Recall that if $z \in B^M(x,\epsilon)$, $J_t s(z) = \frac{\mu(z)}{(\alpha_t)^{1/2}} \exp\left(\frac{-t|z|^2}{2}\right)\rho_x \otimes h$. Since the support of μ is contained in $B^M(x,\epsilon)$, the support of $J_t s$ is also contained in $B^M(x,\epsilon)$. Then

$$\begin{aligned}
\int_{\overline{W}^u(x)} e^{t(f-f(x))} J_t s &= \int_{\overline{W}^u(x)} \exp\left(-\frac{t}{2}\sum_{i=1}^{\operatorname{ind}(x)} |y^i|^2\right)(\alpha_t)^{-1/2} \exp\left(-\frac{t}{2}\sum_{i=1}^{\operatorname{ind}(x)} |y^i|^2\right)\rho_x \otimes h \\
&= \left(\left(\frac{\pi}{t}\right)^{-n/4} + O(e^{-ct})\right)\left(\int_{\overline{W}^u(x)} \exp\left(-\frac{1}{2}\cdot 2t \sum_{i=1}^{\operatorname{ind}(x)} |y^i|^2\right)\rho_x\right) h \\
&= \left(\left(\frac{\pi}{t}\right)^{-n/4} + O(e^{-ct})\right)\left(\frac{\pi}{t}\right)^{\operatorname{ind}(x)/2} h \\
&= \left(\frac{\pi}{t}\right)^{\frac{\operatorname{ind}(x)}{2}-\frac{n}{4}} (1+O(e^{-ct}))h.
\end{aligned} \tag{6.50}$$

Now suppose that $y \in B$. That is, y is a critical point of f. It is proven in [Lau92], the limit points of $W^u(y)$ consist of the union of some $\overline{W}^u(y')$, where each y' is a critical point of f such that $\operatorname{ind}(y') < \operatorname{ind}(y)$. In particular, if $y \in B$ satisfies $\operatorname{ind}(y) = \operatorname{ind}(x)$ then $x \notin \overline{W}^u(y)$.

There exists some $c > 0$ such that if $y \in B$ so that $y \neq x$ and $\operatorname{ind}(y) = \operatorname{ind}(x)$, then $x \notin \overline{W}^u(y)$ and so

$$J_t s = O(e^{-ct}) \tag{6.51}$$

on $\overline{W}^u(y)$.

Note that

$$\begin{aligned}
\sum_{y \in B^{\operatorname{ind}(x)}} & e^{tf(y)} W^u(y)^* \otimes \int_{\overline{W}^u(y)} e^{t(f-f(y))} \tilde{e}_t s \\
&= e^{tf(x)} W^u(x)^* \otimes \left(\frac{\pi}{t}\right)^{\frac{\operatorname{ind}(x)}{2}-\frac{n}{4}} (1+O(e^{-ct})) + O(e^{-ct}) \\
&= e^{tf(x)} W^u(x)^* \otimes \left(\frac{\pi}{t}\right)^{\frac{\operatorname{ind}(x)}{2}-\frac{n}{4}} (1+O(e^{-ct})).
\end{aligned} \tag{6.52}$$

Therefore, $P_{\infty,t} = e^{t\mathcal{F}}(\frac{\pi}{t})^{N/2-n/4}(1+O(e^{-ct}))$. □

6.3 Two Identities

Note that $\Omega_{sm,t}$ is finite dimensional, so we may apply the Knudsen-Mumford map to obtain a metric on $\det H^\bullet(\Omega_{sm,t})$.

Definition 6.3.1. Let $\|\cdot\|_{\det H^\bullet(M,F),t}$ be the metric on $H^\bullet(M,F)$ associated to $\langle\cdot,\cdot\rangle_{\Omega_{sm,t},t}$ by the Knudsen-Mumford map and the canonical isomorphism $H^\bullet(\Omega_{sm,t}) \cong H^\bullet(M,F)$. Let $|\cdot|^{Hodge}_{\det H^\bullet(M,F),t}$ be the metric associated to $\langle\cdot,\cdot\rangle_{\Omega_{sm,t},t}$ by the Hodge isomorphism and the canonical isomorphism $H^\bullet(\Omega_{sm,t}) \cong H^\bullet(C^\bullet(W^u,F)) \cong H^\bullet(M,F)$.

The following proposition follows immediately from Proposition 2.3.3.

Proposition 6.3.2. *The following identity holds*

$$\log(\tau^2_{RS,sm(f,t)}) + \log((|\cdot|^{Hodge}_{\det(H^\bullet(M,F)),t})^2) = \log((\|\cdot\|_{\det(H^\bullet(M,F)),t})^2). \tag{6.53}$$

By Theorems 6.1.7 and 6.2.2, if $t \geq 0$ is large enough, $P_{\infty,t}$ is invertible. Then for $t \geq 0$ large enough, $P^*_{\infty,t}P_{\infty,t}$ is invertible.

Theorem 6.3.3. *For $t \geq 0$ large enough,*

$$\log\left(\frac{\|\cdot\|^{\mathcal{M}}_{\det(H^\bullet(M,F))}}{\|\cdot\|_{\det(H^\bullet(M,F)),t}}\right)^2 = \mathrm{Tr}_s[\log(P^*_{\infty,t}P_{\infty,t})]. \tag{6.54}$$

Proof. For $t \geq 0$ large enough, the map $P_{\infty,t} : \Omega_{sm,t} \to C^\bullet(W^u,F)$ is a one-to-one quasi-isomorphism, which induces the canonical isomorphism

$$H^\bullet(\Omega_{sm,t}) \cong H^\bullet(C^\bullet(W^u,F)). \tag{6.55}$$

Thus, $\det^{-1}(\Omega_{sm,t}) \otimes \det(C^\bullet(W^u,F))$ is canonically trivial, and $\det P_{\infty,t}$ is precisely the canonical section which trivializes $\det^{-1}(\Omega_{sm,t}) \otimes \det(C^\bullet(W^u,F))$. Then it is immediately clear that

$$\log\left(\frac{\|\cdot\|^{\mathcal{M}}_{\det(H^\bullet(M,F))}}{\|\cdot\|_{\det(H^\bullet(M,F)),t}}\right)^2 = \log(\|\det P_{\infty,t}\|^2_{\det^{-1}(\Omega_{sm,t})\otimes\det(C^\bullet(W^u,F))}). \tag{6.56}$$

However, $\|\det P_{\infty,t}\|^2_{\det^{-1}(\Omega_{sm,t})\otimes\det(C^\bullet(W^u,F))} = \det(P^*_{\infty,t}P_{\infty,t})$, thus

$$\begin{aligned}\log(\|\det P_{\infty,t}\|^2_{\det^{-1}(\Omega_{sm,t})\otimes\det(C^\bullet(W^u,F))}) &= \log(\det(P^*_{\infty,t}P_{\infty,t})) \\ &= \mathrm{Tr}_s[\log(P^*_{\infty,t}P_{\infty,t})].\end{aligned} \tag{6.57}$$

□

6.4 Proof of the Main Theorem

Now we prove the desired theorem.

