Concordance Invariants from Grid Homology

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Contents

Chapter 1

Introduction

The aim of this thesis is to construct several homology theories associated to grid diagrams, and to find new ways to calculate knot invariants from them. More concretely, we find new ways to compute $\Upsilon(K)$, $\Upsilon^2(K)$ and $\mathcal{G}_1(K;-,-)$ from a grid diagram representing some knot K. We focus primarily on grid homology, however, in what follows we also give a brief discussion of the pseudo-holomorphic theory, as this is a useful way to orient oneself when thinking about grid diagrams. In fact, most constructions in grid homology are combinatorial analogues of constructions in the pseudo-holomorphic theory.

The algebraic machinery underlying grid homology originates in *Floer* homology. Floer homology originated as *Lagrangian Floer homology* in symplectic geometry. In this formulation, Floer constructed a functional $\mathcal{A}$ on a covering space of the path space from one Lagrangian submanifold L_0 to a second Lagrangian submanifold L_1 for a symplectic manifold M. $\mathcal{A}$ is chosen so that its critical points correspond to points in $L_0 \cap L_1$. Floer was able to determine relative indices for these critical points, which allowed him to construct a form of Morse homology on the covering space.

In [1], Ozsváth and Szabó make modifications to the Lagrangian Floer homology of a *Heegaard diagram* (Σ, α, β) for a 3-manifold Y. A Heegaard diagram for Y consists of a triple like the one above, where Σ is a genus g surface and α and β are disjoint collections of circles $\{\alpha_1, \dots, \alpha_g\}$ and $\{\beta_1, \dots, \beta_g\}$. These circles must be chosen so that they form a symplectic basis for $H_1(\Sigma; \mathbb{Z})$. Let U_α be the handlebody with boundary Σ, in which each α-circle bounds a disk, and let U_β be the handlebody with boundary Σ, in which each β-circle bounds a disk. If we glue our two handlebodies along their common boundary, we obtain a closed 3-manifold Y. We say our Heegaard diagram (Σ, α, β) *represents* Y. The *Heegaard Floer homology* of (Σ, α, β) is the Lagrangian Floer homology of $(Sym_g\Sigma, T_{\alpha_1} \times \cdots \times T_{\alpha_g}, T_{\beta_1} \times \cdots \times T_{\beta_g})$. Because the circles are disjoint, both of the submanifolds $T_{\alpha_1} \times \cdots \times T_{\alpha_g} = T_\alpha$ and $T_{\beta_1} \times \cdots \times T_{\beta_g} = T_\beta$ are tori consisting of configurations of basepoints on each α-circle or β-circle, respectively. The differential counts pseudo-holomorphic disks between points in $T_\alpha \cap T_\beta$.

Unfortunately, this homology is not quite an invariant of the 3-manifold Y. To fix that, Ozsváth and Szabó add the requirement that (Σ, α, β) is *admissible*, they add a basepoint $z \in \Sigma \backslash (\alpha \cup \beta)$, alter the differential to record the intersection of holomorphic disks with $V_z = \{z\} \times$

$Sym_{g-1}\Sigma$, and finally, equip Y with a spin^c-structure $\mathfrak{s}$. With these modifications, Ozsváth and Szabó obtain the chain complexes $CF^\infty(Y,\mathfrak{s})$, $CF^+(Y,\mathfrak{s})$, $CF^-(Y,\mathfrak{s})$ and $\widehat{CF}(Y,\mathfrak{s})$. Treating the choice of spin^c-structure as a grading, we obtain:

$$CF^\dagger(Y) = \bigoplus_{\mathfrak{s}\in \mathrm{spin}^c(Y)} CF^\dagger(Y,\mathfrak{s}),$$

where $\dagger$ is one of $+,-,\infty,\widehat{}$. Their homologies are denoted $HF^\infty(Y,\mathfrak{s})$, $HF^+(Y,\mathfrak{s})$, $HF^-(Y,\mathfrak{s})$ and $\widehat{HF}(Y,\mathfrak{s})$. We also form:

$$HF^\dagger(Y) = \bigoplus_{\mathfrak{s}\in \mathrm{spin}^c(Y)} HF^\dagger(Y,\mathfrak{s}).$$

Each $HF^\dagger(Y)$ is an invariant of Y. Moreover, they are related to oneanother by long exact sequences:

$$\cdots \longrightarrow \widehat{HF}(Y,\mathfrak{s}) \longrightarrow HF^+(Y,\mathfrak{s}) \longrightarrow HF^+(Y,\mathfrak{s}) \longrightarrow \cdots,$$

$$\cdots \longrightarrow HF^-(Y,\mathfrak{s}) \longrightarrow HF^-(Y,\mathfrak{s}) \longrightarrow \widehat{HF}(Y,\mathfrak{s}) \longrightarrow \cdots,$$

$$\cdots \longrightarrow HF^-(Y,\mathfrak{s}) \longrightarrow HF^\infty(Y,\mathfrak{s}) \longrightarrow HF^-(Y,\mathfrak{s}) \longrightarrow \cdots.$$

These modules contain a large mount of geometric and topological information about the 3-manifold Y. For example, in [2], Ozsváth and Szabó prove that $\widehat{HF}(Y,\mathfrak{s})$ detects the Thurston norm of Y.

By introducing a filtration, we can use the machinery of Heegaard Floer homology to study a null-homologous link K in Y. In [3], Ozsváth and Szabó lay out modifications one can make to a Heegaard diagram (Σ,α,β) for Y to compute the Heegaard Floer homology of the exterior of some *nullhomologous* knot K (that is, K has a Seifert surface). The extra data consists of a second marked point w, so that z and w do not lie in the same component of $\Sigma \setminus \alpha \cup \beta$. To associate $(\Sigma,\alpha,\beta,z,w)$ to a link, first, choose disks D_{α_i} bounded by α_i in U_α and D_{β_i} bounded by β_i in U_β, then, connect basepoints in handlebodies U_α and U_β with unknotted arcs which avoid the disks we chose. Joining up these arcs forms a link L. To form a chain complex from this data, start by forming $CF^\dagger(\Sigma,\alpha,\beta,z)$, where $\dagger$ is any of $+,-,\infty,\widehat{}$. Ozsváth and Szabó construct a function $Alex : T_\alpha \cap T_\beta \to \mathbb{Q}$, which is chosen in such a way that the $\widehat{}$ flavour of knot Floer homology (defined below) categorifies the Alexander polynomial of K. They prove that this function induces a filtration on $CF^\dagger(\Sigma,\alpha,\beta,z)$, and call the new filtered complex $CFK^\dagger(\Sigma,\alpha,\beta,w,z)$. As above, this yields the complexes $CFK^\dagger(Y,K,\mathfrak{s})$ where $(\Sigma,\alpha,\beta,w,z)$ represents (Y,K), and $\mathfrak{s}$ is a spin^c-structure on Y. Their homologies are similarly denoted $HFK^\dagger(Y,K,\mathfrak{s})$. This was also independently discovered by Rasmussen in his thesis [4]. When doing classical knot theory (*i.e.* in S^3), we often suppress the 3-manifold and spin^c-structure, as S^3 admits a unique spin^c-structure.

As above, we can form spin^c-graded complexes:

$$CFK^{\dagger}(Y,K) = \bigoplus_{\mathfrak{s}\in \mathrm{spin}^c(Y)} CFK^{\dagger}(Y,K,\mathfrak{s}),$$

and their spin^c-graded homologies:

$$HFK^{\dagger}(Y,K) = \bigoplus_{\mathfrak{s}\in \mathrm{spin}^c(Y)} HFK^{\dagger}(Y,K,\mathfrak{s}).$$

We denote Alexander filtration levels by $CFK^{\dagger}(Y,K,p)$ and $HFK^{\dagger}(Y,K,p)$ for $p \in \mathbb{Q}$. These split along spin^c-structure in the following manner:

$$CFK^{\dagger}(Y,K,p) = \bigoplus_{\mathfrak{s}\in \mathrm{spin}^c} CFK^{\dagger}(Y,K,\mathfrak{s},p),$$

$$HFK^{\dagger}(Y,K,p) = \bigoplus_{\mathfrak{s}\in \mathrm{spin}^c} HFK^{\dagger}(Y,K,\mathfrak{s},p).$$

The Knot Floer chain complexes interact very well with operations on their underlying knot. For example, in [3], Ozsváth and Szabó prove that for a pair of null-homologous knots (Y_1,K_1) and (Y_2,K_2), and spin^c structures $\mathfrak{s}_i$ on Y_i, there is a filtered chain homotopy equivalence:

$$CFK^{\infty}(Y_1,K_1,\mathfrak{s}_1)\otimes CFK^{\infty}(Y_2,K_2,\mathfrak{s}_2) \to CFK^{\infty}(Y_1\#Y_2,K_1\#K_2,\mathfrak{s}_1\#\mathfrak{s}_2). \tag{1.1}$$

Moreover, if (Y,K) is a null-homologous knot, and $(-Y,m(K))$ denotes its *mirror*, then we have chain homotopies:

$$CFK^{\dagger}(-Y,m(K)) \simeq CFK^{\dagger}(Y,K)^*, \tag{1.2}$$

where the right hand side denotes the dual complex of $CFK(Y,K)$.

Many geometric properties of null-homologous knots can be detected with this machinery. The *Seifert genus*, denoted $g_3(K)$, of a nullhomologous knot $K\subset Y$ is the minimal genus of oriented surfaces in Y which bound K. In [2], Ozsváth and Szabó prove that $\widehat{HFK}(Y,K)$ detects the Seifert genus of K. A knot is *fibred* if its complement $Y\setminus K$ is an S^1-bundle. In the series of papers: [5], [6], [7] and [8], it is confirmed that a knot is fibred if and only if $\widehat{HFK}(Y,K,g_3(K))$ has rank 1.

At this point, we have not discussed the computability of Knot Floer homology. Having effective techniques to calculate these modules, and numerical invariants associated to them, is crucial for using these tools to study knots. Unfortunately, computing the boundary map in $CFK^{\dagger}(Y,K)$ requires one to solve nonlinear PDEs, so in general this formulation of Knot Floer homology is not computable. Two developments solved this for knots in lens spaces or S^3. First, in [9], Lipshitz reformulated Heegaard Floer homology by choosing a new differential which counts pseudo-holomorphic curves in $\Sigma\times[0,1]\times\mathbb{R}$. This reduction in dimension is accounted for by allowing higher genus curves. Then, in [10] and [11], Manolescu, Ozsváth, and Sarkar use this reformulation, alongside a very rigid type of Heegaard diagram, to further simplify the boundary map to one which counts rectangles. From here, we assume $Y = S^3$. The immediate advantage of this formulation is that its boundary map is computable (there are massive computational improvements to be made here, as doing this naively requires $2 \cdot n!\binom{n-1}{2}$ calculations, so, in practise, one tries to avoid this calculation). This allows for much larger scale tabulation of invariants arising from Knot Floer homology with the aid of computers. For example, consider the Υ-invariant and d-invariants of a knot. These are defined in terms of CFK^∞, and in [12], Sano and Sato use grid diagrams to tabulate the Υ-invariants and d-invariants of knots with up to 11 crossings.

Another advantage of this combinatorial boundary map is that it streamlines the construction of spectra and pro-spectra for a stable homotopy theory of Knot Floer homology. In [13], Manolescu and Sarkar use the grid complex of domains associated to a grid diagram to construct framed flow categories, from which they obtain a stable homotopy theory for HFK^+. They also note that this construction works for all flavours of Knot Floer homology.

So far, we have discussed how $HFK^\dagger$ can be used to detect geometric properties of knots. We can also use $HFK^\dagger$ to study larger scale objects. Suppose we have two links K and J in S^3. Then we say K is *concordant* to J if there is an embedding:

$$C : S^1 \times [0,1] \to S^3 \times [0,1],$$

so that $C|_{S^1\times\{0\}}$ is $K \times \{0\}$ and $C|_{S^1\times\{1\}}$ is $J \times \{1\}$. This is clearly an equivalence relation. If $\mathcal{C}$ is the set of these equivalence classes, connected sum induces a well defined operation on $\mathcal{C}$, so that the inverse of $[K]$ is given by $[-m(K)]$. Clearly (1.1) and (1.2) have useful interactions here. In fact, recent work by Sato in [14] use these, plus Hom's ν^+-equivalence classes defined in [15], to produce an algebraic model of the concordance group. Although $\mathcal{C}$ has been studied for many years, suprisingly little is known about the group. A wide array of techniques have been used to study the structure of $\mathcal{C}$, so in the interest of conciseness, we will focus on those which arise from Knot Floer theory.

In [16], Hom introduces the ϵ-invariant of a knot. This invariant records the interaction between vertical and horizontal differentials in the doubly filtered complex $CFK^\infty(K)$. As demonstrated in [17], $\epsilon(K)$ is an invariant up to concordance. This has been studied using grid diagrams by Dey and Doğa in [18]. In this paper, they use properties of grid diagrams under mirroring to extract a new definition for $\epsilon(K)$. With this, and a generic grid presentation for positive braids, Dey and Doğa show that all positive braids have $\epsilon(K) = 1$.

Ozsváth and Szabó also define the concordance invariant $\tau(K)$ in [19]. This actually descends to a homomorphism $\mathcal{C} \to \mathbb{Z}$, via (1.1) and (1.2). Further, $|\tau(K)|$ gives a lower bound for the 4-ball genus of a knot K. These facts allowed Ozsváth and Szabó to give a new proof of the Milnor conjecture. Using specialisations of Knot Floer theory, more powerful concordance invariants can be constructed.

The *Upsilon invariant* of a knot K, denoted Υ_K, is a piecewise linear function $\Upsilon_K : [0,2] \to \mathbb{R}$. This invariant was initially introduced by Ozsváth, Stipsicz, and Szabó in [20].

In [20], Ozsváth, Stipsicz, and Szabó construct a continuous family of chain complexes parameterised by $t \in [0,2]$, $tCFK^-(K)$. $\Upsilon_K(t)$ records -2 times the height of $H(tCFK^-(K))$ with respect to the t-modified grading. This was made combinatorial by Földvári in her thesis [22], and subsequent paper [23]. The construction is formally very similar to that of [20], with the major difference that the initial function extracted is only a knot invariant for $t \in [1,2]$. However, it is demonstrated in [20] that for all $t \in [0,1]$, $\Upsilon_K(t) = \Upsilon_K(2-t)$, so this version recovers the full invariant. Another approach to this construction was outlined in [22]. In this paper, Livingston places a special family of filtrations, $\mathcal{F}_t$ for $t \in [0,2]$, on the double filtered complex $CFK^\infty(K)$. Using the behaviour of this filtration on $HFK^\infty(K)$, Livingston was able to recover $\Upsilon_K(t)$. This construction was generalised to arbirtrary *south-western* regions in $\mathbb{R}^2$ in [24], which allowed Alfieri to recover Rasmussen's h_i invariants. Livingston's construction has recently been made combinatorial in [12], in which Sano and Sato determine a reasonably efficient way to compute Υ_K from $\mathcal{GC}^-(\mathbb{G})$, where $\mathbb{G}$ is a grid diagram representing K. Note that, although constructions of $\mathcal{GC}^\infty(\mathbb{G})$ exist, they are not suited to computation, as the rank of $\mathbb{G}$ is at least two, so the set of generators for a given filtration level may be infinite. See the orange parts of Figure 1.(1) for a visual description of these relationships.

The crucial property of Υ_K is that it induces a homomorphism:

$$\Upsilon : \mathcal{C} \to \{f | f : [0,2] \to \mathbb{R} \text{ is piecewise linear}\}.$$

Observing how singularities interact with addition, this group of piecewise linear functions clearly has uncountable rank, so Υ is potentially a very sensitive homomorphism. We can think of Υ_K as a significantly more sensitive version of $\tau(K)$, given $\Upsilon'_K(0) = -\tau(K)$. Because torus knots and quasi-alternating knots have very well understood knot complexes, computations of their Υ-invariants allowed Ozsváth, Stipsicz, and Szabó to determine that $T_{2,3}$ is linearly independent from all alternating knots. Moreover, Ozsváth, Stipsicz, and Szabó use the change in derivative function $\Delta\Upsilon'_K(t)$ to find a $\mathbb{Z}^\infty$-summand of $\mathcal{C}$. Ozsváth, Stipsicz, and Szabó are also able to replicate a result of Hom by finding a family of topologically slice knots $\{K_n\}_{n=2}^\infty$ which are a basis for a direct summand of $\mathcal{C}$ which is isomorphic to $\mathbb{Z}^\infty$.