Theorem 6.4.1. *Let M be a closed manifold Riemannian of dimension n, and let $F \to M$ be a vector bundle associated to a representation $\pi_1(M) \to \mathrm{GL}(n, \mathbb{R})$ equipped with a metric g^F whose induced metric on the line bundle $\det F$ is flat. Let (g^{TM}, f) be a generalized triangulation for M. Then the following identity holds:*

$$\lim_{t\to+\infty}\left[\log(\tau_{RS,sm}(f,t)^2) + \log\left(\frac{|\cdot|^{Hodge}_{\det H^\bullet(M,F),t}}{|\cdot|^{Hodge}_{\det H^\bullet(M,F)}}\right)^2 + \log\left(\frac{t}{\pi}\right)\left(\frac{n}{2}\chi(F) - \tilde{\chi}'(F)\right) + 2t\,\mathrm{rank}(F)\,\mathrm{Tr}_s^{\mathrm{Cr}(f)}[f]\right] = \log\left(\frac{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}}{|\cdot|^{Hodge}_{\det H^\bullet(M,F)}}\right)^2. \tag{6.58}$$

Proof. By Proposition 6.3.2 and Theorem 6.3.3, for $t \geq 0$ large enough,

$$\log(\tau_{RS,sm}(f,t)^2) + \log(|\cdot|^{Hodge}_{\det(H^\bullet(M,F)),t})^2 + \mathrm{Tr}_s[\log(P^*_{\infty,t}P_{\infty,t})] = \log(\|\cdot\|^{\mathcal{M}}_{\det(H^\bullet(M,F))})^2. \tag{6.59}$$

For $t \geq 0$ large enough,

$$\mathrm{Tr}_s[\log(P^*_{\infty,t}P_{\infty,t})] = \mathrm{Tr}_s[\log((P_{\infty,t}e_t)^*P_{\infty,t}e_t)] - \mathrm{Tr}_s[\log(e_t^*e_t)]. \tag{6.60}$$

Then by Theorem 6.1.7, there is some $c > 0$ such that as $t \to \infty$

$$\mathrm{Tr}_s[\log(e_t^*e_t)] = O(e^{-ct}). \tag{6.61}$$

Also, by Theorem 6.2.2,

$$(P_\infty, te_t)^*P_{\infty,t}e_t = (1 + O(e^{-ct}))^*\left(\frac{\pi}{t}\right)^{N-n/2} e^{2t\mathcal{F}}(1 + O(e^{-ct})). \tag{6.62}$$

Therefore,

$$\mathrm{Tr}_s[\log((P_{\infty,t}e_t)^*P_{\infty,t}e_t)] = \mathrm{Tr}_s\left[\log\left(\left(\frac{\pi}{t}\right)^{N-n/2} e^{2t\mathcal{F}}\right)\right] + O(e^{-ct}). \tag{6.63}$$

Moreover,

$$\operatorname{Tr}_s\left[\log\left(\left(\frac{\pi}{t}\right)^{N-n/2}e^{2t\mathcal{F}}\right)\right] = 2\operatorname{Tr}_s[\mathcal{F}]t = \operatorname{Tr}_s\left[\left(\frac{n}{2}-N\right)\right]\log\left(\frac{t}{\pi}\right). \tag{6.64}$$

Note that

$$\begin{aligned}\operatorname{Tr}_s[\mathcal{F}] &= \operatorname{Tr}_s^B[f],\\ \operatorname{Tr}_s\left[\left(\frac{n}{2}-N\right)\right] &= \frac{n}{2}\chi(F) - \tilde{\chi}'(F).\end{aligned} \tag{6.65}$$

From the above considerations, we find that as $t \to \infty$,

$$\begin{aligned}&\log(\tau_{RS,sm}(f,t)^2) + \log(|\cdot|^{Hodge}_{\det(H^\bullet(M,F)),t})^2\\ &\quad + 2\operatorname{Tr}_s^B[f]t + \left(\frac{n}{2}\chi(F) - \tilde{\chi}'(F)\right)\log\left(\frac{t}{\pi}\right)\\ &\quad = \log(\|\cdot\|^{\mathcal{M}}_{\det(H^\bullet(M,F))})^2 + O(e^{-ct}),\end{aligned} \tag{6.66}$$

which implies the claim. □

Now we will obtain Theorem 6.0.1 from Theorem 6.4.1. First note that since M is compact and odd dimensional, $\chi(F) = 0$ and

$$\log\left(\frac{|\cdot|^{Hodge}_{\det H^\bullet(M,F),t}}{|\cdot|^{Hodge}_{\det H^\bullet(M,F)}}\right)^2 = 0. \tag{6.67}$$

Then multiply both sides by $-\frac{1}{2}$ to obtain as $t \to \infty$

$$\begin{aligned}\log\frac{|\cdot|^{Hodge}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}} &= -\log\tau_{RS,sm}(f,t) - t\operatorname{rank}(F)\operatorname{Tr}_s^{\mathrm{Cr}(f)}[f]\\ &\quad + \frac{1}{2}\tilde{\chi}'(F)\log\left(\frac{t}{\pi}\right) + O(1).\end{aligned} \tag{6.68}$$

By adding $\log\tau_{RS,sm}(f,t)$ to both sides we obtain Theorem 6.0.1.

Chapter 7

Asymptotic Expansion of the Large Eigenvalues

In this section we examine the large t asymptotics of $\tau_{RS,la}(f,t)$. Rather than computing it directly, we will instead prove a *comparison theorem*, which describes the asymptotics of the difference of the Ray-Singer torsion for two different manifolds equipped with Morse functions that have the same critical point structure. Using this, we will show that $\|\cdot\|^{RS}_{\det H^\bullet(M,F)} = \|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}$ by considering the product manifolds $M \times S^1 \times S^1$ and $M \times S^2$. This chapter follows [Bra03].

7.1 Statement of the Comparison Theorem

Let M, $\tilde{M}$ be closed Riemannian manifolds of odd dimension n. Let F, $\tilde{F}$ be real flat vector bundles over M, $\tilde{M}$, respectively, with $\dim F = \dim \tilde{F}$. We assume that F and $\tilde{F}$ are equipped with Hermitian metrics so that the induced metrics on the determinant lines $\det F$, $\det \tilde{F}$ are flat. Let $f : M \to \mathbb{R}$ and $\tilde{f} : \tilde{M} \to \mathbb{R}$ be Morse functions.

Definition 7.1.1. We say the Morse functions f, $\tilde{f}$, *have the same critical point structure* if there exist open neighborhoods $U \subset M$, $\tilde{U} \subset \tilde{M}$ of the sets of critical points of f, $\tilde{f}$ respectively, and an isometry $\phi : U \to \tilde{U}$, such that $f = \tilde{f} \circ \phi$.

Definition 7.1.2. We say that a function $l : \mathbb{R} \to \mathbb{R}$ *has a nice asymptotic expansion as*

$t \to \pm\infty$ if

$$l(t) = \sum_{j=0}^{d} a_j(t/|t|)t^j + \sum_{k=0}^{d} b_k(t/|t|)t^k \log(|t|) + O(1) \tag{7.1}$$

and the coefficient a_0 satisfies $a_0(1) = a_0(-1) = 0$.

In the first half of this chapter we prove the following theorem, which we will then use to prove the Cheeger-Müller theorem.

Theorem 7.1.3. *Suppose $f : M \to \mathbb{R}$ and $\tilde{f} : \tilde{M} \to \mathbb{R}$ are Morse functions with the same critical point structure, and U, $\tilde{U}$ are isometric neighborhoods around the critical points as in the above definition. Then $\log \tau_{RS,la}(f,t) - \log \tau_{RS,la}(\tilde{f},t)$ has a nice asymptotic expansion.*

7.2 Determinant of an Almost Elliptic Operator with Parameter

We will work in a more general situation. Let F be a real vector bundle over a closed Riemannian manifold M of dimension n. As in previous chapters, suppose that g^{TM} is a Riemannian metric on M and suppose that g^F is a metric on F. Consider the following operator, acting on smooth sections of F:

$$H_t = A + tB + t^2V : C^\infty(M,F) \to C^\infty(M,F), \quad t \in \mathbb{R}, \tag{7.2}$$

where $A : C^\infty(M,F) \to C^\infty(M,F)$ is a second-order self-adjoint elliptic differential operator with positive definite leading symbol, and where $B, V = B(x), V(x) : F \to F$ are self-adjoint bundle maps with $V(x) \geq 0$ for all $x \in M$. Recall in Chapter 5 we expanded the Witten Laplacian so it does take this form.

Suppose there exists constants $t_0, c_1, c_2 > 0$ so that for all $|t| > t_0$, there are finitely many eigenvalues of H_t which are smaller than $e^{-c_1|t|}$, and all other eigenvalues of H_t are larger than $c_2|t|$. Let $P_{la,t}$ be the orthogonal projection onto the subspace of smooth sections to E spanned by eigensections corresponding to eigenvalues greater than 1. Again, this assumption is satisfied by the Witten Laplacian due to the spectral gap theorem.

Recall that $P_{la,t}$ and $(\mathrm{id} - P_{la,t})$ are idempotent, so $\mathrm{rank}(\mathrm{id} - P_{la,t})$ is equal to the number of eigenvalues smaller than 1, counting multiplicites. Hence the mapping $t \mapsto \mathrm{rank}(\mathrm{id} - P_{la,t})$ is locally constant for $|t| > \max\{t_0, 1/c_2\}$. Let $|t| > \max\{t_0, 1/c_2\}$ and set

$$m_\pm = \mathrm{rank}(\mathrm{id} - P_{la,t}). \tag{7.3}$$

Also assume that there exist constants $k > 2n + 1$ and $C > 0$ so that

$$\mathrm{Tr}\left[(H_t^k + \epsilon)^{-1} P_{la,t}\right] < C \tag{7.4}$$

for all $\epsilon > 0$ and $|t| \gg 0$. We wish to prove that the Witten Laplacian satisfies this assumption. To do so, we will use the following from [BFK96, Lemma 3.3].

Let $N(q, \lambda)$ denote the *counting function* for the large eigenvalues of $\Delta^q_{f,t}$. More precisely, $N(q, \lambda)$ is the number of eigenvalues (counting multiplicities) between 1 and λ.

Lemma 7.2.1. *There exists a constant $C > 0$ which is independent of t such that for all $\lambda \in \operatorname{spec}(\Delta^q_{f,t}) \cap [1, \infty)$,*

$$N(q, \lambda) \leq C\lambda^n. \tag{7.5}$$

The following proposition was stated in [Bra03, Section 5], however the proof given here is original.

Proposition 7.2.2. *There exists a constant $C > 0$ which is independent of t such that for all $k > 2n + 1$, $\epsilon > 0$,*

$$\operatorname{Tr}\left[((\Delta_{f,t})^k + \epsilon)^{-1} P_{la,t}\right] < C. \tag{7.6}$$

Proof. Fix some $\epsilon > 0$, $k > 2n + 1$, and some $q = 0, ..., n$. Let $\operatorname{spec}(\Delta^q_{f,t}) \cap [1, \infty) = \{\lambda_{q,t,1}, \lambda_{q,t,2}, ...\}$ so that $\lambda_{q,t,i} \leq \lambda_{q,t,j}$ for $i < j$. We need to show that there exists some constant $C_q > 0$ which is independent of t so that

$$\sum_{i=1}^{\infty} \frac{1}{\lambda^k_{q,t,i} + \epsilon} < C. \tag{7.7}$$

To do so, use Lemma 7.2.1 to choose some $C_q > 0$ which is independent of t such that for all $\lambda \in \operatorname{spec}(\Delta^q_{f,t}) \cap [1, \infty)$,

$$N(q, \lambda) \leq C_q\lambda^n. \tag{7.8}$$

In particular,

$$i \leq C_q\lambda^n_{q,t,i} \tag{7.9}$$

so

$$\frac{1}{\lambda^{2n}_{q,t,i}} \leq C_q\frac{1}{i^2}. \tag{7.10}$$

Then,

$$\sum_{i=1}^{\infty} \frac{1}{\lambda^k_{q,t,i} + \epsilon} < \sum_{i=1}^{\infty} \frac{1}{\lambda^{2n}_{q,t,i}} \leq C_q\left(\sum_{i=1}^{\infty} \frac{1}{i^2}\right) = C_q\frac{\pi^2}{6}. \tag{7.11}$$

Set

$$C = \sum_{q=0}^{n} C_q\frac{\pi^2}{6} \tag{7.12}$$

□

Let $\tilde{F}$ be a real vector bundle over another compact Riemannian manifold $\tilde{M}$. Assume $\operatorname{rank}\tilde{F} = \operatorname{rank} F$. Let

$$\tilde{H}_t = \tilde{A} + t\tilde{B} + t^2\tilde{V} : C^\infty(\tilde{M}, \tilde{F}) \to C^\infty(\tilde{M}, \tilde{F}) \tag{7.13}$$

be defined with the same assumptions as above. Similarly, let $\tilde{P}_{la,t}$ be the orthogonal projection onto the span of eigensections of $\tilde{H}_t$ with eigenvalues greater than 1. We wish to prove the following theorem.

Theorem 7.2.3. *Suppose there exist open sets $U \subset M$ and $\tilde{U} \subset \tilde{M}$ such that $V(x) > 0$ for all $x \in M \setminus U$ and $\tilde{V}(x) > 0$ for all $x \in \tilde{M} \setminus \tilde{U}$. Let $\phi : U \to \tilde{U}$ be a diffeomorphism which preserves the metric. Assume the induced map $\psi : \phi^*\tilde{E}|_{\tilde{U}} \to E|_U$ is an isometry, so we may identify the restriction of H_t to U with the restriction of $\tilde{H}_t$ to $\tilde{U}$. Then $\log\det' H_t P_{la,t} - \log\det' \tilde{H}_t\tilde{P}_{la,t}$ has a nice asymptotic expansion.*

This theorem implies Theorem 7.1.3 as follows. By definition

$$\begin{aligned}
\log\tau_{RS,la}(f,t) - \log\tau_{RS,la}(\tilde{f},t) &= \frac{1}{2}\sum_{i=0}^{n}(-1)^i i\log\det'[\Delta^i_{f,t}P^i_{la,t}] \\
&\quad - \frac{1}{2}\sum_{j=0}^{n}(-1)^j j\log\det'[\Delta^i_{\tilde{f},t}\tilde{P}^i_{la,t}] \\
&= \frac{1}{2}\sum_{i=0}^{n}(-1)^i i\Big(\log\det'[\Delta^i_{f,t}P^i_{la,t}] \\
&\quad - \log\det'[\Delta^i_{\tilde{f},t}\tilde{P}^i_{la,t}]\Big).
\end{aligned} \tag{7.14}$$

Provided each $\log\det'[\Delta^i_{f,t}P^i_{la,t}] - \log\det'[\Delta^i_{\tilde{f},t}\tilde{P}^i_{la,t}]$ has a nice asymptotic expansion, then it is clear that $\log\tau_{RS,la}(f,t) - \log\tau_{RS,la}(\tilde{f},t)$ also must have a nice asymptotic expansion.

Before proving Theorem 7.2.3 we will prove the following proposition.

Proposition 7.2.4. *For each $k > 2n+1$ the following equality holds*

$$\begin{aligned}
k\log\det' H_t P_{la,t} &= \log\det' H^k P_{la,t} \\
&= \log\det'[(H_t^k + t^{2k})P_{la,t}] - t^{2k}\int_0^1 \operatorname{Tr}[(H_t^k + \tau t^{2k})^{-1}P_{la,t}]d\tau.
\end{aligned} \tag{7.15}$$

Proof. Recall for $k > 2n+1$ the operator $[(H_t^k + \tau t^{2k})P_{la,t}]^{-1}$ has a well defined trace for each t. The first equality is obvious. For the second equality consider

$$\begin{aligned}\frac{d}{d\tau}\log\det\text{'}[(H_t^k+\tau t^{2k})P_{la,t}] &= \mathrm{Tr}\left[(H_t^k+\tau t^{2k})^{-1}\frac{d}{d\tau}(H_t^k+\tau t^{2k})P_{la,t}\right]\\ &= \mathrm{Tr}\left[(H_t^k+\tau t^{2k})^{-1}t^{2k}P_{la,t}\right]\\ &= t^{2k}\,\mathrm{Tr}\left[(H_t^k+\tau t^{2k})^{-1}tP_{la,t}\right]\end{aligned} \tag{7.16}$$

Now we will integrate both sides of this equality. For the left hand side we have

$$\int_0^1 \frac{d}{d\tau}\log\det\text{'}[(H_t^k+\tau t^{2k})P_{la,t}]d\tau = \log\det\text{'}[(H_t^k+t^{2k})P_{la,t}] - \log\det\text{'}[H_t^kP_{la,t}]. \tag{7.17}$$

Thus we see that

$$\log\det\text{'}[(H_t^k+t^{2k})P_{la,t}] - \log\det\text{'}[H_t^kP_{la,t}] = t^{2k}\int_0^1 \mathrm{Tr}\left[(H_t^k+\tau t^{2k})^{-1}tP_{la,t}\right]d\tau. \tag{7.18}$$

□

From now on, fix a $k > 2n+1$ so that there exists a constant $C > 0$ such that

$$\mathrm{Tr}\left[(H_t^k+\epsilon)^{-1}P_{la,t}\right] < C \tag{7.19}$$

for all $\epsilon > 0$, $|t| \gg 0$.