As we noted above, an efficient way of computing Υ_K is outlined in [12]. This is because Υ_K can be recovered from the set $\mathcal{G}_0(K)$. The invariant $\mathcal{G}_0(K)$ fits into the larger sequence of invariants $\{\mathcal{G}_n\}_{n=0}^\infty$, which are constructed in [14]. These are chain homotopy invariants of *formal knot complexes.* Loosely, a formal knot complex is an algebraic model of $CFK^\infty(K)$. We discuss formal knot complexes in much more detail in chapters 4 and 6. Of the invariants $\mathcal{G}_n$, $\mathcal{G}_0$ has been shown to contain a lot of useful information. In [12], Sano and Sato demonstrate how one can easily recover Υ_K and d-invariants from this set.

In their paper [25], Kim and Livingston extend the work of [22] to construct a second concordance invariant, Υ_K^2. This invariant essentially looks for filtration levels with a specific kind of homology at each singularity of Υ_K. Alfieri also generalises this in [24], defining for a triple of south-western regions C, C^+, C^- $\Upsilon_{K,C,C^\pm}$. Unfortunately, it is not clear that the work of Sato in [14] allows one to compute Υ_K^2 from $\mathcal{G}_1(K;-,-)$. Sato is only able to

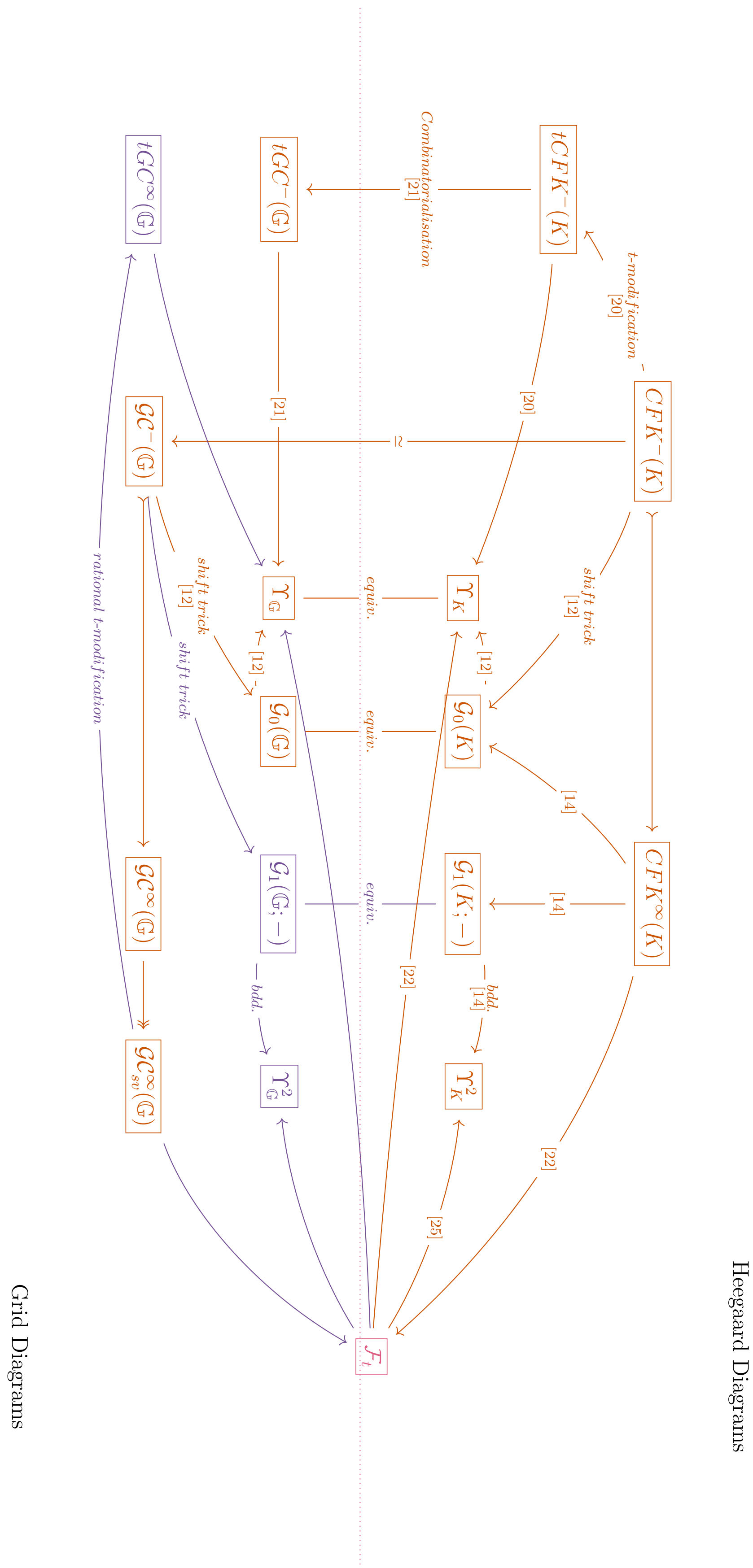


Figure 1.1: Relationships between chain complexes, filtrations and invariants. Existing work is coloured orange, new work from this thesis is coloured purple.

demonstrate that one can bound Υ_K^2 with $\mathcal{G}_1(K;-,-)$.

In chapter 2, we outline the basics of knot theory. In chapter 3 we develop the basic theory and combinatorial topology of grid diagrams, and discuss most of the basic objects we will need to construct our chain complexes. In chapter 4, we finally introduce the various chain complexes from which our grid homologies arise. In chapter 5, we show that the approach taken by Livingston in [22] also works in the setting of grid diagrams. To do this, we use the new chain complexes constructed in chapter 4 to give two new constructions of Υ_K, and prove that both compute the same Υ-invariant as Földvári in [21]. These two new constructions confirm that the work in [22] is mirrored in the world of grid diagrams, and the main theorem of chapter 5 confirms that this reflection is consistent with [21]. In chapter 6, we survey some very recent work done by Sano and Sato in [12]. In this paper, Sano and Sato demonstrate how one can use a *shift trick* to compute the knot invariant $\mathcal{G}_0(K)$ from the grid complex $\mathcal{GC}^-(\mathbb{G})$. As we will discuss later, there are major advantages to this approach, so in the second half of chapter 6 we alter the shift trick to yield a new, combinatorial, way to compute $\mathcal{G}_1(K;-,-)$. Because this new method is combinatorial, we see that $\mathcal{G}_1(K;-,-)$ is computable.

All of the new work in this thesis is done for knots in S^3. However, knots in lens spaces also admit representations by *twisted toroidal grid diagrams*, so a natural direction for future research is to extend the work in this thesis to twisted toroidal grid diagrams. Preliminary calculations indicate there is no obvious obstruction to generalising the results in chapter 6 to twisted toroidal grid diagrams. This is exciting, because any attempt to study the effect of stabilisations on $\mathcal{GC}^-_{sv}(\mathbb{G})$ or $tGC^\infty(\mathbb{G})$ in the twisted setting must at some point grapple with a very difficult counting problem. By finding a way to avoid this obstruction, we conjecture that all of the approaches to constructing invariants from toroidal grid diagrams outlined in this thesis can be generalised to twisted toroidal grid diagrams.

Chapter 2

Topology

2.1 Basic Definitions

Definition 2.1.1. A *knot* $K \subset S^3$ is a circle smoothly embedded into S^3. An *oriented knot* is a knot with a choice of orientation.

Definition 2.1.2. Let $K, J \subset S^3$ be knots. We say that K and J have the same *knot type* if there exists an orientation preserving diffeomorphism $D : S^3 \to S^3$ which sends K to J. If the knots are oriented we also require that D preserves the orientations.

In practice, when we are working with a knot, we are actually working with its knot type. This definition generalises immediately to the following.

Definition 2.1.3. A *link* $L \subset S^3$ with n components is a disjoint union of n circles, smoothly embedded into S^3. An *oriented link* is a link with an orientation on each component.

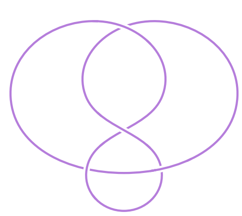
Link equivalence is almost identical to knot equivalence.

Definition 2.1.4. Two n-component links $L, M \subset S^3$ have the same *link type* if there exists an orientation preserving diffeomorphism of S^3 which sends L to M.

2.2 Diagrams

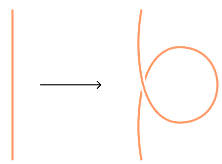
Definition 2.2.1. Let $K \subset S^3$ be a knot, and let $S^2 \subset S^3$, such that the projection of K onto S^2 is an immersion with finitely many double points. Denote this projection by D. Then, after we add crossing information, D is a *diagram* for K

Example 2.2.1. The following is a diagram for the figure eight knot.



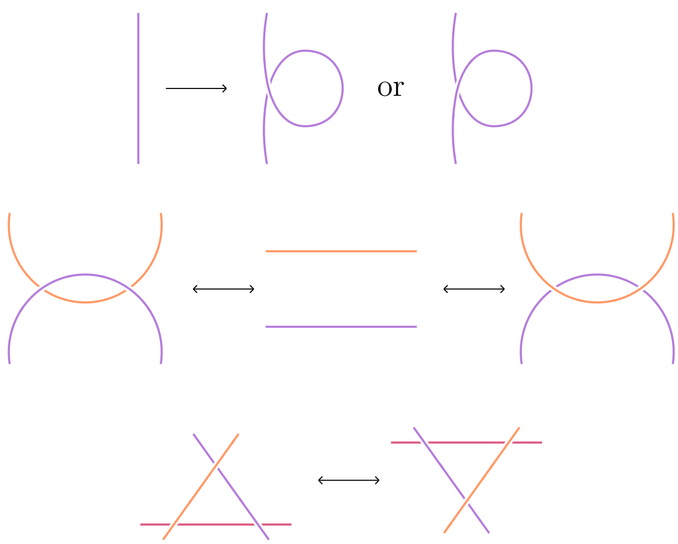
Note that, if we are given a diagram D, (which is really just a special kind of 4-valent graph), then we can recover a knot of the same type as that which produced D. Start by choosing a ball about each double point sufficiently small that it does not intersect any other ball about any other double point, and only contains the "cross" at the double point. Then within each ball remove the cross and replace this with a pair of strands chosen to respect crossing information (that is, the overstrand should be above the understrand, with respect to S^2).

Clearly there is more than one diagram for a given knot, and by adding *nugatory crossings*,



we can produce infinitely many different diagrams for a given knot type. The obvious question is then: *when do two diagrams represent the same knot type?* This is answered by Reidemeister's theorem.

Theorem 2.2.1 (Reidemeister)**.** Two diagrams D and D' represent the same knot type if and only if they can be connected by a sequence of the following three moves.



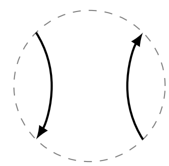
We call these Reidemeister type 1, 2 and 3 moves respectively.

2.3 Cobordisms and Concordance

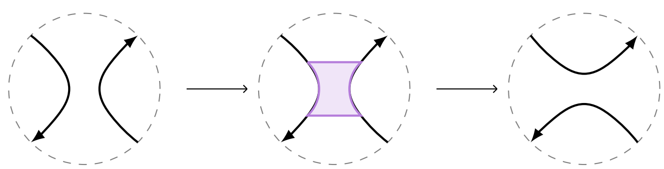
When studying manifolds, cobordisms are often a very useful relation. The same holds true for knots. We are doing classical knot theory, so the ambient cobordism on S^3 needs to be trivial.

Definition 2.3.1. Let L_0 and L_1 be oriented links in S^3. A *cobordism* from L_0 to L_1 is a compact oriented surface W with boundary, smoothly embedded in $S^3 \times [0, 1]$ so that $W \cap (S^3 \cap \{i\}) = L_i$.

Definition 2.3.2. Let L be an oriented link in S^3. Choose a ball in S^3 which intersects L in two unknotted, unlinked arcs, that is, we can find a diagram with the same link type which locally looks like (possibly adding a nugatory crossing to obtain correct orientation).



We can produce a new oriented link by glueing a thin rectangle r along L and then replacing its intersection with L with the remainder of ∂r. Again this is easier to see with a local picture.



This is called a *saddle move*.

Saddle moves correspond to crossing an index 1 critical point in our cobordism. We also have *birth* and *death* moves.

Definition 2.3.3. Let L be a link. A *birth move* on L consists of adding an unknotted, unlinked component to L. These correspond to index 0 critical points. If b is a positive integer, we denote by $\mathcal{U}_b(L)$ the link obtained by performing b birth moves simultaneously on L.

Definition 2.3.4. Let L be a link. A *death move* corresponds to the removal of an unknotted, unlinked component. These correspond to index 2 critical points on a cobordism.

The following theorem gives a normal form for cobordisms of *knots*.

Theorem 2.3.1. Let K_1 and K_2 be a pair of knots connected by a genus g cobordism W. Then there exist knots K_1', K_2' and integers b, d such that:

1. $\mathcal{U}_b(K_1)$ can be obtained from K_1' by b simultaneous saddle moves.

2. K_1' and K_2' can be connected by a sequence of $2g$ saddle moves.

3. $\mathcal{U}_d(K_2)$ can be obtained form K_2' by d simultaneous saddle moves.

Of particular interest are minimal genus cobordisms. These are, in some sense, the correct notion of homotopy equivalence in this setting. It is immediate that if L_0 and L_1 are knots then a minimal genus cobordism between the two must be an embedded cylinder. We have already discussed concordance briefly in the introduction, but for completeness we restate it as a definition.

Definition 2.3.5. Let K_0, K_1 be oriented knots in S^3. A *concordance* from K_0 to K_1 is a smoothly embedded cylinder $C \subset S^3 \times [0,1]$ so that $C \cap (S^3 \times i) - K_i$. If a concordance exists between K_0 and K_1 we say the knots are *concordant*.

This is clearly an equivalence relation, and with a bit of work we can actually endow the set of these equivalence classes with a group structure.

Definition 2.3.6. The *concordance group* $\mathcal{C}$ consists of the set of concordance classes of oriented knots in S^3, with group operation given by:

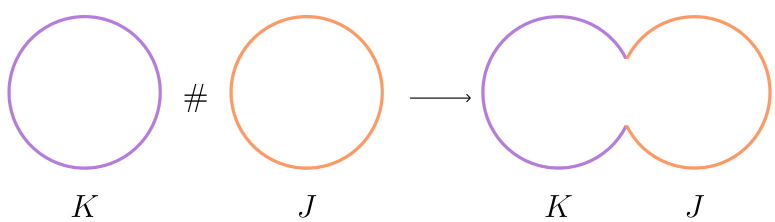
$$[K_1] + [K_2] = [K_1 \# K_2],$$

and inverses given by:

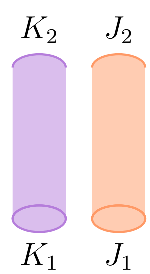
$$-[K] = [-m(K)].$$

Where $-m(K)$ is the mirror of K with reversed orientation.

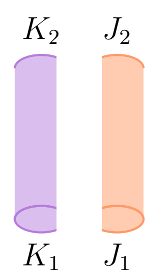
We conclude the section by sketching the proof that $\mathcal{C}$ is a group. Begin by noting that the connected sum of two knots K and J corresponds to an operation on their *codomains*, *i.e.* the abstract circle before it is embedded in S^3. That is, we perform the following operation:



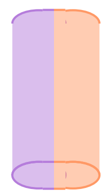
This is also the case for concordances. Suppose we have four knots K_1, K_2, J_1, J_2 so that K_1 is concordant to K_2 and J_1 is concordant to J_2. Then we have abstract cylinders:



We cut out squares to get a pair of disks:



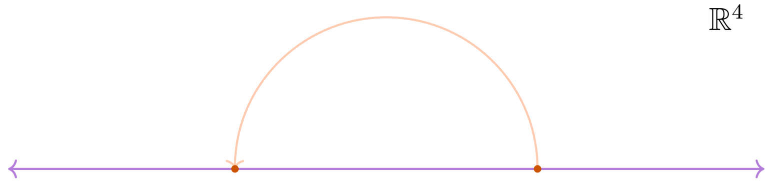
Which finally we join to see that $K_1\#J_1$ is concordant to $K_2\#J_2$, hence the group operation is well defined.



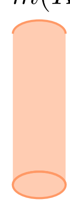
We can do this operation on abstract cylinders by thinking of our operation as a type of connected sum of embeddings:

$$i, j : S^1 \times [0,1] \to S^3 \times [0,1].$$

Moreover, if $\mathcal{U}$ is the unknot, then $\mathcal{U}\#K = K$, so $[\mathcal{U}]$ is the identity element of $\mathcal{C}$.
To see that the mirror reverse provides an inverse, start by embedding S^3 as a great sphere in S^4. We can remove a point to think of our knot K as lying in a hyperplane of $\mathbb{R}^4$. Rotate K up and out of the hyperplane, and then back down into it.



This traces out a cylinder, and we see that K is concordant to $m(K)$. Thus we have the cylinder:



Cutting out a strip, we obtain a disk bounded by $K\# - m(K)$ whose interior lies in a single component of $S^4 \setminus S^3 = B^4$. Pushing the boundary of this disk into S^3, we see that $K\# - m(K)$ bounds a disk in D^4. This, if we cut a ball out of D^4, chosen sufficiently small that it also cuts a ball out of the cylinder, we obtain a concordance from $K\# - m(K)$ to the unknot, hence $\mathcal{C}$ is a group.

Chapter 3

Grid Diagrams

To begin with, we define the basic combinatorial structure which will represent a knot or a link.

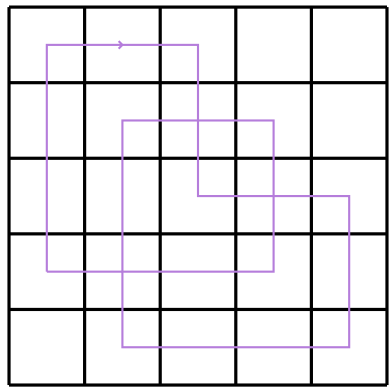
Definition 3.0.1. Let $n \in \mathbb{N}$. A *grid diagram* of *rank* n is an $n \times n$ grid $\mathbb{G}$ alongside disjoint sets of *grid markings* $\mathbb{X}$ and $\mathbb{O}$, where $\mathbb{X}$ and $\mathbb{O}$ have been chosen such that there is exactly one $X \in \mathbb{X}$ (resp. $O \in \mathbb{O}$) per column per row in $\mathbb{G}$.

Given a grid diagram $\mathbb{G}$, we can produce an oriented link diagram as follows. Begin by connecting grid markings which lie in the same row or column with arcs. Then add crossing data by declaring that vertical arcs go over horizontal arcs. This produces a link diagram which we orient by declaring horizontal arcs to point from O-marking to X-marking, and vertical arcs to point from X-marking to O-marking. We say that the grid diagram $\mathbb{G}$ *represents* the link which corresponds to this diagram, and we say the *link type* of $\mathbb{G}$ is the link type of the link which $\mathbb{G}$ represents.

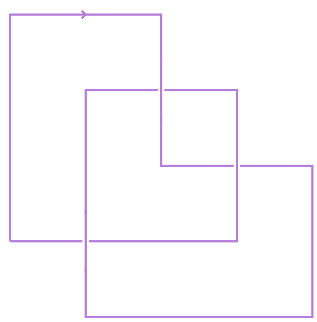
Example 3.0.1. Consider the following grid diagram.

O		X		
	O		X	
		O		X
X			O	
	X			O

Adding oriented arcs, and suppressing markings, we obtain:

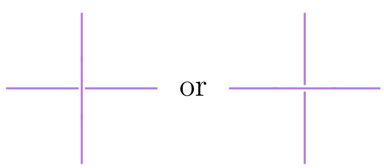


Finally, we add crossing information, orientation and suppress the grid to see this grid represents the trefoil.

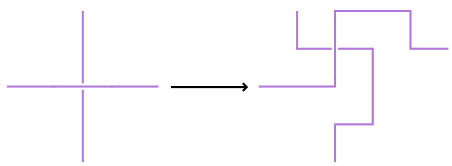


We can also associate a grid diagram to a link diagram in the following way.

Example 3.0.2. Let $L \subset S^3$ be a link. Pick a diagram D for L which we view as lying on $\mathbb{R}^2$. Isotope the diagram in $\mathbb{R}^2$ so that it is piecewise linear, all corners have angle $\pi/2$ and all edge segments lie on either a horizontal or vertical line. We call this *rectilinearisation* and diagrams which satisfy this property are *rectilinear*. At this point, crossings in D look like one of:

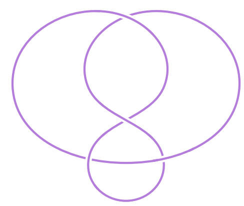


In the first configuration, the vertical strand crosses over the horizontal strand, which is what the crossings coming from a grid diagram *must* look like, so we need to isotope the second configuration to look (locally) like the first configuration. We do this by performing the following local isotopy.

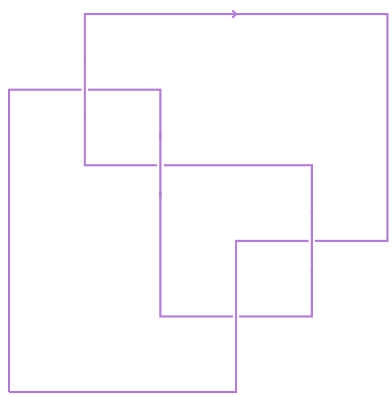


Next, we isotope the diagram so that, if e and f are distinct edge segments, then the lines through them are distinct. Finally, we place X and O markings on corners of the diagram to align with the orientation on L, and place a grid around them.

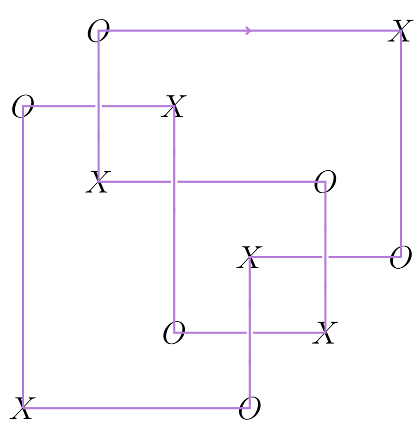
Example 3.0.3. Consider the following diagram for the figure eight knot.



We start by rectilinearising this diagram.



Then we can place X and O-markings to align with the orientation on our rectilinear diagram.



Finally, we remove the curve and overlay a grid.

	O				X
O		X			
	X			O	
			X		O
		O		X	
X			O		

Using these constructions, we can treat grid diagrams and links interchangeably.

Example 3.0.4. The following are grid diagrams which represent the unknot:

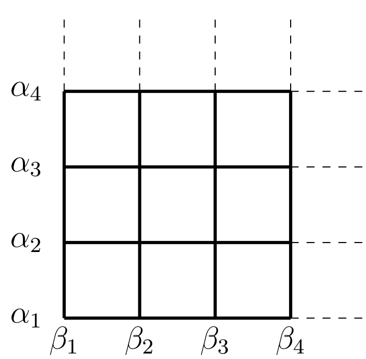
O	X
X	O

O		X
	X	O
X	O	

O			X
		X	O
	X	O	
X	O		

3.1 Toroidal Grid Diagrams

In S^3, our grid diagrams are *toroidal*. This means that we identify opposite outer edges of the grid $\mathbb{G}$ and treat the grid lines as meridian/equatorial circles on the surface of a torus. Conventionally, we will denote the horizontal circles $\overline{\alpha} := \{\alpha_1, \ldots, \alpha_n\}$ and the vertical circles $\overline{\beta} := \{\beta_1, \ldots, \beta_n\}$, with numbering as depicted below:



More formally, we have the following.

Definition 3.1.1. Let T be a torus and $n \geq 2$. Let $[0, n) \times [0, n)$ be a fundamental domain for T. Then a *toroidal grid diagram* $\mathbb{G}$ consists of the following data:

1. A collection of circles $\overline{\alpha} = \{\alpha_1, \dots, \alpha_n\}$, where α_i is given by the line $y = i - 1$ in $[0, n) \times [0, n)$. Components of $T \setminus \bigcup \overline{\alpha}$ are called rows. We number these rows based upon the index of the lower boundary circle.

2. A collection of circles $\overline{\beta} = \{\beta_1, \dots, \beta_n\}$, where β_i is given by $x = i - 1$ in $[0, n) \times [0, n)$. The components of $T \setminus \bigcup \overline{\beta}$ are called columns. We number these columns based upon the index of the left hand boundary circle.

3. A collection $\mathbb{X} = \{X_1, \dots, X_n\}$ of points lying in $T \setminus (\bigcup \overline{\alpha} \cup \bigcup \overline{\beta})$ such that each element of $\mathbb{X}$ lies in exactly one row and column.

4. A collection $\mathbb{O} = \{O_1, \dots, O_n\}$ of points lying in $T \setminus (\bigcup \overline{\alpha} \cup \bigcup \overline{\beta})$ such that each element of $\mathbb{O}$ lies in exactly one row and column, and never lies in the same component of $T \setminus (\bigcup \overline{\alpha} \cup \bigcup \overline{\alpha})$ as an element of $\mathbb{X}$.

Treating our grid diagrams as toroidal, we have a second (equivalent) way of associating a grid diagram to a link. First, we need to construct a genus 1 Heegaard decomposition of S^3. We treat S^3 as the set:

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

Within S^3, we have the Clifford torus $T = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1, |z_1| = |z_2| = 1/\sqrt{2}\}$. We can decompose S^3 along T into the two sets:

$$T_\beta = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1, |z_1| \geq 1/\sqrt{2}\},$$

$$T_\alpha = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1, |z_1| \leq 1/\sqrt{2}\}.$$

These are both solid tori, and clearly if we glue along T we recover S^3. This is called a *genus* 1 *Heegaard decomposition* of S^3. Now suppose we have a toroidal grid diagram lying on T. Connect grid markings which lie in the same column with an unknotted arc in T_β, and grid markings which lie in the same row with an unkotted arc in T_β. Orient the arcs so that arcs in T_β point from X-marking to O-marking, and those in T_α point from O-marking to X-marking. This clearly produces a link L. Note that, if we place the diagram in a sufficiently small ball on T, then projecting these arcs onto the small ball we obtain the same link diagram as the one we obtained earlier from a grid diagram.

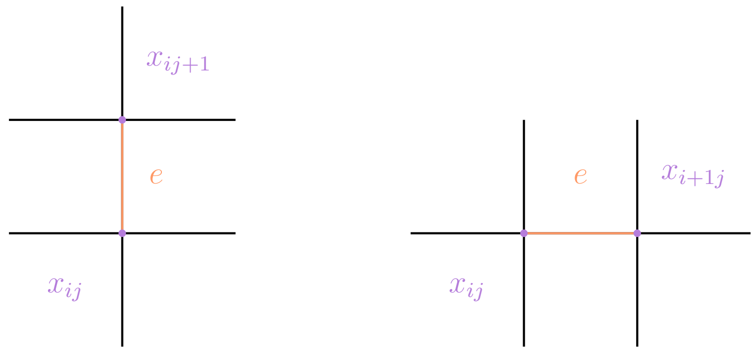
3.1.1 Algebraic Topology of Toroidal Grid Diagrams

In this section, we build some simplicial structures for the torus T from grid diagrams and pairs of grid diagrams. These will be fundamental to constructing our chain complexes and checking the topological invariance of their homologies in the next chapter. In particular, we will introduce *domains*, we will count different types of domains to construct our chain complexes and chain maps. To begin with, we want to use a toroidal grid diagram $\mathbb{G}$ to produce a simplicial structure on the torus. Fix a diagram $\mathbb{G}$ and let $\mathbb{G}$ have α-circles $\overline{\alpha} = \{\alpha_1, \dots, \alpha_n\}$ and β-circles $\overline{\beta} = \{\beta_1, \dots \beta_n\}$. Let $L = L(\mathbb{G})$ be the set of points $(\bigcup \overline{\alpha}) \cap (\bigcup \overline{\beta})$.

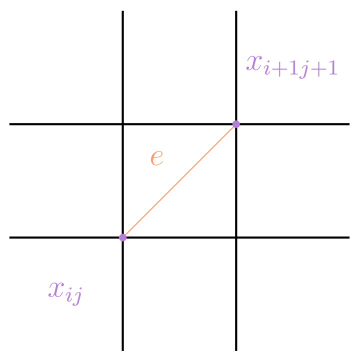
That is, L consists of points which are the intersection of two grid circles. We call $L(\mathbb{G})$ the *lattice points* of $\mathbb{G}$. We can parameterise these points by writing:

$$x_{ij} = \alpha_i \cap \beta_j.$$

For each adjacent pair of lattice points, there is a 1-cell e which lies in the grid circle spanned by the two points (and stays inside the fundamental domain). That is, e is one of:



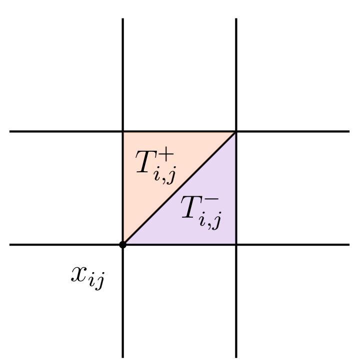
Collect these cells into sets E_α and E_β, depending upon the type of the corresponding circle. We also have a diagonal edge e between x_{ij} and x_{i+1j+1} (indices modulo n) whose interior lies in $T^2 \setminus \bigcup \overline{\alpha} \cup \bigcup \overline{\beta}$. That is, we have the following configuration:



Collect these edges into a set E_Δ and set:

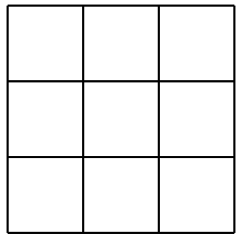
$$E = E_\alpha \cup E_\beta \cup E_\Delta.$$

These will be the edges for our simplicial structure. Finally, note that in the local picture below, we have 2 embedded simplices, one upper and one lower:

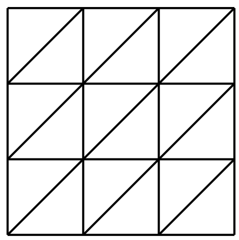


Collect these triangles into $F = F^+ \cup F^-$. Clearly setting lattice points as 0-simplices, E as 1-simplices and F as 2-simplices we obtain a simplicial structure on T^2. We call this the simplicial structure *associated* to $\mathbb{G}$ and denote it $\Delta_{\mathbb{G}}$.

Example 3.1.1. Consider the following rank 3 grid:

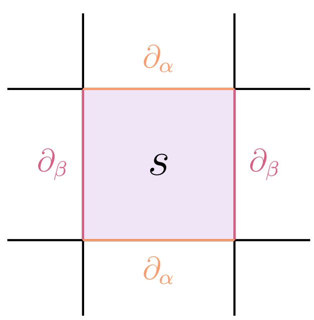


The corresponding simplicial structure on T^2 is then:



Definition 3.1.2. A *small square* of a grid diagram $\mathbb{G}$ is a formal sum $T^+_{ij} + T^-_{ij}$ for any $i, j \in \{1, \ldots, n\}$. We call the the $(i,j)^{th}$ small square, and denote it T_{ij}.