Lemma 7.2.5. *As $t \to \infty$ we have*

$$\int_0^1 \mathrm{Tr}\left[(H_t^k+\tau t^{2k})^{-1}P_{la,t}\right]d\tau = \int_0^1 \mathrm{Tr}\left[(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}P_{la,t}\right]d\tau + O(t^{-2k}). \tag{7.20}$$

Proof. Since large eigenvalues of H_t are greater than $c_2|t|$ and since $P_{la,t}$ is the projection onto the subspace spanned by eigensections corresponding to large eigensections, we have $\left\|(H_t^k+\tau t^{2k})^{-1}P_{la,t}\right\| \le ((c_2t)^k+\tau t^{2k})^{-1}$, where we are using the operator norm. Recall Hilbert's resolvent identity:

$$(A-z)^{-1}-(A-w)^{-1} = (z-w)(A-z)^{-1}(A-w)^{-1} \tag{7.21}$$

for any operator A and any z, w in the resolvent set of A. Using this identity we have

$$\begin{aligned}&\int_0^1 \mathrm{Tr}\left[(H_t^k+\tau t^{2k})^{-1}P_{la,t}\right]d\tau - \int_0^1 \mathrm{Tr}\left[(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}P_{la,t}\right]d\tau\\ &\qquad = |t|^{-k}\int_0^1 \mathrm{Tr}\left[(H_t^k+\tau t^{2k})^{-1}(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}P_{la,t}\right]d\tau.\end{aligned} \tag{7.22}$$

Then using the above and the inequalities $\left\|(H_t^k + \tau t^{2k})^{-1} P_{la,t}\right\| \leq ((c_2 t)^k + \tau t^{2k})^{-1}$ and $\operatorname{Tr}\left[(H_t^k + \epsilon)^{-1} P_{la,t}\right] < C$ we have

$$\begin{aligned}
&|t|^{-k} \int_0^1 \operatorname{Tr}\left[(H_t^k + \tau t^{2k})^{-1}(H_t^k + \tau t^{2k} + |t|^{-k})^{-1} P_{la,t}\right] d\tau \\
&\leq |t|^{-k} \int_0^1 \left\|(H_t^k + \tau t^{2k})^{-1} P_{la,t}\right\| \operatorname{Tr}\left[(H_t^k + \tau t^{2k} + |t|^{-k})^{-1} P_{la,t}\right] d\tau \\
&\leq C|t|^{-k} \int_0^1 \frac{1}{(c_2 t)^k + \tau t^{2k}} d\tau \\
&= C|t|^{-3k} \log(1 + t^k / c_2^k) = O(t^{-2k}).
\end{aligned} \tag{7.23}$$

□

Next consider the following proposition.

Proposition 7.2.6. *As* $t \to \pm\infty$

$$\begin{aligned}
&\int_0^1 \operatorname{Tr}\left[(H_t^k + \tau t^{2k} + |t|^{-k})^{-1} P_{la,t}\right] d\tau \\
&= \int_0^1 \operatorname{Tr}\left[(H_t^k + \tau t^{2k} + |t|^{-k})^{-1}\right] d\tau - 3km_{\pm} \log(|t|) + O(t^{-2k}),
\end{aligned} \tag{7.24}$$

and

$$\log \det{}' \left[(H_t^k + t^{2k}) P_{la,t}\right] = \log \det{}' \left[H_t^k + t^{2k}\right] - 2km_{\pm} \log(|t|) + O(1). \tag{7.25}$$

Proof. For the first equality note that

$$\begin{aligned}
&\int_0^1 \operatorname{Tr}\left[(H_t^k + \tau t^{2k} + |t|^{-k})^{-1} P_{la,t}\right] d\tau \\
&= \int_0^1 \operatorname{Tr}\left[(H_t^k + \tau t^{2k} + |t|^{-k})^{-1}\right] d\tau \\
&- \int_0^1 \operatorname{Tr}\left[(H_t^k + \tau t^{2k} + |t|^{-k})^{-1}(\operatorname{id} - P_{la,t})\right] d\tau.
\end{aligned} \tag{7.26}$$

By our assumptions on the small eigenvalues of H_t we have that

$$\begin{aligned}
&\int_0^1 \operatorname{Tr}\left[(H_t^k + \tau t^{2k} + |t|^{-k})^{-1}(\operatorname{id} - P_{la,t})\right] d\tau \\
&\geq \int_0^1 m_{\pm}(e^{-c_1|t|k} + \tau t^{2k} + |t|^{-k})^{-1} d\tau.
\end{aligned} \tag{7.27}$$

Thus

$$\begin{aligned}&\int_0^1 \operatorname{Tr}\left[(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}\right]d\tau-\int_0^1 \operatorname{Tr}\left[(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}(\operatorname{id}-P_{la,t})\right]d\tau\\&\leq\int_0^1 \operatorname{Tr}\left[(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}\right]d\tau-\int_0^1 m_\pm(e^{-c_1|t|k}+\tau t^{2k}+|t|^{-k})^{-1}d\tau.\end{aligned}\tag{7.28}$$

So it remains to compute the integral $\int_0^1 m_\pm(e^{-c_1|t|k}+\tau t^{2k}+|t|^{-k})^{-1}d\tau$. Note that

$$\begin{aligned}\int_0^1 m_\pm(e^{-c_1|t|k}+\tau t^{2k}+|t|^{-k})^{-1}d\tau&=m_\pm\left[\frac{\log(e^{-c_1|t|k}+\tau t^{2k}+|t|^{-k})}{t^{2k}}\right]_0^1\\&=m_\pm t^{-2k}\log\left(1+\frac{t^{2k}}{e^{-c_1|t|k}+|t|^{-k}}\right)\\&=m_\pm t^{-2k}\Bigg(\log\left(|t|^{3k}\right)\\&+\log\left(|t|^{-k}+\frac{|t|^{-k}}{e^{-c_1|t|k}}+|t|^{-k}\right)\Bigg).\end{aligned}\tag{7.29}$$

Therefore

$$\begin{aligned}\int_0^1 m_\pm(e^{-c_1|t|k}+\tau t^{2k}+|t|^{-k})^{-1}d\tau&=3km_\pm t^{-2k}\log|t|\\&+m_\pm t^{-2k}\log\left(|t|^{-3k}+\frac{|t|^{-k}}{e^{c_1|t|k}+|t|^{-k}}\right).\end{aligned}\tag{7.30}$$

However

$$\lim_{t\to\pm\infty}\log\left(|t|^{-3k}\,\frac{|t|^{-k}}{e^{c_1|t|k}+|t|^{-k}}\right)=0,\tag{7.31}$$

so

$$\int_0^1 m_\pm(e^{-c_1|t|k}+\tau t^{2k}+|t|^{-k})^{-1}d\tau=3km_\pm t^{-2k}\log|t|+O(t^{2k}),\tag{7.32}$$

which implies the first equation. For the second equation, note that

$$\log\det{}'\left[(H_t^k+t^{2k})P_{la,t}\right]=\log\det{}'\left[H_t^k+t^{2k}\right]-\log\det{}'\left[(H_t^k+t^{2k})(\operatorname{id}-P_{la,t})\right].\tag{7.33}$$

By the assumptions on the small eigenvalues of H_t we have that

$$\begin{aligned}\log\det\text{'}\left[(H_t^k + t^{2k})(\mathrm{id} - P_{la,t})\right] &\leq m_\pm \log\left(e^{-c_1|t|k} + t^{2k}\right) \\ &= m_\pm\left(\log(t^{2k}) + \log\left(1 + \frac{e^{-c_1|t|k}}{t^{2k}}\right)\right) \\ &= 2km_\pm \log|t| + O(1),\end{aligned} \tag{7.34}$$

since

$$\lim_{t\to\pm\infty} \log\left(1 + \frac{e^{-c_1|t|k}}{t^{2k}}\right) = 0. \tag{7.35}$$

This implies the second equation. □

Note that $H_t^k + t^{2k}$ is an *elliptic differential operator with parameter* t *and weight* $\chi > 0$. For more details on elliptic operators with parameter, see for example, [Shu96, Chapter 2]. We will utilize the following Theorem from [BFK92, Theorem A.3].

Theorem 7.2.7. *Suppose* $P(t)$ *is an elliptic differential operator of order* $m \geq 1$ *with parameter* t *and weight* $\chi > 0$ *such that there exists a solid angle* L *in the complex plane with the property that any* $\theta \in L$ *is a principal angle for* $P(t)$*.*

Then $\log\det\text{'}(P(t))$ *has a nice asymptotic expansion as* $|t| \to \infty$*.*

Since the spectrum of $H^k + t^{2k}$ is entirely real, $H^k + t^{2k}$ satisfies the conditions of Theorem 7.2.7. So $\log\det\text{'}(H^k + t^{2k})$ has a nice asymptotic expansion.

By Proposition 7.2.4,

$$\begin{aligned}&k\log\det\text{'}\, H_t P_{la,t} - k\log\det\text{'}\, \tilde{H}_t\tilde{P}_{la,t} \\ &\quad= \log\det\text{'}\left[(H_t^k + t^{2k})P_{la,t}\right] - \log\det\text{'}\left[(\tilde{H}_t^k + t^{2k})\tilde{P}_{la,t}\right] \\ &\quad- t^{2k}\int_0^1 \mathrm{Tr}\left[(H_t^k + \tau t^{2k})^{-1}P_{la,t}\right] - \mathrm{Tr}\left[(\tilde{H}_t^k + \tau t^{2k})^{-1}\tilde{P}_{la,t}\right] d\tau.\end{aligned} \tag{7.36}$$

By Proposition 7.2.6,

$$\begin{aligned}&\log\det\text{'}\left[(H_t^k + t^{2k})P_{la,t}\right] - \log\det\text{'}\left[(\tilde{H}_t^k + t^{2k})\tilde{P}_{la,t}\right] \\ &\quad= \log\det\text{'}\left[(H_t^k + t^{2k})\right] - \log\det\text{'}\left[(\tilde{H}_t^k + t^{2k})\right] \\ &\quad- 2km_\pm \log|t| + 2k\tilde{m}_\pm \log|t| + O(1).\end{aligned} \tag{7.37}$$

Since $\log\det'(H_t^k+t^{2k})$ and $\log\det'(\tilde{H}_t^k+t^{2k})$ have nice asymptotic expansions, then the above equation indicates that $\log\det'\left[(H_t^k+t^{2k})P_{la,t}\right]-\log\det'\left[(\tilde{H}_t^k+t^{2k})\tilde{P}_{la,t}\right]$ must also have a nice asymptotic expansion. Thus Theorem 7.2.3 is true provided $t^{2k}\int_0^1 \operatorname{Tr}\left[(H_t^k+\tau t^{2k})^{-1}P_{la,t}\right]-\operatorname{Tr}\left[(\tilde{H}_t^k+\tau t^{2k})^{-1}\tilde{P}_{la,t}\right]d\tau$ has a nice asymptotic expansion. By Lemma 7.2.5,

$$\begin{aligned}&\int_0^1 \operatorname{Tr}\left[(H_t^k+\tau t^{2k})^{-1}P_{la,t}\right]-\operatorname{Tr}\left[(\tilde{H}_t^k+\tau t^{2k})^{-1}\tilde{P}_{la,t}\right]d\tau\\&\quad=\int_0^1 \operatorname{Tr}\left[(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}P_{la,t}\right]-\operatorname{Tr}\left[(\tilde{H}_t^k+\tau t^{2k}+|t|^{-k})^{-1}\tilde{P}_{la,t}\right]d\tau+O(t^{-2k}).\end{aligned}\tag{7.38}$$

Then by Proposition 7.2.6

$$\begin{aligned}&\int_0^1 \operatorname{Tr}\left[(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}P_{la,t}\right]-\operatorname{Tr}\left[(\tilde{H}_t^k+\tau t^{2k}+|t|^{-k})^{-1}\tilde{P}_{la,t}\right]d\tau\\&\quad=\int_0^1 \operatorname{Tr}\left[(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}\right]-\operatorname{Tr}\left[(\tilde{H}_t^k+\tau t^{2k}+|t|^{-k})^{-1}\right]d\tau\\&\quad-3km_\pm t^{-2k}\log|t|+3k\tilde{m}_\pm t^{-2k}\log|t|+O(t^{-2k}).\end{aligned}\tag{7.39}$$

Thus, $t^{2k}\int_0^1 \operatorname{Tr}\left[(H_t^k+\tau t^{2k})^{-1}P_{la,t}\right]-\operatorname{Tr}\left[(\tilde{H}_t^k+\tau t^{2k})^{-1}\tilde{P}_{la,t}\right]d\tau$ has a nice asymptotic expansion if $\int_0^1 \operatorname{Tr}\left[(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}\right]-\operatorname{Tr}\left[(\tilde{H}_t^k+\tau t^{2k}+|t|^{-k})^{-1}\right]d\tau$. Theorem 7.2.3 follows from the following proposition.

Proposition 7.2.8. *Under the assumptions of Theorem 7.2.3 the function*

$$t^{2k}\int_0^1 \operatorname{Tr}\left[(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}\right]-\operatorname{Tr}\left[(\tilde{H}_t^k+\tau t^{2k}+|t|^{-k})^{-1}\right]d\tau\tag{7.40}$$

has a nice asymptotic expansion.

We will prove Proposition 7.2.8 in the remainder of this section. Recall that the sets U and $\tilde{U}$ are defined in the statement of Theorem 7.2.3. Let $W\subset M$ be an open subset such that $\overline{W}\subset U$ and such that $V(x)>0$ for all $x\notin W$. This choice is possible since the points x where $V(x)=0$ is a closed subset of the open set U. Fix an $\epsilon>0$ such that $V(x)>\epsilon$ for all $x\notin W$. We will assume without a loss of generality that $\epsilon<1$. Let $v:M\to[0,\epsilon]$ be a smooth function such that $\operatorname{supp} v\subset U$ and $v|_W=\epsilon$. Let

$$\begin{aligned}A_{t,\tau}&:=H_t^k+\tau t^{2k}+|t|^{-k},\\A_{t,\tau,v}&:=H_t^k+\tau t^{2k}+|t|^{-k}+v^2t^{2k}.\end{aligned}\tag{7.41}$$

To simplify our notation we will identify U and $\tilde{U}$ via the diffeomorphism $\phi : U \to \tilde{U}$ which was defined in the statement of Theorem 7.2.3. In particular, we will also view v as a function on $\tilde{M}$. We will define $\tilde{A}_{t,\tau}$ and $\tilde{A}_{t,\tau,v}$ by replacing H_t^k with $\tilde{H}_t^k$ in the definitions for $A_{t,\tau}$ and $A_{t,\tau,v}$.

Lemma 7.2.9. *Let $K_{\tau,v}(t,x,y)$ denote the Schwartz kernel of the operator $A_{t,\tau,v}^{-1}$. For each $N \in \mathbb{N}$*

$$K_{\tau,v}(t,x,x) = \sum_{j=0}^{N} \alpha_j(\tau, t/|t|, x) t^{n-j-2k} + r_{N,\tau}(t,x), \tag{7.42}$$

where $t^N r_{N,\tau}(t,x) \to 0$ as $t \to \pm\infty$ uniformly in $\tau \in [0,1]$ and $x \in M$. The coefficients $\alpha_j(\tau, \pm 1, x)$ depend continuously on $\tau \in [0,1]$ and can be expressed in terms of the full symbol of H_t and a finite number of its derivatives. If $j = 2i$ is even, then $\alpha_j(\tau,1,x) + \alpha_j(\tau,-1,x) = 0$.

Proof. Note that the operator $A_{t,\tau,v}$ is an elliptic operator with parameter. Then this lemma is a consequence of the standard contruction of the parametrix of an elliptic operator with parameter. For more details on this construction, see for example, [Shu96, Section 9]. □

By definition $\operatorname{Tr} A_{t,\tau,v}^{-1} = \int_M K_{\tau,v}(t,x,x)dx$, so we have the following corollary.

Corollary 7.2.10. *The function $t^{2k} \int_0^1 \operatorname{Tr} A_{t,\tau,v}^{-1} d\tau$ has a nice asymptotic expansion.*

The following lemma will be used to obtain Proposition 7.2.8.