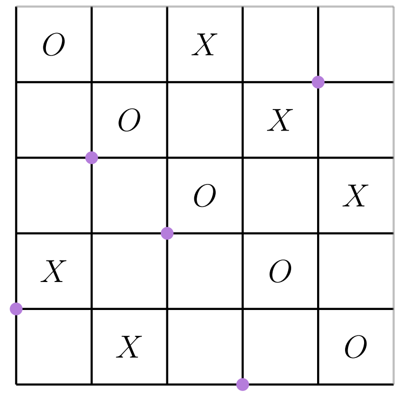
Now consider the simplicial boundary map $\partial : C_2(\Delta_{\mathbb{G}}) \to C_1(\Delta_{\mathbb{G}})$. If $s \in C_2(\Delta_{\mathbb{G}})$ is a small square, $\partial(s)$ splits as $\partial_\alpha(s) + \partial_\beta(s)$, where summands of ∂_α come from E_α and those of ∂_β come from E_β. Diagrammatically, this looks like



We can use this splitting to associate a category with $\mathbb{G}$. Our objects are grid states and our morphisms between grid states are *domains.*

Definition 3.1.3. Let $\mathbb{G}$ be a grid diagram. A *grid state* is a collection x of lattice points, so that each α-circle and each β-circle contains exactly one element of x.

Example 3.1.2. The following is a grid state in a grid diagram which represents the trefoil.



Definition 3.1.4. Let x, y be grid states of $\mathbb{G}$. A *domain* ψ from x to y is a 2-chain of the simplicial structure described above which is a formal sum of small squares c with positive coefficients, such that:

$$\partial(\partial_\alpha c) = y - x.$$

That is, horizontal boundary segments of c point from x to y. We collect these in the set $Dom(x, y)$. Note that a domain consists of three pieces of data, the *initial state* x, the *terminal state* y and its *support*, denoted $\operatorname{supp}\psi$. A domain ψ from x to y is empty if its support only intersects each element of $x \cap y$ with multiplicity zero. We collect empty domains into the set $Dom^o(x, y)$. If all multiplicities of ψ at small squares are nonnegative, we say ψ is a *positive domain*. We call the points in $(x \cup y) \setminus (x \cap y)$ the *corners* of ψ.

One particular type of domain we are interested in is a *rectangle*.

Definition 3.1.5. Let x, y be grid states of $\mathbb{G}$. A *rectangle* from x to y is an embedded rectangle r in T^2, with counterclockwise orientation, whose boundary lies on $\bigcup \alpha \cup \bigcup \beta$, such that the horizontal components of r point from x to y. We identify r with the corresponding domain ψ who has multiplicity one at all squares which intersect $int(r)$ and zero everywhere else. We collect these into the set $Rect(x, y)$. We say r is *empty* if its corresponding domain is empty. Collect these rectangles into the set $Rect^o(x, y)$.

Definition 3.1.6. Suppose $\psi \in Dom(x, y)$ and $\phi \in Dom(y, z)$. Let $\psi * \phi$ be the domain in $Dom(x, z)$ with support $\operatorname{supp}\psi + \operatorname{supp}\phi$. We call this domain the *juxtaposition* of ψ and ϕ.

This operation produces a domain from x to z, as:

$$\partial\partial_\alpha(\operatorname{supp}\psi + \operatorname{supp}\phi) = z - y + y - x = z - x.$$

This is all the information we need to define our category.

Definition 3.1.7. Let $\mathcal{C}_\mathbb{G}$ be the category given by the following data:

Objects: Grid States($S(\mathbb{G})$)
Morphisms: $\mathcal{C}_\mathbb{G}(x, y) = Dom(x, y)$
Composition: $\phi \circ \psi = \psi * \phi$

We call $\mathcal{C}_{\mathbb{G}}$ the *category associated to* $\mathbb{G}$. We also have the *empty category associated to* $\mathbb{G}$, denoted $\mathcal{C}^o_{\mathbb{G}}$, and defined by:

$$\begin{aligned} &\text{Objects: Grid States}(S(\mathbb{G}))\\ &\text{Morphisms: } \mathcal{C}_{\mathbb{G}}(x,y) = Dom^o(x,y)\\ &\text{Composition: } \phi \circ \psi = \psi * \phi \end{aligned}$$

One question we can ask is: If p and q are both positive formal sums of squares, which are both the supports of domains, when is $p+q$ the support of a domain? More concisely, when is it reasonable to juxtapose the supports of positive domains?

Definition 3.1.8. Let $\operatorname{supp} Dom(\mathbb{G})$ be the set of formal linear sums which are the supports of domains in $\mathbb{G}$. For each $a \in \operatorname{supp} Dom(\mathbb{G})$ let $t(a)$ be the lattice points in $\partial\partial_\alpha a$ with positive coefficients, and $s(a)$ be those in $\partial\partial_\alpha a$ with negative coefficient. So $t(a)$ consists of terminal corners, and $s(a)$ consists of initial corners. Then clearly the corners of a can be partitioned as:

$$t(a) \cup s(a) = c(a).$$

Moreover:

$$\partial\partial_\alpha a = \sum_{x \in t(a)} x - \sum_{y \in s(a)} y.$$

To simplify this, we can abuse notation and write $\partial\partial_\alpha a = t(a) - s(a)$. Suppose $a, b \in \operatorname{supp} Dom(\mathbb{G})$. Suppose there exist grid states x', y', z' and domains $\psi \in Dom(x,y)$ and $\phi \in Dom(y,z)$ such that $\operatorname{supp}\psi = a$ and $\operatorname{supp}\phi = b$. Then we can use the partition $c(a) = s(a) \cup t(a)$ to construct minimal sets of lattice points x, y, z such that $x \subset x'$, $y \subset y'$ and $z \subset z'$ Consider the sets of points:

$$x = s(a) \cup s(b) \setminus t(a),$$

$$y = t(a) \cup s(b),$$

$$z = (t(a) \cup t(b)) \setminus s(b).$$

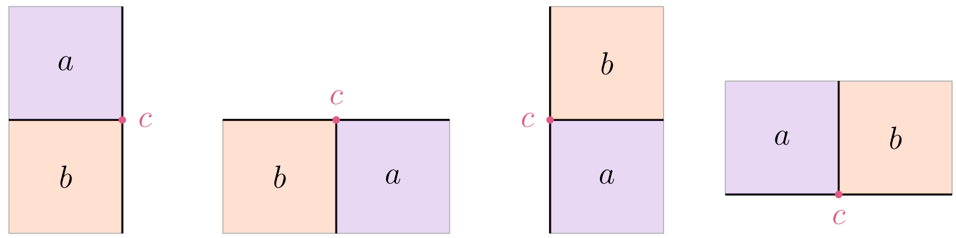
It is clear these are the minimal sets of points described above, so if, for each grid circle γ, we have:

$$|x \cap \gamma|, |y \cap \gamma|, |z \cap \gamma| \in \{0,1\}. \tag{3.1}$$

Then x, y, z can be extended to grid states which satisfy the condition. Hence $a + b \in \operatorname{supp} Dom(\mathbb{G})$ if and only if 3.1 holds. We can translate 3.1 into some more geometric conditions which make this easier to visualise. First note that we have to retain the following weaker combinatorial condition for this to work:

$$|(\gamma \cap (x \cup y \cup z)| \in \{0,1,2,3\} \tag{3.2}$$

If $e = \partial a \cap \gamma$ and $f = \partial b \cap \gamma$ are both nonempty, e and f *must* share an endpoint. Label this corner c. Clearly $c \in y$, so c has a positive coefficient in $\partial\partial_\alpha a$ and a negative coefficient in $\partial\partial_\alpha b$, hence the local structure at c looks like one of:



Notably, if all shared corners look like this locally, we have:

$$t(a) \cap \gamma = s(b) \cap \gamma,$$

where γ is a grid circle through this corner. I claim that if all shared corners look like this locally, and 3.2 holds, then $a + b \in \operatorname{supp} Dom(\mathbb{G})$. We need to check that, if this is the case, then for each grid circle γ,:

$$|x \cap \gamma|, |y \cap \gamma|, |z \cap \gamma| \in \{0, 1\}.$$

Fix a grid circle γ and suppose $|\gamma \cap (s(a) \cup s(b))| = 2$. Then by 3.2 a and b have a shared corner on γ. By our geometric conditions, this implies that $s(b) = t(a)$, hence:

$$|\gamma \cap ((s(a) \cup s(b)) \setminus t(a))| = 1$$

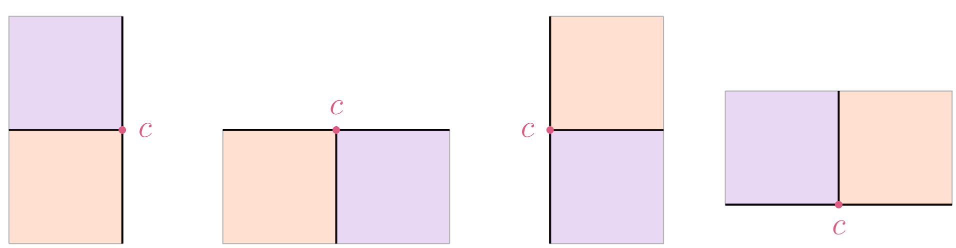
Hence $|\gamma \cap x| = 0$ or 1. Now, suppose $|\gamma \cap t(a)| = |\gamma \cap s(b)| = 1$. Then by 3.2 and our geometric conditions this implies that $t(a) = s(b)$, hence:

$$|\gamma \cap y| = 1.$$

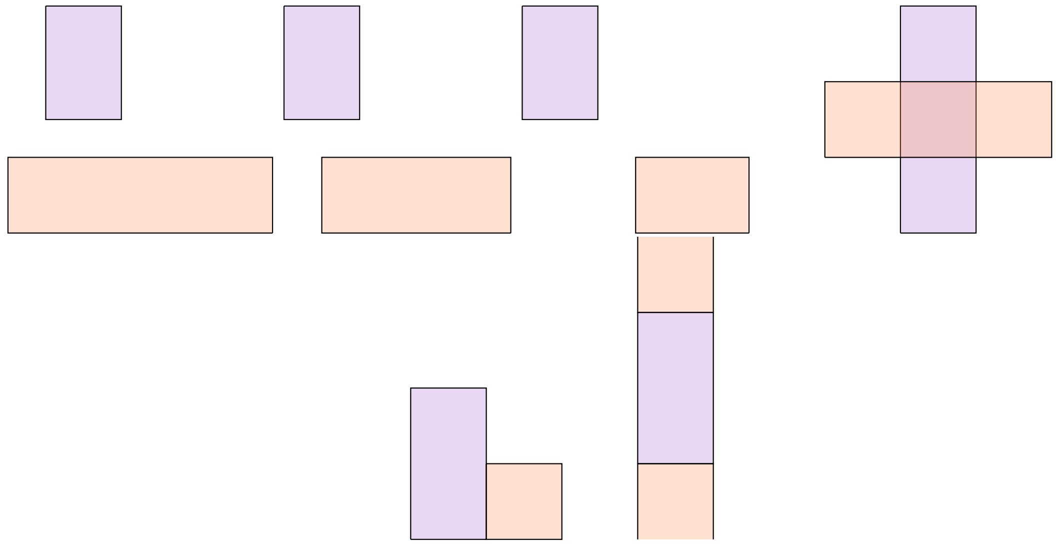
So $|\gamma \cap y| = 0$ or 1. Finally, suppose $|\gamma \cap (t(a) \cup t(b))| = 2$. Then clearly $t(a) = s(b)$, so:

$$|\gamma \cap z| = |\gamma \cap (t(a) \cup t(b)) \setminus s(b)| = 1.$$

Thus all of $|x \cap \gamma|, |y \cap \gamma|, |z \cap \gamma| \in \{0, 1\}$, so 3.2 and local geometric conditions are sufficient and necessary for 3.1. Note that to have $a + b \in \operatorname{supp} Dom(\mathbb{G})$, we could also exchange the roles of a and b in the working above, so to obtain full conditions we need to swap a and b in the working above. Noting that the geometric conditions are symmetric if we remove labels:



and that 3.2 is symmetric in a and b, we obtain complete conditions to determine if $a + b$ lies in $\operatorname{supp} Dom(\mathbb{G})$. Note that these conditions place very tight constraints on how we can form juxtapositions of rectangles. Up to combinatorial structure (and rotating/reflecting), we only get the following arrangements:



3.1.2 Grid Moves

It should be reasonably clear from the manner in which we construct grid diagrams that there are many grid diagrams which correspond to a given link type. For example, 3.0.4 demonstrates a few different diagrams of the unknot. Thus, as with link diagrams, we need to introduce some moves which preserve link type and can be used to obtain any grid diagram of a fixed link type from any other diagram of that type. We will call these moves grid moves and Cromwell moves interchangeablely.

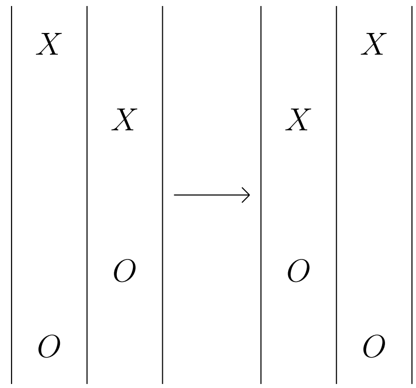
Cyclic Permutation

Definition 3.1.9. Let $\mathbb{G}$ be a toroidal grid diagram. Then a cyclic permutation of $\mathbb{G}$ is a grid move which corresponds to a rotation of the torus which sends adjacent columns to adjacent columns.

In view of the Heegaard decomposition approach of associating a link to a toroidal grid diagram, these moves don't change the type of the link corresponding to the cyclically permuted grid diagram. Moreover, cyclic permutations give equivalences between the category of domains for the pre and post permuted grid diagrams, so preserves all counts of domains.

Commutation

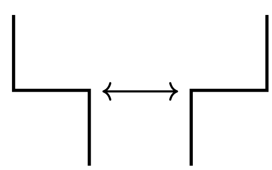
Suppose $\mathbb{G}$ is a grid diagram. Associate to each row and column of $\mathbb{G}$ an interval by connecting the two markings in that row or column with a straight line and then projecting onto the vertical or horizontal axis respectively. If $\mathbb{G}$ contains a pair of adjacent rows or columns such that either: one interval contains the other in its interior, or the two intervals do not intersect, we can form a new diagram $\mathbb{G}'$ by swapping the two rows or columns. If $\mathbb{G}'$ is obtained from $\mathbb{G}$ in this way, we say the two differ by a *commutation move*. For example:

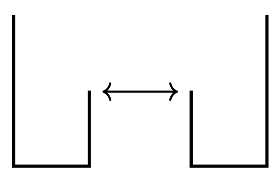


In fact, up to cyclic permutation, all commutations of columns look like this, or its inverse.

Switches

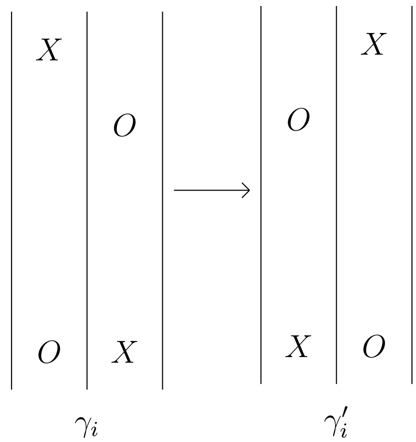
Switch moves model, up to mirroring along α lines, β lines or rotations, the following moves:





Definition 3.1.10. We introduce these moves so that we only need to consider one type of stabilisation in our proofs of invariance. Let $\mathbb{G}$ be a grid diagram. Let $k \in \{1, \ldots, n\}$. Suppose the X marking in either the k^{th} or $k+1^{th}$ column lies in the same row as the O-marking in the respective $k+1^{th}$ or k^{th} column. Let $\mathbb{G}'$ be the diagram obtained by swapping the k^{th} and $k+1^{th}$ column. We say $\mathbb{G}$ and $\mathbb{G}'$ differ by a *column switch*. Vertical switches are defined by interchanging rows and columns in the definition above. In either case $\mathbb{G}$ and $\mathbb{G}'$ differ by a switch.