Lemma 7.2.11. *Under the assumptions of Theorem 7.2.3 we have*

$$\operatorname{Tr}\left[A_{t,\tau}^{-1} - A_{t,\tau,v}^{-1}\right] - \operatorname{Tr}\left[\tilde{A}_{t,\tau}^{-1} - \tilde{A}_{t,\tau,v}^{-1}\right] = O(t^{-2k}) \tag{7.43}$$

as $t \to \infty$ uniformly in $\tau \in [0,1]$.

Proof. We have

$$A_{t,\tau}^{-1} - A_{t,\tau,v}^{-1} = A_{t,\tau}^{-1} v^2 t^{2k} A_{t,\tau,v}^{-1} = A_{t,\tau,v}^{-1} v^2 t^{2k} A_{t,\tau}^{-1}. \tag{7.44}$$

Therefore,

$$\begin{aligned} \operatorname{Tr}\left[A_{t,\tau}^{-1} - A_{t,\tau,v}^{-1}\right] &= \operatorname{Tr}\left[A_{t,\tau}^{-1} v^2 t^{2k} A_{t,\tau,v}\right] \\ &= t^{2k} \operatorname{Tr}\left[v A_{t,\tau,v}^{-1} A_{t,\tau}^{-1} v\right] \\ &= t^{2k} \operatorname{Tr}\left[v A_{t,\tau,v}^{-2} v\right] + t^{4k} \operatorname{Tr}\left[v A_{t,\tau,v}^{-2} v^2 A_{t,\tau}^{-1} v\right]. \end{aligned} \tag{7.45}$$

We may use the same argument to obtain a similar equality for $\operatorname{Tr}\left[\tilde{A}_{t,\tau}^{-1} - \tilde{A}_{t,\tau,v}^{-1}\right]$.

Let $\tilde{K}_{\tau,v}(t,x,y)$ denote the Schwartz kernel of the operator $\tilde{A}_{t,\tau,v}^{-1}$. By Lemma 7.2.9, for all $x \in \operatorname{supp} v \subset U$ and all $N \in \mathbb{N}$, we have $K_{\tau,v}(t,x,x) - \tilde{K}_{\tau,v}(t,x,x) = O(t^{-N})$ as $t \to \infty$ uniformly in $\tau \in [0,1]$. Therefore

$$\operatorname{Tr}\left[vA^{-2}v\right] - \operatorname{Tr}\left[v\tilde{A}_{t,\tau,v}^{-2}\right] = \int_M v(K_{\tau,v}(t,x,x) - \tilde{K}_{\tau,v}(t,x,x))v dx = O(t^{-N}) \tag{7.46}$$

as $t \to \infty$ uniformly in $\tau \in [0,1]$.

Let $I_{t,\tau,v} = \operatorname{Tr}\left[A_{t,\tau}^{-1} - A_{t,\tau,v}^{-1}\right] - \operatorname{Tr}\left[\tilde{A}_{t,\tau}^{-1} - \tilde{A}_{t,\tau,v}^{-1}\right]$. Thus we have that

$$\begin{aligned}
I_{t,\tau,v} &= \operatorname{Tr}\left[A_{t,\tau}^{-1} - A_{t,\tau,v}^{-1}\right] - \operatorname{Tr}\left[\tilde{A}_{t,\tau}^{-1} - \tilde{A}_{t,\tau,v}^{-1}\right] \\
&= t^{2k}\left(\operatorname{Tr}\left[vA_{t,\tau,v}^{-2}v\right] - \operatorname{Tr}\left[v\tilde{A}_{t,\tau,v}^{-2}v\right]\right) \\
&+ t^{4k}\left(\operatorname{Tr}\left[vA_{t,\tau,v}^{-2}v^2A_{t,\tau}^{-1}v\right] - \operatorname{Tr}\left[v\tilde{A}_{t,\tau,v}^{-2}v^2\tilde{A}_{t,\tau}^{-1}v\right]\right) \\
&= t^{2k}\int_M v(K_{\tau,v}(t,x,x) - \tilde{K}_{\tau,v}(t,x,x))v dx \\
&+ t^{4k}\left(\operatorname{Tr}\left[vA_{t,\tau,v}^{-2}v^2A_{t,\tau}^{-1}v\right] - \operatorname{Tr}\left[v\tilde{A}_{t,\tau,v}^{-2}v^2\tilde{A}_{t,\tau}^{-1}v\right]\right) \\
&= t^{4k}\left(\operatorname{Tr}\left[vA_{t,\tau,v}^{-2}v^2A_{t,\tau}^{-1}v\right] - \operatorname{Tr}\left[v\tilde{A}_{t,\tau,v}^{-2}v^2\tilde{A}_{t,\tau}^{-1}v\right]\right) + O(t^{2k-N})
\end{aligned} \tag{7.47}$$

as $t \to \infty$ uniformly in $\tau \in [0,1]$.

Using the isometry $\psi : \phi^*\tilde{E}|_{\tilde{U}} \to E|_U$ which was defined in the statement of Theorem 7.2.3, we may view $v\tilde{A}_{t,\tau,v}^{-2}v$ and $v\tilde{A}_{t,\tau}v$ as operators acting on the space of sections of the bundle E. Then

$$\begin{aligned}
&\left|\operatorname{Tr}\left[vA_{t,\tau,v}^{-2}v^2A_{t,\tau}^{-1}v\right] - \operatorname{Tr}\left[v\tilde{A}_{t,\tau,v}^{-2}v^2\tilde{A}_{t,\tau}^{-1}v\right]\right| \\
&\le \left|\operatorname{Tr}\left[(vA_{t,\tau,v}^{-2}v - v\tilde{A}_{t,\tau,v}^{-2}v)vA_{t,\tau}^{-1}v\right]\right| \\
&+ \left|\operatorname{Tr}\left[v\tilde{A}_{t,\tau,v}^{-2}v(vA_{t,\tau}^{-1}v - v\tilde{A}_{t,\tau}^{-1}v)\right]\right| \\
&\le \left\|vA_{t,\tau}^{-1}v\right\| \cdot \left|\operatorname{Tr}\left[vA_{t,\tau,v}^{-2}v - v\tilde{A}_{t,\tau,v}^{-2}v\right]\right| \\
&+ \left\|v\tilde{A}_{t,\tau,v}^{-2}v\right\| \cdot \left|\operatorname{Tr}\left[vA_{t,\tau}^{-1}v - v\tilde{A}_{t,\tau}^{-1}v\right]\right|.
\end{aligned} \tag{7.48}$$

Note that

$$\left\|vA_{t,\tau}^{-1}v\right\| \le |t|^k \tag{7.49}$$

and

$$\left\| v\tilde{A}_{t,\tau,v}^{-2} v \right\| \leq \epsilon^{-2} t^{-4k}. \tag{7.50}$$

Now fix $N > 7k$. We see that

$$\begin{aligned}
|I_{t,\tau,v}| &= \left| t^{4k} \operatorname{Tr}\left[vA_{t,\tau,v}^{-2} v^2 A_{t,\tau}^{-1} v \right] - t^{4k} \operatorname{Tr}\left[v\tilde{A}_{t,\tau,v}^{-2} v^2 \tilde{A}_{t,\tau}^{-1} v \right] \right| + O(t^{2k-N}) \\
&\leq t^{4k} \left\| vA_{t,\tau}^{-1} v \right\| \cdot \left| \operatorname{Tr}\left[vA_{t,\tau,v}^{-2} v - v\tilde{A}_{t,\tau,v}^{-2} v \right] \right| \\
&+ t^{4k} \left\| v\tilde{A}_{t,\tau,v}^{-2} v \right\| \cdot \left| \operatorname{Tr}\left[vA_{t,\tau}^{-1} v - v\tilde{A}_{t,\tau}^{-1} v \right] \right| + O(t^{2k-N}) \\
&\leq |t|^{5k} O(t^{-N}) + \epsilon^{-2} \left| \operatorname{Tr}\left[vA_{t,\tau}^{-1} v - v\tilde{A}_{t,\tau}^{-1} v \right] \right| + O(t^{2k-N}) \\
&= \epsilon^{-2} \left| \operatorname{Tr}\left[vA_{t,\tau}^{-1} v - v\tilde{A}_{t,\tau}^{-1} v \right] \right| + O(t^{-2k}).
\end{aligned} \tag{7.51}$$