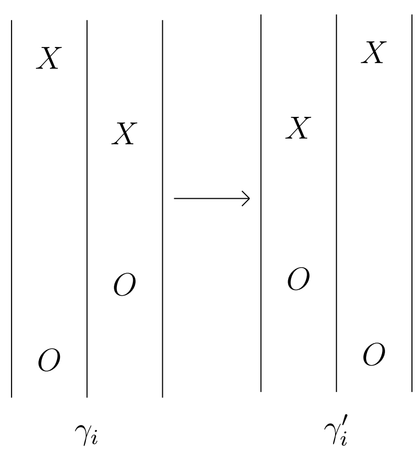
Switches are very similar to commutation. Up to cyclic permutation, a switch will look like the following, or its inverse.



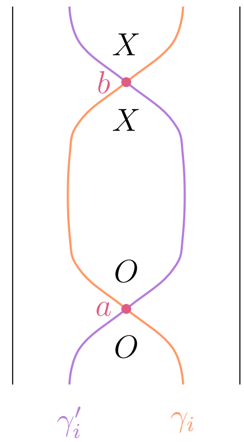
Note that this move does not change the knot type of $\mathbb{G}$.

Augmented Diagrams

If $\mathbb{G}$ and $\mathbb{G}'$ differ by a switch or a commutation, we can form an *augmented diagram* by "drawing both diagrams on the same torus". These are how we study the relationship between the two grid diagrams. Our diagrams will all be for column commutations, but the discussion also applies to row commutations, row switches and column switches. To begin, let γ_i be the grid circle in $\mathbb{G}$ across which the commutation/switch occurs. Let γ_i' be that of $\mathbb{G}'$. If we have a commutation, we assume $\mathbb{G}$ and $\mathbb{G}'$ are arranged like the following (rotate by $\pi/2$ to see the row version):



The case for switches is very similar, importantly, the perturbation below works identically. By perturbing markings in the two columns, and the grid circles γ_i and γ_i', we obtain the *augmented diagram* by drawing both diagrams on the same torus. Locally, this looks like:



As with an ordinary grid diagram, we can define a CW-structure on T^2 and use this to define categories of domains. The CW-structure is determined by the following:

1. 0-cells are elements of the set:
$$\left(\bigcup \alpha\right) \cap \left(\bigcup \beta \cup \gamma_i'\right),$$
or:
$$\left(\bigcup \alpha \cup \gamma_i'\right) \cap \left(\bigcup \beta\right),$$
Depending upon if γ_i lies in β or in α.

2. 1-cells are the connected components of:
$$\left(\bigcup \alpha \cup \bigcup \beta \cup \gamma_i'\right) \setminus \{\text{0-cells}\}.$$

3. 2-cells are the connected components of:
$$T^2 \setminus \left(\bigcup \alpha \cup \bigcup \beta \cup \gamma_i'\right).$$

Fix the notation developed above for the remainder of this section.

Definition 3.1.11. A *state* in the augmented grid diagram is an element of $S(\mathbb{G}) \cup S(\mathbb{G}')$ embedded into the augmented diagram.

We can clearly treat these as 0-chains in the CW chain complex associated to the augmented diagram. As a result we can define domains, and the associated category.

Definition 3.1.12. Let $x, y \in S(\mathbb{G}) \cup S(\mathbb{G}')$. A *domain* ψ from x to y is a 2-chain $\operatorname{supp}\psi$ such that one of the following holds:

$$\partial\partial_\alpha(\operatorname{supp}\psi) = y - x \qquad \text{(if } \gamma_i \text{ is a } \beta\text{-circle)}$$
$$\partial\partial_\beta(\operatorname{supp}\psi) = x - y \qquad \text{(if } \gamma_i \text{ is an } \alpha\text{-circle)}$$

If $\operatorname{supp}\psi$ has multiplicity zero on all elements of $x \cap y$, we say ψ is *empty*. We collect these in sets $Dom_{aug}(x, y)$ and $Dom^o_{aug}(x, y)$ respectively.

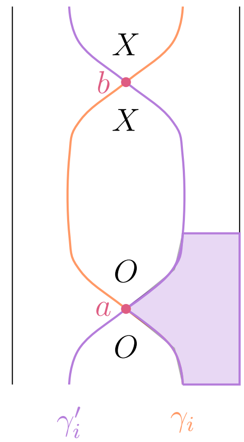


Figure 3.1: A pentagon

Clearly we can juxtapose domains, and the conditions to composing domains are similar to those discussed earlier. Juxtaposition is trivially associative, so we have a category of domains.

Definition 3.1.13. The *category of domains* corresponding to the augmented diagram is given by the following data:

Objects: Elements of $\mathbb{G} \cup \mathbb{G}'$
Morphisms: $Mor(x, y) := Dom(x, y)$
Composition: $\psi \circ \phi = \phi * \psi$

There are a few specific types of domains we will find useful when working with the augmented diagram.

Definition 3.1.14. Let $x, y \in S(\mathbb{G}) \cup S(\mathbb{G}')$. A *pentagon* from x to y is a disk p embedded into T^2 such that:

1. $\partial p \subset \bigcup \alpha \cup \bigcup \beta \cup \gamma_i'$
2. ∂p has 5 singular points, one of which lies on $\gamma_i \cap \gamma_i'$ and four of which lie on both an α-circle and a β-circle. The distinguished point is called the *vertex* of p
3. Equipped with counterclockwise orientation, and starting at the vertex, the next point along the boundary is an element of x.

If no elements of $x \cup y$ lie in the interior of p, we say that pentagon is *empty*.

Definition 3.1.15. Let $x, y \in S(\mathbb{G} \cup \mathbb{G}')$. A *hexagon* from x to y is a disk h embedded into T^2 so that:

1. $\partial h \subset \bigcup \alpha \cup \bigcup \beta \cup \gamma_i'$.

2. ∂h has six singular points, two of which are a and b, the other four are $(x \cup y) \setminus (x \cap y)$.

3. If $\mathbb{G}$ differs from $\mathbb{G}'$ by a column commutation, then $\partial\partial_\alpha h = y - x$, if the pair differ by a row commutation then $\partial\partial_\beta h = x - y$.

We collect these in the set $Hex(x, y)$. If h intersects no point of x or y in its interior, we say it is *empty*. We collect empty hexagons from x to y into the set $Hex^o(x, y)$.

Definition 3.1.16. Let $x, y \in S(\mathbb{G}) \cup S(\mathbb{G}')$. Then, a *triangle* t from x to y is a disk embedded in T^2 such that:

1. $\partial t \subset \bigcup \alpha \cup \bigcup \beta \cup \gamma_i'$.

2. ∂p has 3 singularities, one of which lies on $\gamma_i \cap \gamma_i'$ and 2 of which lie on both an α-circle and a β-circle. We call the distinguished point the *vertex* of t.

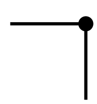
3. $\partial\partial_\alpha t = y - x$ or $\partial\partial_\beta t = x - y$.

We say a triangle is *empty* if its interior contains no element of x or y. Noting that triangles always lie in one of the bigons cut out by γ_i and γ_i', all triangles are empty.

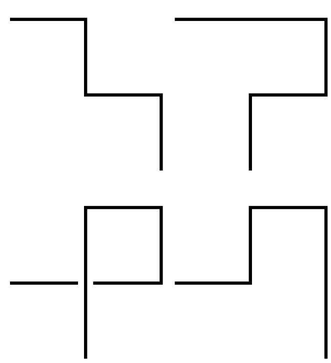
Note that the conditions defining triangles are actually very strong. Because the grid states can only differ on one point, x uniquely determines y and y uniquely determines x. As a result we will label the unique triangle from x by t_x. The points they differ on must lie on $\gamma_i \cup \gamma_i$, so the support of each t_x must lie between γ_i and γ_i'.

Stabilisation

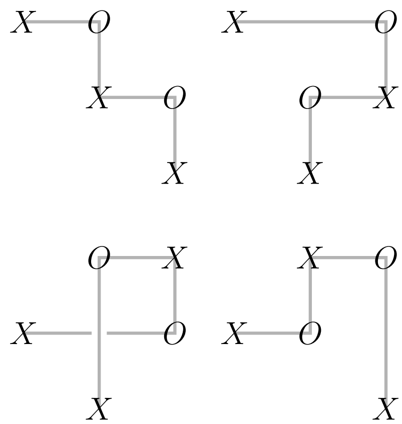
Up to rotations, and mirror symmetries along α or β lines, we want to isotope the following local feature in a rectilinear diagram:



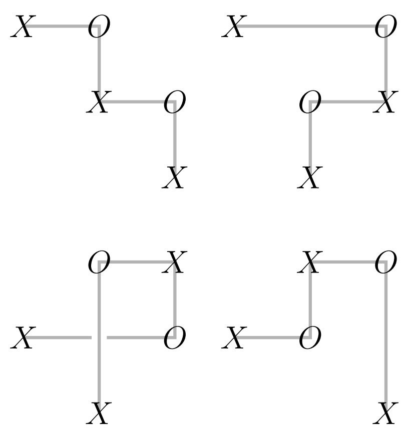
Into one of the following four configurations:



To turn these into pieces of grid diagrams, we have two choices for the marking on the distinguished point in the first diagram. If it is an X-marking, we can fill in the second diagrams to obtain:



If it is an O-marking, then we have:



We can describe these more formally by the following:

Definition 3.1.17. Let $\mathbb{G}$ be a grid diagram. We obtain a stabilisation of $\mathbb{G}$ at a grid marking by the following. Select a column and a row through that grid marking. Erase the markings in these rows and split them down the middle to produce a new column and row. Then there are 4 different ways to fill this in to produce a new grid diagram. All of these diagrams are *stabilizations* of $\mathbb{G}$. We say $\mathbb{G}$ is obtained from any of these diagrams by a *destabilization*.

The diagrams above give the various combinatorial structures of the new rows after stabilization. The different types of stabilisations are denoted with one label indicating the type of marking the stabilisation occurs at, and one indicating the location of the unmarked square relative to the distinguished point. So, for example, if we have the following distinguished 2×2 square, then we have an $X : SW$ stabilisation.

X	O
	X

Cromwell's Theorem

Having defined our grid moves, we state the following fundamental theorem, which relates grid moves to the link type of a grid diagram.

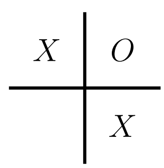
Theorem 3.1.1 (Cromwell)**.** Let $\mathbb{G}$ and $\mathbb{G}'$ be a pair of grid diagrams. Then $\mathbb{G}$ has the same link type as that of $\mathbb{G}'$ if and only if there exists a sequence of stabilisations and commutations which transforms $\mathbb{G}$ into $\mathbb{G}'$.

Proof. See [26]. □

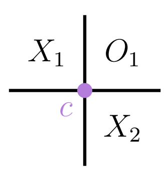
In fact, [27, Lemma 3.2.2] simplifies the moves we need to X-stabilisations and commutations. Noting that all X-stabilisations differ by a sequence of switch moves, two grids represent the same knot if and only if they are connected by a sequence of switches, commutations and $X : SW$-stabilisations. So there is a bijective correspondence between equivalence classes of grid diagrams (with respect to commutations and stabilisations) and knot types. This allows us to study knots by studying these equivalence classes.

3.1.3 Destabilization Domains

In this section, we discuss the structure of destabilization domains. These are a type of positive domain which are nessecary in proving stabilisation invariance on the level of homology for grid complexes where the boundary map is not obstructed by X or O-markings. This is intended to supplement the discussion of destabilization domains in [27], in particular, verifying some properties which the authors take for granted. To begin, we lay out some notation. Let $\mathbb{G}$ be a grid diagram. Suppose we perform an X:SW destabilisation domain on some X-marking in $\mathbb{G}$ to obtain a new grid diagram $\mathbb{G}'$. By construction, $\mathbb{G}'$ has a distinguished 2×2 square:



We label points on this square as below, and let O_2 be the O-marking in the same row as X_2.



Note that grid states in $\mathbb{G}'$ fall into one of two types. Denote the set of states who contain c by $I(\mathbb{G}')$ and those who do not by $N(\mathbb{G}')$. We can think of the set $I(\mathbb{G}')$ as representing $S(\mathbb{G})$ by sending $x \in S(\mathbb{G})$ to $x \cup c \in I(\mathbb{G}')$. This suggests that domains with an initial state in $N(\mathbb{G}')$ and final state in $I(\mathbb{G}')$ should tell us something about the relationship between $\mathbb{G}$ and $\mathbb{G}'$.

Definition 3.1.18. Fix a state $x \in S(\mathbb{G}')$ and a state $y \in I(\mathbb{G}')$. Then a domain ϕ from x to y is *of type iL* if it follows the following conditions:

1. All local multiplicities of ϕ are non-negative.
2. At each corner point of ϕ, apart from c, at least three of the four adjacent squares have multiplicity 0
3. ϕ has the same local multiplicity k at three of the four squares who share the corner c, and has multiplicity $k-1$ at the southwest square.
4. $|x \setminus (x \cap y)| = 2k+1$

Similarly:

Definition 3.1.19. Fix a state $x \in S(\mathbb{G}')$ and a state $y \in I(\mathbb{G}')$. Then a domain ϕ from x to y is *of type iR* if it follows the following conditions:

1. All local multiplicities of ϕ are non-negative.
2. At each corner point of ϕ, apart from c, at least three of the four adjacent squares have multiplicity 0
3. ϕ has the same local multiplicity k at three of the four squares who share the corner c, and has multiplicity $k+1$ at the southeast square.
4. $|x \setminus (x \cap y)| = 2k+1$

We collect these domains in the sets $\pi^{iL}(x,y)$ and $\pi^{iR}(x,y)$ respectively. If a domain ϕ from x to y lies in either of these sets, then we say ϕ is a *destabilization domain*. We collect destabilization domains into the set $\pi^{D}(x,y)$.

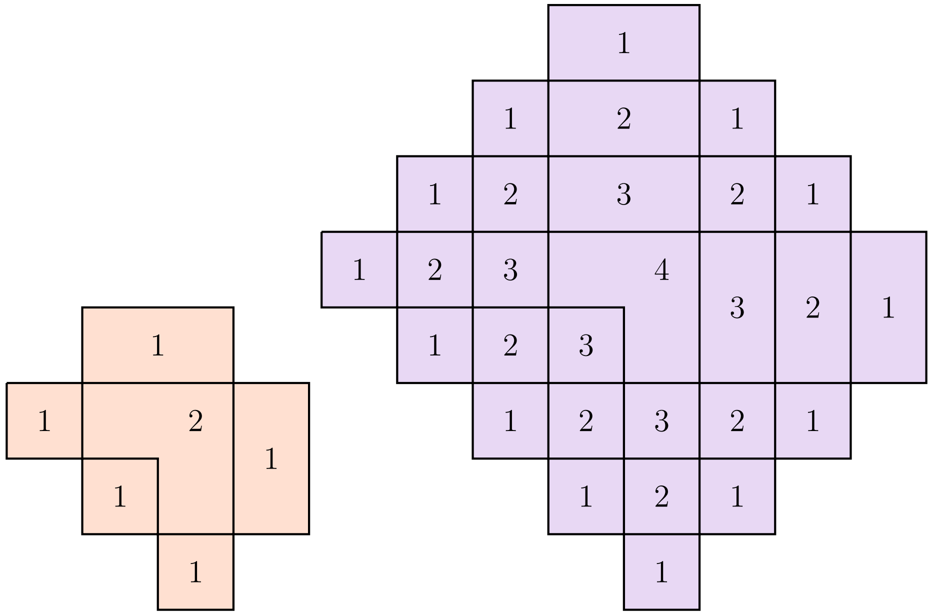
Definition 3.1.20. Let ϕ be a destabilisation domain from x to y. Then the *complexity* of ϕ is the number of horizontal segments in its boundary.

Definition 3.1.21. Let $\phi \in \pi^{D}(x,y)$. Let x_1 be the intersection point of x and β_i, where β_i is the β-circle through c. Then the *innermost width* of ϕ is the length of the horizontal segment of $\partial\phi$ containing x.