Similar to the beginning of the proof, we have

$$\begin{aligned}
vA_{t,\tau}^{-1} v - v\tilde{A}_{t,\tau}^{-1} &= t^{2k}(vA_{t,\tau,v}^{-1} v^2 A_{t,\tau}^{-1} v - v\tilde{A}_{t,\tau,v}^{-1} v^2 \tilde{A}_{t,\tau}^{-1} v) \\
&+ (vA_{t,\tau,v}^{-1} v - v\tilde{A}_{t,\tau,v}^{-1} v) \\
&= t^{2k} vA_{t,\tau,v}^{-1} v (vA_{t,\tau}^{-1} v - v\tilde{A}_{t,\tau}^{-1} v) \\
&+ t^{2k}(vA_{t,\tau,v}^{-1} v - v\tilde{A}_{t,\tau,v}^{-1} v) v\tilde{A}_{t,\tau}^{-1} v \\
&+ (vA_{t,\tau,v}^{-1} v - v\tilde{A}_{t,\tau,v}^{-1} v).
\end{aligned} \tag{7.52}$$

Lemma 7.2.9 implies that for all $N \in \mathbb{N}$ the traces of the second and third summands in the right-hand side of the above equation behave as $O(t^{-N})$ when $t \to \infty$ uniformly in $\tau \in [0, 1]$. Therefore

$$\left| \operatorname{Tr}\left[vA_{t,\tau}^{-1} v - v\tilde{A}_{t,\tau}^{-1} v \right] \right| \leq t^{2k} \left\| vA_{t,\tau,v}^{-1} v \right\| \cdot \left| \operatorname{Tr}\left[vA_{t,\tau}^{-1} v - v\tilde{A}_{t,\tau}^{-1} v \right] \right| + O(t^{-N}). \tag{7.53}$$

Also,

$$t^{2k} \left\| vA_{t,\tau,v}^{-1} v \right\| \leq \frac{\epsilon^2 t^{2k}}{\epsilon^2 t^{2k} + |t|^{-k}} \leq 1 - \frac{\epsilon^{-2} |t|^{-3k}}{2}. \tag{7.54}$$

Hence conclude that

$$\left| \operatorname{Tr}\left[vA_{t,\tau}^{-1} v - v\tilde{A}_{t,\tau}^{-1} v \right] \right| \leq O(t^{3k-N}). \tag{7.55}$$

Since we assumed $N > 7k$,

$$|I_{t,\tau,v}| \leq \epsilon^{-2} O(t^{-4k}) + O(t^{-2k}) = O(t^{-2k}), \tag{7.56}$$

so $I_{t,\tau,v} = O(t^{-2k})$ uniformly in $\tau \in [0, 1]$. □

Now we will show how to obtain Proposition 7.2.8 from what we have proven. Note that

$$\begin{aligned}
&t^{2k}\int_0^1 \mathrm{Tr}\left[(H_t^k+\tau t^{2k}+|t|^{-k})^{-1}\right]-\mathrm{Tr}\left[(\tilde{H}_t^k+\tau t^{2k}+|t|^{-k})^{-1}\right]d\tau\\
&=t^{2k}\int_0^1 \mathrm{Tr}\left[A_{t,\tau}^{-1}\right]-\mathrm{Tr}\left[\tilde{A}_{t,\tau}^{-1}\right]d\tau\\
&=t^{2k}\int_0^1 \mathrm{Tr}\left[A_{t,\tau}^{-1}-A_{t,\tau,v}^{-1}\right]-\mathrm{Tr}\left[\tilde{A}_{t,\tau}^{-1}-\tilde{A}_{t,\tau,v}^{-1}\right]d\tau\\
&+t^{2k}\int_0^1 \mathrm{Tr}\left[A_{t,\tau,v}^{-1}\right]d\tau+t^{2k}\int_0^1 \mathrm{Tr}\left[\tilde{A}_{t,\tau,v}^{-1}\right]d\tau.
\end{aligned} \tag{7.57}$$

Note that the first summand on the right hand side of the above equation is $O(1)$ by Lemma 7.2.11, and the last two summands have nice asymptotic expansions by Corollary 7.2.10. Therefore, we have proven Proposition 7.2.8.

7.3 Proof of the Cheeger-Müller Theorem

We will restate Corollary 6.0.2 for convenience. For notational convenience, let

$$R(M,F,f)=\log\frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}}. \tag{7.58}$$

$R(M,F,f)$ is a constant independent of f, however it is convenient to keep track of f in the notation. To prove the Cheeger-Müller theorem we need to show that $R(M,F,f)=0$.

Corollary 7.3.1. *As $t\to\infty$,*

$$\log\tau_{RS,la}(f,t)=R(M,F,f)+t\,\mathrm{rank}(F)\,\mathrm{Tr}_s^{\mathrm{Cr}(f)}[f]-\frac{1}{2}\tilde{\chi}'(F)\log\left(\frac{t}{\pi}\right)+O(1). \tag{7.59}$$

Furthermore, $R(M,F,f)$ is independent of f.

Note that since $e^{-(-t)(-f)}=e^{-tf}$ and $e^{(-t)(-f)}=e^{tf}$, then $\Delta_{f,t}=\Delta_{-f,-t}$. Therefore

$$\tau_{RS,la}(f,-t)=\tau_{RS,la}(-f,t). \tag{7.60}$$

Suppose we have chosen some closed Riemannian manifold $\tilde{M}$, some flat vector bundle $\tilde{F}$, and some Morse function $\tilde{f}$ such that the hypothesis of Theorem 7.2.3 is satisfied. Then

$\mathrm{rank}(F) = \mathrm{rank}(\tilde{F})$, $\tilde{\chi}'(F) = \tilde{\chi}'(\tilde{F})$, and the Morse functions f and $\tilde{f}$ have the same critical point structure. Therefore $\mathrm{Tr}_s^{\mathrm{Cr}(f)}[f] = \mathrm{Tr}_s^{\mathrm{Cr}(f)}[\tilde{f}]$. So by Corollary 6.0.2,

$$R(M,F,f) - R(\tilde{M},\tilde{F},\tilde{f}) = \log \tau_{RS,la}(f,t) - \log \tau_{RS,la}(\tilde{f},t) + O(1)$$

as $t \to \infty$. However by Theorem 7.1.3 $\log \tau_{RS,la}(f,t) - \log \tau_{RS,la}(\tilde{f},t)$ has a nice asymptotic expansion, so $R(M,F,f) - R(\tilde{M},\tilde{F},\tilde{f})$ also has a nice asymptotic expansion. In particular, since $R(M,F,f) - R(\tilde{M},\tilde{F},\tilde{f})$ is constant, it must be equal to the free term of the asymptotic expansion of $\log \tau_{RS,la}(f,t) - \log \tau_{RS,la}(\tilde{f},t)$. By the definition, the free term satisfies $a_0(1) + a_0(-1) = 0$. Therefore

$$[R(M,F,f) - R(\tilde{M},\tilde{F},\tilde{f})] + [R(M,F,-f) - R(\tilde{M},\tilde{F},\tilde{-f})] = 0. \tag{7.61}$$

Since $R(M,F,f)$ is independent of f, we have that

$$\begin{aligned}[R(M,F,f) - R(\tilde{M},\tilde{F},\tilde{f})] + [R(M,F,-f) - R(\tilde{M},\tilde{F},\tilde{-f})]\\ = 2R(M,F,f) - 2R(\tilde{M},\tilde{F},\tilde{f}) = 0,\end{aligned} \tag{7.62}$$

so $R(M,F,f) = R(\tilde{M},\tilde{F},\tilde{f})$.

Now we will consider the product manifolds $M \times S^1 \times S^1$ and $M \times S^2$. In order to define the Ray-Singer and Milnor torsion on these manifolds, we need to choose a Morse function and vector bundle for both.

More generally, let N be an even dimensional manifold with the vector bundle $F_{\rho_2} \to N$ associated to the trivial representation $\rho_2 : \pi_1(N) \to 0$. Equip N with a generalised triangulation (g^{TN}, f_N) and let $\bar{f}$ be the Morse function on the product manifold $M \times N$ defined by $\bar{f}(x,y) = f(x) + f_N(y)$. Then by Theorem 4.1.3, $\log \tau_{RS}(N, f_N) = 0$. Additionally, it is clear that the pullback bundle $p_1^*(F) \otimes p_2^*(F_{\rho_2})$ is isomorphic to $p_1^*(F)$.

Equip the product manifolds $M \times S^2$ and $M \times S^1 \times S^1$, with the vector bundles described above. We wish to define two Morse functions $f_1 : S^2 \to \mathbb{R}$ and $f_2 : S^1 \times S^1 \to \mathbb{R}$, which have the same critical point structure. To do so, we will need the following lemma [BFK96, Lemma 4.2].

Lemma 7.3.2. *Suppose M is a closed Riemannian manifold of dimension n, and (g^{TM}, f) is a generalized triangulation of M. Suppose that $0 \leq q_0 \leq n-1$ is an integer, and $x, y \in M \setminus \mathrm{Cr}(f)$. Then there exists a generalized triangulation $(g^{TM'}, f')$ which satisfies the following four properties.*

1. $\mathrm{Cr}_q(f') = \mathrm{Cr}_q(f)$, *for* $q \notin \{q_0, q_0+1\}$,

2. $\mathrm{Cr}_{q_0}(f') = \mathrm{Cr}_q(f) \cup \{x\}$, $\mathrm{Cr}_{q_0+1}(f') = \mathrm{Cr}_{q_0+1}(f) \cup \{y\}$,

3. $(g^{TM'}, f')$ *is a subdivision of* (g^{TM}, f)*,*

4. $W^u(y) \cap W^s(x)$ *is connected.*

To construct the desired generalized triangulation on $S^1 \times S^1$, choose a metric $g^{T(S^1 \times S^1)}$. We may define a Morse function $g_2 : S^1 \times S^1 \to \mathbb{R}$ by perturbing the usual height function so that we obtain a self-indexing Morse function g_2 which satisfies the Thom-Smale transversality conditions. The function g_2 has four critical points, two of which are index 1, one of which is index 2, and one of which is index 0. Define f_2 by deforming g_2 so that one critical points of index 2 and one index 0 are added, while all other critical points are left unchanged. Assume without a loss of generality that f_2 is self-indexing. This may be achieved by "denting" the torus in an open set which does not contain any critical points of g_2. Then $(g^{T(S^1 \times S^1)}, f_2)$ is a generalized triangulation of $S^1 \times S^1$.

To construct the generalized triangulation on S^2, we will begin by defining $g_1 : S^2 \to \mathbb{R}$ to be the usual self-indexing height function and choose a metric h^{TS^2} so that (g^{TS^2}, g_1) is a generalized triangulation. Then apply Lemma 7.3.2 twice to obtain another generalized triangulation (h^{TS^2}, f_1) such that f_1 has two critical points of index 0, two critical points of index 1, and two critical points of index 2. Once again, we will assume without a loss of generality that f_1 is self-indexing.

By construction, it is clear that f_1 and f_2 have the same critical point structure.

Define the Morse functions $\overline{f}_1 : M \times S^2 \to \mathbb{R}$ and $\overline{f}_2 : M \times S^1 \times S^1 \to \mathbb{R}$ by $\overline{f}_1(x, y) = f(x) + f_1(y)$ and $\overline{f}_2(x, y) = f(x) + f_2(y)$. Additionally, equip the product manifolds $M \times S^2$ and $M \times S^1 \times S^1$ with the product metrics, which are denoted by $g^{T(M \times S^2)}$ and $g^{T(M \times S^1 \times S^1)}$ respectively. Then $(\overline{f}_1, g^{T(M \times S^2)})$ and $(\overline{f}_2, g^{T(M \times S^1 \times S^1)})$ are generalized triangulations, and $\overline{f}_1$ and $\overline{f}_2$ have the same critical point structure. This implies that $R(M \times S^2, p_1^*(F), \overline{f}_1) = R(M \times S^1 \times S^1, p_1^*(F), \overline{f}_2)$.

Since S^2 and $S^1 \times S^1$ are even dimensional and equipped with a vector bundle obtained from an orthogonal representation, $\tau_{RS}(S^2) = \tau_{RS}(S^1 \times S^1) = 1$ by Theorem 4.1.3. Recall that $\chi(S^2) = 2$, and $\chi(S^1 \times S^1) = 0$.

The product formula stated in Theorem 4.2.1 also applies to $\tau_{RS,la}(f, t)$ for product manifolds. Thus

$$\log \tau_{RS,la}(f_1, t) = 2 \log \tau_{RS,la}(f, t). \tag{7.63}$$

Let $\chi'(M) = \sum_{x \in \mathrm{Cr}(f)} (-1)^{\mathrm{ind}(x)} \mathrm{ind}(x)$, let $\chi'(S^2) = \sum_{x \in \mathrm{Cr}(f_1)} (-1)^{\mathrm{ind}(x)} \mathrm{ind}(x)$, and let $\chi'(M \times S^2) = \sum_{x \in \mathrm{Cr}(\overline{f}_1)} (-1)^{\mathrm{ind}(x)} \mathrm{ind}(x)$. $\chi'(M)$ is called the *derived Euler characteristic of* M. By definition, $\mathrm{rank}(F)\chi'(M) = \tilde{\chi}'(F)$. The product formula for the derived Euler characteristic gives

$$\chi'(M \times S^2) = \chi(M)\chi'(S^2) + \chi'(M)\chi(S^2). \tag{7.64}$$

Since M is compact and has odd dimension, $\chi(M) = 0$. Additionally, $\chi(S^2) = 2$. Thus

$$\chi'(M \times S^2) = 2\chi'(M). \tag{7.65}$$

Since $\operatorname{rank}(F) = \operatorname{rank}(p_1^*(F))$, we have $2\tilde{\chi}'(F) = \tilde{\chi}'(p_1^*(F))$. Combining the product formula, the above considerations, and Corollary 6.0.2, as $t \to \infty$,

$$\begin{aligned}
\log \tau_{RS,la}(\overline{f}_1, t) &= R(M \times S^2, p_1^*(F), \overline{f}_1) + \Big(t - \frac{1}{2}\log\Big(\frac{t}{\pi}\Big)\Big)\tilde{\chi}'(p_1^*(F)) + O(1) \\
&= 2\log \tau_{RS,la}(f, t) \\
&= 2R(M, F, f) + \Big(t - \frac{1}{2}\log\Big(\frac{t}{\pi}\Big)\Big)2\tilde{\chi}'(F) + O(1) \\
&= 2R(M, F, f) + \Big(t - \frac{1}{2}\log\Big(\frac{t}{\pi}\Big)\Big)\tilde{\chi}'(p_1^*(F)) + O(1).
\end{aligned} \tag{7.66}$$

Therefore, $R(M \times S^2, p_1^*(F), \overline{f}_1) = 2R(M, F, f)$.

Using Theorem 4.2.1,

$$\begin{aligned}
&\log \tau_{RS}(M \times S^2, p_1^*(F)) = 2\log \tau_{RS}(M, F) \\
&\log \tau_{RS}(M \times S^1 \times S^1, p_1^*(F)) = 0.
\end{aligned} \tag{7.67}$$

Therefore, $R(M \times S^1 \times S^1, p_1^*(F), \overline{f}_2) = 0$. Since $\overline{f}_1$ and $\overline{f}_2$ have the same critical point structure, $R(M \times S^2, p_1^*(F), \overline{f}_1) = R(M \times S^1 \times S^1, p_1^*(F), \overline{f}_2) = 0$, and

$$\begin{aligned}
2R(M, F, f) &= R(M \times S^2, p_1^*(F), \overline{f}_1) \\
&= R(M \times S^1 \times S^1, p_1^*(F), \overline{f}_2) = 0.
\end{aligned} \tag{7.68}$$

It follows that

$$R(M, F, f) = \log \frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}} = 0, \tag{7.69}$$

and

$$\frac{\|\cdot\|^{RS}_{\det H^\bullet(M,F)}}{\|\cdot\|^{\mathcal{M}}_{\det H^\bullet(M,F)}} = 1, \tag{7.70}$$

which completes the proof of the Cheeger-Müller theorem.

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