Definition 3.1.22. Let $\phi \in \pi^{D}(x,y)$. Let x_1 be the intersection point of x and α_k, where α_k is the β-circle through c. Then the *innermost height* of ϕ is the length of the vertical segment of $\partial\phi$ containing x.

From here, it is helpful to dissect a few examples.

Example 3.1.3. Consider the following type iL domains (where the number in a region indicates local multiplicity).

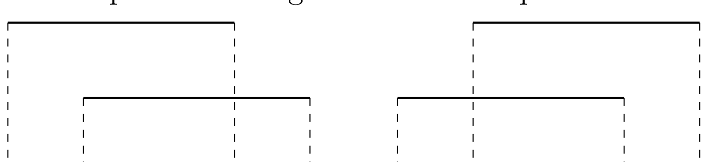


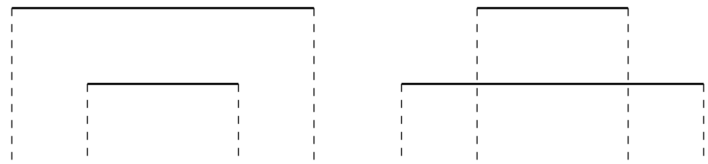
There are a few features to note here. The left hand domain has complexity 5 and the right hand domain has complexity 9. The inner width/height correspond to the shortest horizontal/vertical edges. The vertical and horizontal edges which do not touch c fall into 4 classes, all of whose projections (horizontal/vertical respectively) are nested (based on distance from c). The final interesting feature of these domains is that their boundaries are connected.

I claim the properties we observed above are actually generic properties of all destabilization domains. We start by partitioning horizontal and vertical edges which do not intersect c. Fix a destabilization domain ϕ. Note that all horizontal edges which do not intersect c look like one of the following, by condition 2 of 3.1.19 or 3.1.18.

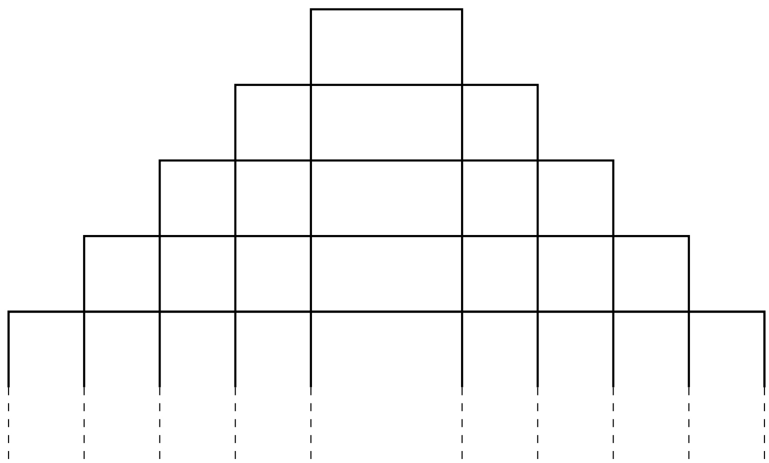
Where the dotted lines indicate the vertical edges connected to the horizontal edge. We denote the first type *South Facing* (which we often abbreviate to SF) and the second *North Facing* (which abbreviates to NF). Clearly SF edges lie above c and NF edges lie below c. We can similarly partition vertical edges which do not intersect c into east and west facing as a consequence of the properties in 3.1.18 and 3.1.19. Above the top horizontal edge of ϕ, ϕ has local multiplicity 0, and edges correspond to pairs of regions with different local multiplicities, so conditions 3 and 4 of 3.1.18 and 3.1.19 guarantee that there are k north facing edges and k south facing edges. Moreover the vertical edges these are connected to must extend at least one square past c. Thus they must intersect all other horizontal edges of the same type.

Now suppose we have a pair of SF edges. We have 4 options for how they are arranged:

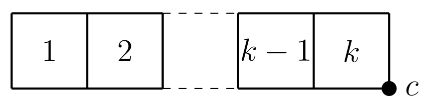




Only one of these does not violate condition 2 of 3.1.18 and 3.1.19, so in general SF edges are arranged like:

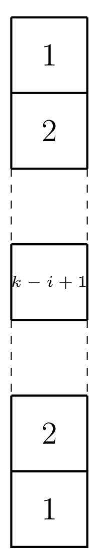


Clearly we have the same structure for NF, EF and WF edges. In the case that ϕ is iL, the local multiplicities above the exceptional horizontal edge look like:



Denoting this edge e_h, we see that e_h intersects all EF edges, so e_h must connect c to the outermost EF edge. Similarly, labelling the vertical exceptional edge e_v, we see that e_v connects c to the outermost NF edge.

Now let e be the ith innermost EF edge. By a similar argument to above, we find that multiplicities to the left of e look like:



Thus e connects the ith outermost NF edge to the $i+1$th SF edge. Similar analysis can be applied to the NF,SF,WF edges to completely describe the combinatorial structure of ϕ. Importantly, note that the complexity of ϕ determines its combinatorial structure. An analogous approach determines the structure of type iR domains in terms of their complexity. Note that, with the rules derived above, we see that the boundary of ϕ is connected. Thus we have described the structure of ϕ. We can now prove an important structural lemma about destabilisation domains.

Lemma 3.1.2. Let $p \in \pi(x, y)$ be a destabilisation domain with complexity k. Then there is a sequence of states $\{x_i\}_{i=1}^{k}$ and rectangles $\{r_i\}_{i=1}^{k-1}$ such that each $r_i \in Rect^o(x_i, x_{i+1})$ and:

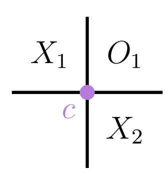
$$p = r_1 * \cdots * r_{k-1}.$$

Among such sequences of rectangles, there is exactly one with the property that each r_i has an edge on the β-circle through c.

Proof. We proceed by induction. Suppose the result holds for all domains with complexity in $\{1, \ldots, n\}$. Suppose ϕ has complexity $n+1$ or $n+2$. Let α_j be the horizontal circle through c and β_i be the vertical circle through c. Let x_1 be the point in x which lies on β_i. For each m, let y_m be the element of y which lies on the same horizontal circle as x_m, and x_{m+1} be that the element of x which lies on the same vertical circle as y_m. Label the rest of the elements of x and y (which coincide) randomly. Let t be the intersection of β_i with the edge between x_2 and y_2, and s be that of β_i with the edge between x_3 and y_3. Then, set $x_1 = x$, $x_2 = \{y_1, t, x_3, \ldots, x_n\}$ and $x_3 = \{y_1, y_2, s, x_4, \ldots, x_n\}$. There are exactly two rectangles in $Rect(x_1, x_2)$ and $Rect(x_2, x_3)$. Moreover, exactly one of these each lies in the support of ϕ. Thus $\phi = r_1 * r_2 * \psi$. Moreover, r_1 intersects the NE and SE squares about c with multiplicity 1, and r_2 intersects the NW and SW squares about c with multiplicity 1, hence ψ is also in π^D and has the same type as ϕ. Thus the result follows by induction. □

3.1.4 Stabilization Domains

Having defined destabilisation domains, we can also define *stabilisation domains*. As before assume we perform an $X : SW$ stabilisation on some grid diagram $\mathbb{G}$ to produce $\mathbb{G}'$. Then we have a distinguished 2×2 square in $\mathbb{G}'$:



Splitting $S(\mathbb{G}')$ as usual we have the following definition:

Definition 3.1.23. Fix $x \in I(\mathbb{G}')$ and $y \in S(\mathbb{G}')$. A domain p from x to y is said to be *out of L* or *out of R* if it is trivial (in which case it is oL) or if it satisfies all of the following:

1. All local multiplicites are non-negative.

2. At each corner in $x \cup y \setminus \{c\}$ at least 3 of the adjoining squares have vanishing local multiplicity.

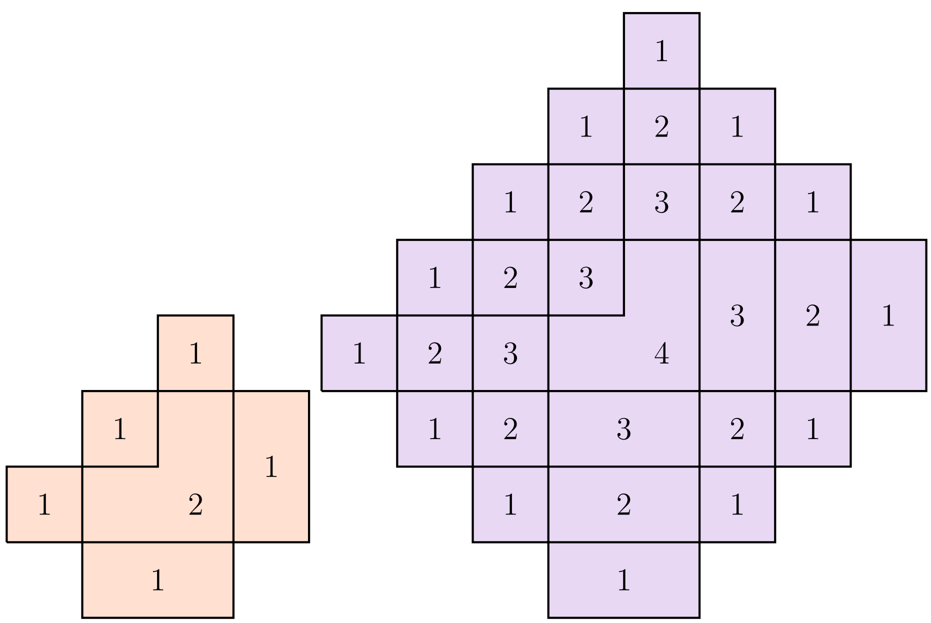
3. In a neighbourhood of c, the local multiplicities in three of the adjoining squares are the same number k. If p is type oL then the domain has local multiplicity $k-1$ at the NW square meeting c. If p is type oR then the domain has local multiplicity $k+1$ at the NE square meeting c.

4. If the domain is type oL then y has $2K+1$ elements not in x, if the domain is type oR then y has $2k+2$ elements not in x.

This definition is also very unwieldy, however, if we note that reflecting along the α-circle through c we preserve conditions $1, 2$ and 4, and transform 3 into the third condition for destabilisation domains. Thus stabilisation domains are really just destabilisation domains reflected across the α-circle through c. The dissection of destabilisation domains above then applies, so the combinatorial structure of a stabilisation domain is determined by its complexity. Moreover, we can perform the same decomposition into rectangles.

Lemma 3.1.3. Let p be a stabilisation domain with complexity k. Then there are grid states $\{x_i\}_{i=1}^{k}$ and rectangles $\{r_i\}_{i=1}^{k-1}$, where $r_i \in Rect^o(x_i, x_{i+1})$ for which $p = r_1 * \cdots * r_{k-1}$. Moreover if we add the condition that each rectangle must have an edge lying in the β-circle through c, this decomposition is unique.

We can reflect our examples from earlier to see how these look.



Chapter 4

Grid Homology

In this chapter, we construct numerous chain complexes from which we will extract a few different definitions of Υ_K in the next chapter. We are interested in applying the techniques of [22] to the setting of grid diagrams. For technical reasons, Livingston uses the construction of Υ_K (given in [3]) from a chain complex which is a $\mathbb{F}[v^{1/n}]$-module. The existing construction of Υ_K via grid diagrams was laid out by Földvári in [21], and uses a chain complex which is a $\mathbb{F}[v^t, v^{2-t}]$-module, where t is any element of $[0, 2]$. To account for this difference of ring, we construct two different *t-modified* chain complexes, we also construct two different $\mathbb{Z}^2$-filtered chain complexes to obtain a grid analogue for other chain complexes used in [22]. In the next chapter, we will relate these different chain complexes, and show that their homologies allow one to compute the same Υ-invariant. We are working with chain complexes over many different rings, so to keep our notation clear, we tabulate notations for the rings we are interested in below ($\mathbb{F}$ is always the field with two elements):

Ring	Notation
$\mathbb{F}[U, U^{-1}]$	Λ
$\mathbb{F}[U]$	Λ^-
$\mathbb{F}[V_1, V_1^{-1}, \ldots, V_n, V_n^{-1}]$	$\mathcal{R}_n^\infty$
$\mathbb{F}[V_1, \ldots, V_n]$	$\mathcal{R}_n^-$
$\mathbb{F}[v^{1/n}, v^{-1/n}]$	$\mathcal{S}_{1/n}$
$\mathbb{F}[v^{1/n}]$	$\mathcal{S}_{1/n}^-$
$\mathbb{F}[v^t, v^{2-t}]$	$\mathcal{S}_t^-$

4.1 Graded Grid Homology Theories

4.1.1 $GC^-(\mathbb{G})$

Gradings

We want to form a bigraded chain complex $(GC^-(\mathbb{G}), \partial_{\mathbb{X}}^-)$ generated over grid states, so we begin by defining *Maslov* (which will give the homological grading) and *Alexander* functions for grid states. The Maslov function on $S(\mathbb{G})$ is determined by the following:

Proposition 4.1.1. For any toroidal grid diagram $\mathbb{G}$, there exists a function:

$$M_{\mathbb{O}} : S(\mathbb{G}) \to \mathbb{Z},$$

called the *Maslov function on grid states*, which is uniquely determined by the following two properties:

1. Let x^{NWO} be the grid state whose components are the upper left corners of squares containing O-markings. Then $M_{\mathbb{O}}(x) = 0$.
2. If $x, y \in S(\mathbb{G})$ and $r \in Rect(x, y)$, then $M_{\mathbb{O}}(x) - M_{\mathbb{O}}(y) = 1 - 2|r \cap \mathbb{O}| + 2|x \cap int(r)|$.

Proof. See [27, Proposition 4.3.1]. □

Exchanging the roles of X and O markings, we also have:

Proposition 4.1.2. For any toroidal grid diagram $\mathbb{G}$, there exists a function:

$$M_{\mathbb{X}} : S(\mathbb{G}) \to \mathbb{Z},$$

which is uniquely determined by the following two properties:

1. Let x^{NWX} be the grid state whose components are the upper left corners of squares containing X-markings. Then $M_{\mathbb{X}}(x) = 0$.
2. If $x, y \in S(\mathbb{G})$ and $r \in Rect(x, y)$, then $M_{\mathbb{X}}(x) - M_{\mathbb{X}}(y) = 1 - 2|r \cap \mathbb{X}| + 2|x \cap int(r)|$.

We can also define these functions in a non-recursive way. Start with the following.

Definition 4.1.1. Let P be a poset and $A, B \subset P \times P$ be finite subsets. Define:

$$\mathcal{I}(A, B) = |\{(a, b) \in A \times B : a_1 < b_1 \text{ and } a_2 < b_2\}|.$$

We can make this symmetric by defining:

$$\mathcal{J}(a, b) = \frac{1}{2}(\mathcal{I}(A, B) + \mathcal{I}(B, A)).$$

Then we can compute $M_{\mathbb{O}}$ and $M_{\mathbb{X}}$ by the following. Let G be a toroidal grid diagram. Choose $[0, n) \times [0, n)$ as a fundamental domain for the corresponding torus. Then we can think of grid states as collections of points with integer coordinates, and grid markings as points in the fundamental domain with coordinates in $(\mathbb{Z} + \frac{1}{2}) \times (\mathbb{Z} + \frac{1}{2})$. Treating grid states and marking in this manner, we obtain the following.

Lemma 4.1.3. If $x \in S(\mathbb{G})$, then:

- $M_{\mathbb{O}}(x) = \mathcal{J}(x, x) - 2\mathcal{J}(x, \mathbb{O}) + \mathcal{J}(\mathbb{O}, \mathbb{O}) + 1,$
- $M_{\mathbb{X}}(x) = \mathcal{J}(x, x) - 2\mathcal{J}(x, \mathbb{X}) + \mathcal{J}(\mathbb{X}, \mathbb{X}) + 1.$

Proof. See [27, Lemma 4.3.5]. □

We can use these two functions to define the following.

Definition 4.1.2. The *Alexander function on grid states* is given by:

$$Alex(x) = \frac{1}{2}(M_{\mathbb{O}}(x) - M_{\mathbb{X}}(x)) - \left(\frac{n-l}{2}\right).$$

Where l is the number of components of the link corresponding to $\mathbb{G}$.

We will almost never use $M_{\mathbb{X}}$, so we abbreviate $M_{\mathbb{O}}$ as M.

Proposition 4.1.4. Let $\mathbb{G}$ be a toroidal grid diagram. Then the function $Alex$ is characterised (up to additive constant) by the following property. For any $x, y \in S(\mathbb{G})$, and $r \in Rect(x, y)$:

$$Alex(x) - Alex(y) = |r \cap \mathbb{X}| - |r \cap \mathbb{O}|$$

Proof. The idea for this proof is to combine the two recursive formulae above, see [27, Proposition 4.3.3] for details. □

For this subsection, fix a grid diagram $\mathbb{G}$ of rank n and recall $\mathcal{R}_n^- = \mathbb{F}[V_1, \dots, V_n]$. Label the O-markings as $\{O_i\}_{i=1}^n$.

Definition 4.1.3. The *unblocked grid complex* $GC^-(\mathbb{G})$ is a free module over $\mathcal{R}_n^-$, with boundary map given on grid states x by:

$$\partial_{\mathbb{X}}^-(x) = \sum_{y \in S(\mathbb{G})} \sum_{\{r \in Rect^o(x,y) : r \cap \mathbb{X} = 0\}} V_1^{O_1(r)} \dots V_n^{O_n(r)} \cdot y.$$

Where $O_i(r)$ is one if r contains O_i, and zero otherwise.

We can make this bigraded by extending the Maslov and Alexander functions in the following manner, for each $x \in S(\mathbb{G})$:

$$M(V_1^{k_1} \dots V_n^{k_n} x) = M(x) - 2k_1 - 2k_2 - \dots - 2k_n,$$

$$Alex(V_1^{k_1} \dots V_n^{k_n} x) = Alex(x) - k_1 - k_2 - \dots - k_n.$$

This clearly equips $GC^-(\mathbb{G})$ with a bigrading. More explicitly, let:

$$GC_d^-(\mathbb{G}, s),$$

be the vector space spanned by basis vectors $V_1^{k_1}, \dots, V_n^{k_n} x$ ($x \in S(\mathbb{G})$) which satisfy:

$$M(V_1^{k_1} \dots V_n^{k_n} x) = d,$$

$$Alex(V_1^{k_1} \dots V_n^{k_n} x) = s.$$

Theorem 4.1.5. The object $(GC^-(\mathbb{G}), \partial_{\mathbb{X}}^-)$ is a bigraded chain complex over $\mathcal{R}_n^-$.

Proof. See proof of [27, Theorem 4.6.3]. □

Definition 4.1.4. The *unblocked grid homology* of $\mathbb{G}$, denoted $GH^-(\mathbb{G})$ is the homology of $(GC^-(\mathbb{G}), \partial_{\mathbb{X}}^-)$, viewed as a bigraded module over Λ^-, where U is induced by multiplication by V_1.

This is well defined as a consequence of the following.

Lemma 4.1.6. Suppose $\mathbb{G}$ represents a knot. For any pair $i, j \in \{1, \ldots, n\}$, multiplication by V_i is chain homotopic to multiplication by V_j, when thought of as homogeneous maps from $GC^-(\mathbb{G})$ to itself of degree $(-2, -1)$.

Proof. This was proven in [27, Lemma 4.6.9], by showing that, if two O markings share a common X marking (ie. are adjacent to the same X-marking in the corresponding rectilinear diagram), then there is a chain homotopy between the variables in $\mathcal{R}_n^-$ which correspond to those markings. The proof is complete once one notes that, because $\mathbb{G}$ represents a knot, there is a sequence of O-markings so that each consecutive pair of markings share an X-marking, and all O markings appear in the sequence. □

We have the following fundamental invariance theorem:

Theorem 4.1.7. The bigraded $\mathbb{F}[U]$-module $GH^-(\mathbb{G})$ depends only upon the knot type of $\mathbb{G}$.

Proof. See [27, Theorem 5.3.1]. □

Finally, we have the following theorem which connects combinatorial and pseudo-holomorphic theories.

Theorem 4.1.8. If $\mathcal{H}$ is a Heegaard diagram induced from a grid diagram $\mathbb{G}$, then $(CFK^-(\mathcal{H}), \partial_K^-)$ is isomorphic to $(GC^-(\mathbb{G}), \partial_{\mathbb{X}}^-)$.

Proof. See [27, Theorem 16.4]. □

For more details on $CFK^-(\mathcal{H})$, see [27, §16.3].

4.2 t-Modified Homologies

In this section, we construct variants of t-modified grid homology by changing the coefficient ring we use. This is necessary for us to make the construction in [22] combinatorial, as the algebraic filtration induced by the larger coefficient ring Földvári uses in [21], when expanded to include positive algebraic filtration levels, does not separate the corresponding homology in the correct way.

We start with a brief summary of Földvári's chain complex.

4.2.1 $tGC^-(\mathbb{G})$

This is a grid homology theory developed in [21]. We briefly summarise the pertinent properties of $tGC^-(\mathbb{G})$ in this section. To begin with, fix a grid diagram $\mathbb{G}$ and a number $t \in [0,2]$. Recall we have the functions M and $Alex$ defined on $S(\mathbb{G})$. Then the *t-Modified Grid Homology* of $\mathbb{G}$ is given by the following definition.

Definition 4.2.1. Let $tGC^-(\mathbb{G})$ be the free $\mathcal{S}_t^-$-module generated over $S(\mathbb{G})$, equipped with the $\mathcal{S}_t^-$-endomorphism:

$$\partial_t^- : tGC^-(\mathbb{G}) \to tGC^-(\mathbb{G}),$$

given on grid states $x \in S(\mathbb{G})$ by:

$$\partial_t^-(x) = \sum_{y \in S(\mathbb{G})} \sum_{r \in Rect^o(x,y)} v^{t|\mathbb{X}\cap r| + (2-t)|\mathbb{O}\cap r|} \cdot y.$$

We call the pair $(tGC^-(\mathbb{G}), \partial_t^-)$ the *t-Modified Grid Complex* and its homology, $H_*(tGC^-(\mathbb{G}))$, is called the *t-Modified Grid Homology* of $\mathbb{G}$. We denote this $tGH^-(\mathbb{G})$.

We have implicitly assumed in the definition above that $tGC^-(\mathbb{G})$ is a chain complex. The following verifies this.

Theorem 4.2.1. The endomorphism ∂_t^- given above satisfies $\partial_t^- \circ \partial_t^- = 0$.

Proof. See [21, Theorem 3.22]. □

$\mathbb{R}$-grading

The homological grading on $tGC^-(\mathbb{G})$ is an $\mathbb{R}$-grading. This is necessary, as we want to use this grading to construct $\Upsilon_K(t)$, which can take on non integer values.

Definition 4.2.2. Let $x \in S(\mathbb{G})$ and $U^\alpha \in \mathcal{S}_t^-$. Then the *t-grading* on $U^\alpha \cdot x$ is given by the following:

$$gr_t(U^\alpha \cdot x) = M(x) - tAlex(x) - \alpha.$$

It makes sense to treat this as the homological grading of $tGC^-(\mathbb{G})$, as a consequence of the following.

Proposition 4.2.2. ∂_t^- is homogeneous of degree -1 with respect to gr_t.

Proof. See [21, Proposition 3.24]. □

Effect of Grid Moves

Recall that two grid diagrams have the same link type if and only if there exists a sequence of Cromwell moves connecting the two diagrams. In [21], Földvári proves the following.

Theorem 4.2.3. If $\mathbb{G}$ differs from $\mathbb{G}'$ by a commutation move or a switch move, then $tGH^-(\mathbb{G})$ and $tGH^-(\mathbb{G}')$ are isomorphic as $\mathbb{R}$-graded $\mathcal{S}_t^-$-modules.

Proof. See [21, Theorem 3.34]. □

Theorem 4.2.4. If $\mathbb{G}'$ is obtained from $\mathbb{G}$ by a stabilisation move, then we have the following isomorphism of $\mathbb{R}$-graded $\mathcal{S}_t^-$-modules:

$$tGH^-(\mathbb{G}') \cong tGH^-(\mathbb{G}) \oplus tGH^-(\mathbb{G})[1-t].$$

Proof. See [21, §3.5]. □

4.2.2 $tGC^\infty(\mathbb{G})$

In this subsection we construct a grid analogue of the t-modified homology used in [22]. To begin, fix $t \in \mathbb{Q} \cap [0,2]$ and $n \in \mathbb{Z}$ for which $t = \frac{m}{n}$ and $\gcd(m,n) = 1$. Recall $\mathcal{S}_{1/n}^\infty$ is the ring of Laurent polynomials modulo 2 in the variable $v^{\frac{1}{n}}$. Similarly $\mathcal{S}_{1/n}^-$ is the ring of polynomials modulo 2 in the variable $v^{\frac{1}{n}}$.

Definition 4.2.3. For t, n as defined above, and a grid diagram $\mathbb{G}$, the t-modified grid complex is the free $\mathcal{S}_{1/n}^\infty$-module $tGC^\infty(\mathbb{G})$ generated over $S(\mathbb{G})$, equipped with the differential:

$$\partial_t^\infty(x) = \sum_{y \in S(\mathbb{G})} \sum_{r \in Rect^o(x,y)} v^{t|\mathbb{X} \cap r| + (2-t)|\mathbb{O} \cap r|} y \qquad x \in S(\mathbb{G}).$$

We also have a grading and a filtration:

$$gr_t(v^\alpha x) = M(x) - tAlex(x) - \alpha \qquad x \in S(\mathbb{G}), v^\alpha \in \mathcal{S}_{1/n}^\infty.$$
$$alg(v^\alpha x) = -\alpha \qquad x \in S(\mathbb{G}), v^\alpha \in \mathcal{S}_{1/n}^-.$$

Our first task is to check that this forms a graded, filtered chain complex.

Theorem 4.2.5. The tuple $(tGC^\infty(\mathbb{G}), \partial_t^\infty, gr_t, alg)$ is a gr_t-graded, alg-filtered chain complex.

Proof. To begin with, we check that $(\partial_t^\infty)^2 = 0$. Fix grid states $x, z \in S(\mathbb{G})$, and for each $\psi \in Dom(x,y)$, let $N(\psi)$ be the number of ways ψ can be expressed as a juxtaposition of two empty rectangles. Then clearly we have:

$$\left(\partial_t^\infty\right)^2(x) = \sum_{z \in S(\mathbb{G})} \sum_{\psi \in Dom(x,z)} N(\psi) v^{t|\mathbb{X} \cap \psi| + (2-t)|\mathbb{O} \cap \psi|} z.$$

Based upon our discussion of juxtapositions of rectangles in §3.1, if $N(\psi) \neq 0$, then ψ falls into one of three cases. Decompose ψ as $\psi = r * s$ where $r \in Rect^o(x,y)$ and $s \in Rect^o(y,z)$. Then ψ falls into one of three cases. If r and s share 0 or 1 corner, then by the proof of [27, Lemma 4.6.7] $N(\psi) = 2$, so these cases cancel in $(\partial_t^\infty)^2$. If r and s share 4 corners, then we must have $z = x$. Moreover, ψ is empty so must be a *thin annulus* (a domain corresponding to a component of $T \setminus \bar{\alpha}$ or $T \setminus \bar{beta}$). There are exactly 2rank($\mathbb{G}$) of these thin annuli which contribute to $(\partial_t^\infty)^2$, each contributing $v^2 x$, so this case also contributes zero. Thus $(\partial_t^\infty)^2 = 0$.

Next, we need to check that ∂_t^∞ drops gr_t by 1. Let $x, y \in S(\mathbb{G})$ and $r \in Rect^o(x, y)$. Recall we have the recursive formulae:

$$M(x) - M(y) = 1 - 2|\mathbb{O} \cap r|,$$

$$Alex(x) - Alex(y) = |\mathbb{X} \cap r| - |\mathbb{O} \cap r|.$$

Thus:

$$\begin{aligned} gr_t(x) - gr_t(y) &= M(x) - M(y) - t(Alex(x) - ALex(y)) && (4.1)\\ &= 1 - 2|\mathbb{O} \cap r| - t(|\mathbb{X} \cap r| - |\mathbb{O} \cap r|) && (4.2)\\ &= 1 - t|\mathbb{X} \cap r| - (2-t)|\mathbb{O} \cap r|. && (4.3) \end{aligned}$$

Hence:

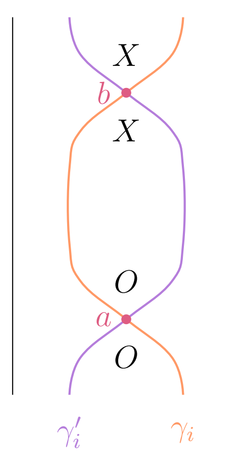
$$gr_t(x) - gr_t(v^{t|\mathbb{X}\cap r| + (2-t)|\mathbb{O}\cap r|}y) = 1,$$

so the tuple is gr_t-graded. Finally, noting that all coefficients of the summands in $\partial_t^\infty(x)$ have a positive power, ∂_t^∞ is trivially filtered. □

As per [27, Corollary 3.2.3] to understand how tGC^∞ transforms under grid moves, it is sufficient to understand how it transforms under: commutations, switches and stabilisations of type $X : SW$.

Commutations and Switches

For this section, let $\mathbb{G}$ and $\mathbb{G}'$ be a pair of grid diagrams who differ by a commutation or a switch. Assume without loss of generality that this occurs across γ_i in $\mathbb{G}$ and γ_i' in $\mathbb{G}'$ so that their augmented diagram looks like:



As before rotate by $\pi/2$ to obtain the diagram for row commutations/switches. We can use this diagram to construct the *pentagon counting maps*.

Definition 4.2.4. For $\mathbb{G}$ and $\mathbb{G}'$ as above, define the following pair of maps.

$$\begin{aligned}
&P : tGC^\infty(\mathbb{G}) \to tGC^\infty(\mathbb{G}') \\
&P(x) = \sum_{y\in S(\mathbb{G}')} \sum_{p\in Pent_a^o(x,y)} v^{t|\mathbb{X}\cap p|+(2-t)|\mathbb{O}\cap p|} y \qquad x \in S(\mathbb{G}) \\
&P : tGC^\infty(\mathbb{G}') \to tGC^\infty(\mathbb{G}) \\
&P(x) = \sum_{y\in S(\mathbb{G})} \sum_{p\in Pent_b^o(x,y)} v^{t|\mathbb{X}\cap p|+(2-t)|\mathbb{O}\cap p|} y \qquad x \in S(\mathbb{G}')
\end{aligned}$$

We call these the *pentagon counting maps.*

Lemma 4.2.6. The maps P and P' are graded and filtered.

Proof. As noted in the proof that ∂_t^∞ is filtered, all coefficients in $P(x)$ and $P'(x')$ have a non-negative power, so the maps are trivially filtered. Checking these maps are graded is much more involved. There is a canonical bijection between $S(\mathbb{G})$ and $S(\mathbb{G}')$ obtained by sending $x \in S(\mathbb{G})$ to the element $x' \in S(\mathbb{G}')$ which agrees with x on $n-1$ lattice points. By the proof of [27, Lemma 5.1.3], we know that, if p is a pentagon from $x \in S(\mathbb{G}) \cup S(\mathbb{G}')$ to $y \in S(\mathbb{G} \cup S(\mathbb{G}'))$, then:

$$\begin{aligned}
M(x) - M(y) &= -2|p \cap \mathbb{O}| + 2|x \cap Intp|, \\
Alex(x) - Alex(y) &= |p \cap X| - |p \cap O|.
\end{aligned}$$

Thus we have:

$$\begin{aligned}
gr_t(x) - gr_t(y) &= M(x) - M(y) - (tAlex(x) - tAlex(y)) \\
&= -2|p \cap \mathbb{O}| + t(|p \cap \mathbb{O}| - |p \cap \mathbb{X}|) \\
&= -t|p \cap \mathbb{X}| - (2-t)|p \cap \mathbb{O}|.
\end{aligned}$$

So:

$$gr_t(x) - gr_t(v^{t|\mathbb{X}\cap p|+(2-t)|\mathbb{O}\cap p|} y) = 0,$$

which means that P and P' are both graded. □

Next we need to check these are chain maps. The proof is combinatorial, so much can be lifted from the proof of [27, Lemma 5.1.4].

Lemma 4.2.7. The maps P and P' are chain maps.

Proof. We prove this for P, the proof for P' is identical. The approach is very similar to the proof that ∂_t^∞ squares to zero. Let $x \in S(\mathbb{G})$ and $y \in \mathbb{G}'$ and ψ be a domain in the augmented diagram connecting x to y. Let $N(\psi)$ be the number of ways that ψ can be decomposed as a the juxtaposition of an empty pentagon and an empty rectangle, or an empty rectangle and an empty pentagon. Clearly, we have, for $x \in S(\mathbb{G})$:

$$\partial_t^\infty \circ P + P \circ \partial_t^\infty(x) = \sum_{y\in S(\mathbb{G}')} \sum_{\psi\in Dom^o(x,y)} N(\psi) v^{t|\mathbb{X}\cap\psi|+(2-t)|\mathbb{O}\cap\psi|} y.$$

In the proof of [27, Lemma 5.1.4] it is verified that $N(\psi) = 2$ for all ψ with $N(\psi) \neq 0$, apart from the case where ψ consists of an annulus supported in between γ_{i-1} and γ_{i+1}. Given we are working modulo 2, we only need to deal with this special case. These domains must have support in the annulus between γ_{i-1} and γ_{i+1}, so their states can differ on at most one point, hence $y = x'$. Thus there are only two of these domains for each x. We have one to the west, which we denote ψ_1 and one to the east which we label ψ_2. To establish the relationship between ψ_1 and ψ_2, let χ_1 be the western annulus and χ_2 be the eastern annulus. If t'_x is the triangle connecting x and $x' = y$, with its third corner at b, then it is clear that $\psi_1 = \chi_1 + t'_x$ and $\psi_2 = \chi_2 + t'_x$. Neither χ_1 nor χ_2 intersects a grid marking, so noting that $\psi_1 - \psi_2 = \chi_1 - \chi_2$, both domains contribute the same summand to the equation. Again given we are working modulo 2, this means that their contributions cancel, so we have proven that:

$$\partial_t^\infty \circ P = P \circ \partial_t^\infty.$$

□

Now we use hexagons to construct chain homotopy equivalences from P to P' and P' to P. Start by defining the hexagon counting maps.

Definition 4.2.5. Let $\mathbb{G}$ and $\mathbb{G}'$ be as above. Then the *hexagon counting maps* are given by:

$$\begin{aligned}
&H : tGC^\infty(\mathbb{G}) \to tGC^\infty(\mathbb{G}) \\
&H(x) = \sum_{y \in S(\mathbb{G})} \sum_{h \in Hex^o(x,y)} v^{t|\mathbb{X}\cap h| + (2-t)|\mathbb{O}\cap h|} y \qquad\qquad x \in S(\mathbb{G}) \\
&H' : tGC^\infty(\mathbb{G}') \to tGC^\infty(\mathbb{G}') \\
&H(x) = \sum_{y \in S(\mathbb{G}')} \sum_{h \in Hex^o(x,y)} v^{t|\mathbb{X}\cap h| + (2-t)|\mathbb{O}\cap h|} y \qquad\qquad x \in S(\mathbb{G}')
\end{aligned}$$

Lemma 4.2.8. Both H and H' raise the grading by 1.

Proof. Note that if $h \in Hex^o(x, y)$ then adding one of the two bigons between a and b results in a rectangle r from x to y. Both of these bigons intersect exactly one X-marking and exactly one O-marking, so we have:

$$\begin{aligned}
gr_t(x) - gr_t(y) &= -t|\mathbb{X} \cap r| - (2-t)|\mathbb{O} \cap r| + 1 \\
&= -t|\mathbb{X} \cap h| - (2-t)|\mathbb{O} \cap h| + 1 - 2 \\
&= -t|\mathbb{X} \cap h| - (2-t)|\mathbb{O} \cap h| - 1.
\end{aligned}$$

Thus:

$$gr_t(x) - gr_t(v^{t|\mathbb{X}\cap h| + (2-t)|\mathbb{O}\cap h|}) = -1,$$

so the result follows. □

Theorem 4.2.9. H is a chain homotopy from $P' \circ P$ to the identity.

Proof. We need to check the following equation holds:

$$P' \circ P + \partial_t^\infty \circ H + H \circ \partial_t^\infty = id.$$

Let $x, y \in S(\mathbb{G})$ and ψ be an empty domain in the augmented diagram from x to y. Let $N(\psi)$ be the number of ways ψ can be decomposed as $h * r$, $r * h$ or $p * q$, for some hexagon and rectangle h and r respectively, or for pentagons p, q. Then:

$$(P' \circ P + \partial_t^\infty \circ H + H \circ \partial_t^\infty)(x) = \sum_{y \in S(\mathbb{G})} \sum_{\psi \in Dom^o_{aug}(x,y)} N(\psi) v^{t|\mathbb{X} \cap \psi| + (2-t)|\mathbb{O} \cap \psi|} y. \tag{4.4}$$

It is verified in the proof of [27, Lemma 5.1.6] that if $x \neq y$, then $N(\psi) = 2$ or 0, so these terms drop out of 4.4. If $x = y$, then there is a unique domain ψ in $Dom^o_{aug}(x, y)$ and as noted in [27, Lemma 5.1.6] $N(\psi) = 1$. Because ψ is empty it must be a thin annulus, one of χ_1 or χ_2. Neither of these intersect a grid marking, so each summand contributes x, giving:

$$(P' \circ P + \partial_t^\infty \circ H + H \circ \partial_t^\infty)(x) = x.$$

And the result follows. □

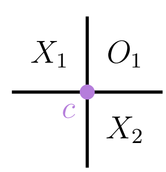
A nearly identical proof verifies this is the case for H', so we have the following:

Theorem 4.2.10. If $\mathbb{G}$ and $\mathbb{G}'$ are grid diagrams who differ by a commutation move or a switch, then there is a filtered chain homotopy equivalence between $tGC^\infty(\mathbb{G})$ and $tGC^\infty(\mathbb{G}')$. In particular, we have a filtered isomorphism:

$$tGH^\infty(\mathbb{G}) \cong tGH^\infty(\mathbb{G}').$$

Stabilisation

Now, suppose $\mathbb{G}$ is a grid diagram of rank n and $\mathbb{G}'$ is a grid diagram obtained from $\mathbb{G}$ by an $X : SW$ stabilisation. Recall we have the distinguished 2×2 square:



We fix a numbering of the markings in $\mathbb{G}'$ for brevity. First label markings as indicated in the figure above. Then let O_2 be the O-making in the same row as X_2. Number the remaining markings arbitrarily. Then the grid states in $\mathbb{G}'$ can be decomposed as $S(\mathbb{G}') = I(\mathbb{G}') \cup N(\mathbb{G}') = I \cup N$, where $x \in I(\mathbb{G}') = I$if $c \in x$. We have a canonical map $e : I(\mathbb{G}') \to S(\mathbb{G})$ defined by sending x to $x \setminus \{c\}$.

Lemma 4.2.11. If $x \in I(\mathbb{G}')$, then:

$$gr_t(x) = gr_t(e(x)) + t - 1.$$

Proof. By [27, Lemma 5.2.4], we know that:

$$M(x) = M(e(x)) - 1$$
$$Alex(x) = Alex(e(x)) - 1$$

So:

$$\begin{aligned} gr_t(x) &= M(x) - tAlex(x) \\ &= M(e(x)) - tAlex(e(x)) + t - 1 \\ &= gr_t(e(x)) + t - 1 \end{aligned}$$

Also note that $alg(x) = alg(e(x))$. Then the result follows. □

Recall that we defined and described destabilisation domains in 3.1.1. We can now use these to construct the following.

Definition 4.2.6. Fix $t \in [0, 2] \cap \mathbb{Q}$. Let $\overline{\mathbb{O}} = \{O_1, O_3, O_4, \dots\}$ and $\overline{\mathbb{X}} = \{X_1, X_3, X_4, \dots\}$. Define:

$$D^{iL} : tGC^\infty(\mathbb{G}') \to tGC^\infty(\mathbb{G})[1-t]$$
$$D^{iL}(x) = \sum_{y \in I(\mathbb{G}')} \sum_{\psi \in \pi^{iL}(x,y)} v^{t|\overline{\mathbb{X}} \cap \psi| + (2-t)|\overline{\mathbb{O}} \cap \psi|} e(y) \qquad x \in S(\mathbb{G}')$$

and:

$$D^{iR} : tGC^\infty(\mathbb{G}') \to tGC^\infty(\mathbb{G})$$
$$D^{iR}(x) = \sum_{y \in I(\mathbb{G}')} \sum_{\psi \in \pi^{iR}(x,y)} v^{t|\overline{\mathbb{X}} \cap \psi| + (2-t)|\overline{\mathbb{O}} \cap \psi|} e(y) \qquad x \in S(\mathbb{G}')$$

Taking $D = D^{iR} \oplus D^{iL}$ we obtain the *t-modified destabilisation map*:

$$D : tGC^\infty(\mathbb{G}') \to tGC^\infty(\mathbb{G}) \oplus tGC^\infty(\mathbb{G})[1-t].$$

Lemma 4.2.12. D^{iL} is filtered and graded.

Proof. Let $x \in S(\mathbb{G}')$ and $y \in I(\mathbb{G}')$. Let $\psi \in \pi^{iL}(x, y)$. Let ψ have complexity k. Then by 3.1.2, ψ decomposes as a juxtaposition of rectangles:

$$r_1 * \cdots * r_{k-1}.$$

Then, recursively applying 4.3 we find:

$$gr_t(x) - gr_t(y) = -t|\mathbb{X} \cap \psi| - (2-t)|\mathbb{O} \cap \psi| + k - 1.$$

Because type iL domains have the same multiplicity at both O_1 and X_2, we have:

$$gr_t(x) - gr_t(y) = -t|\overline{\mathbb{X}} \cap \psi| - (2-t)|\overline{\mathbb{O}}|,$$

hence, applying 4.2.11:

$$gr_t(x) - gr_t(v^{t|\mathbb{X} \cap \psi| + (2-t)|\mathbb{O} \cap \psi|} e(y)) = -1 + t$$

So the map is graded. As usual the map is trivially filtered, so the result follows. □

Lemma 4.2.13. D^{iR} is graded and filtered.

Proof. Let $x \in S(\mathbb{G}')$ and $y \in I(\mathbb{G}')$. Let $\psi \in \pi^{iR}(x, y)$. Let ψ have complexity k. Then by 3.1.2, ψ decomposes as a juxtaposition of rectangles:

$$r_1 * \cdots * r_{k-1}.$$

Then, recursively applying 4.3 we find:

$$gr_t(x) - gr_t(y) = -t|\mathbb{X} \cap \psi| - (2-t)|\mathbb{O} \cap \psi| + k - 1.$$

For type iR domains the multiplicity at X_2 is one larger than at $O - 1$, so:

$$gr_t(x) - gr_t(y) = -t|\overline{\mathbb{X}} \cap \psi| - (2-t)|\overline{\mathbb{O}}| + 1 - t,$$

hence:

$$gr_t(x) - gr_t(v^{t|\overline{\mathbb{X}} \cap \psi| + (2-t)|\overline{\mathbb{O}} \cap \psi|} e(y)) = 0,$$

so the map is graded. As usual the map is clearly filtered, so the result follows. □

Theorem 4.2.14. The map $D : tGC^\infty(\mathbb{G}') \to tGC^\infty(\mathbb{G}) \oplus tGC^\infty(\mathbb{G})[1-t]$ is a chain map.

Proof. The proof of this result involves pairing up certain juxtapositions of destabilisation domains and rectangles. Fortunately, the proof of [27, Lemma 13.3.13] applies with the modification that $V_1 = V_2 = v$. □

Next, we need a little bit of homological algebra. What follows is similar to the approach to stabilisation in [21], the only difference being the ring over which we work.

Lemma 4.2.15. Let C and D be graded chain complexes over $\mathbb{F}[v^{\frac{1}{n}}]$. Further, suppose that the grading on C and D is bounded above. Let v^α be a monomial in $\mathbb{F}[v^{\frac{1}{n}}]$, and f be a graded chain map. Then f is a quasi-isomorphism if and only if it induces a quasi-isomorphism:

$$\bar{f} : C/v^\alpha C \to D/v^\alpha D.$$

Proof. We use a mapping cone argument. If f is a quasi-isomorphism, then $H(Cone(f)) = 0$. C and D are free, so $Cone(f)$ is free. Thus, there exists a short exact sequence:

$$0 \longrightarrow Cone(f) \xrightarrow{v^\alpha} Cone(f) \xrightarrow{q} Cone(f)/v^\alpha Cone(f) \longrightarrow 0$$

Via the snake lemma, we obtain the exact triangle:

$$\begin{array}{ccc} H(Cone(f)) & \longrightarrow & H(Cone(f)) \\ \uparrow & \swarrow & \\ H(Cone(f)/v^\alpha Cone(f)) & & \end{array}$$

So clearly, if $H(Cone(f)) = 0$ then $H(Cone(f)/v^\alpha Cone(f)) = 0$. Noting that $H(Cone(\bar{f})) = H(Cone(f)/v^\alpha Cone(f))$, this implies that $\bar{f}$ is a quasi-isomorphism. Suppose now that $H(Cone(f)) \neq 0$. Again we have the exact triangle (now with morphisms labelled):

$$\begin{array}{ccc} H(Cone(f)) & \xrightarrow{\quad v^\alpha \quad} & H(Cone(f)) \\ \delta \uparrow & \swarrow_{q^*} & \\ H(Cone(f)/v^\alpha Cone(f)) & & \end{array}$$

$H(Cone(f)) \neq 0$, so there is a homogeneous element of $H(Cone(f))$ with maximal grading. If $x = v^\alpha y$, we have a contradiction, as the grading is maximal. Thus $x \notin \mathrm{Im} v^\alpha = \ker q_*$, so x injects into $H(Cone(\bar{f}))$. Then clearly $H(Cone(\bar{f})) \neq 0$. Thus $H(Cone(f)) = 0$ if and only if $H(Cone(\bar{f})) = 0$, so f is a quasi-isomorphism if and only if $\bar{f}$ is a quasi-isomorphism. □

Next, for a grid diagram $\mathbb{G}$, let $\widetilde{GC(\mathbb{G})}$ be the quotient of $tGC^\infty(\mathbb{G})$ obtained by setting $v^{1/n} = 0$. Then the differential of $\widetilde{GC(\mathbb{G})}$ is then given on $span_{\mathbb{F}} S(\mathbb{G})$ by:

$$\widetilde{\partial}(x) = \sum_{y \in S(\mathbb{G})} \sum_{\substack{r \in Rect^o(x,y) \\ r \cap (\mathbb{O} \cup \mathbb{X}) = \emptyset}} y.$$

This is the *full blocked grid complex* from [27, §4.4], with its grading altered and algebraic filtration introduced. We sketch the invariance of this, lifting the combinatorics from [27, §4.4]. Recall we have fixed grid diagrams $\mathbb{G}$ and $\mathbb{G}'$ so that $\mathbb{G}'$ is an $X : SW$-stabilisation of $\mathbb{G}$. Noting that rectangles cannot pass X-markings, $\widetilde{GC(\mathbb{G}')}$ splits. Set $\widetilde{I} = span_{\mathbb{F}} I(\mathbb{G}')$, and $\widetilde{N} = span_{\mathbb{F}} N(\mathbb{G}')$. $\widetilde{GC(\mathbb{G}')}$ is the mapping cone of $\widetilde{\partial}_{\widetilde{I}}^{\widetilde{N}}$. First, we relate $\widetilde{I}$ to $\widetilde{GC(\mathbb{G})}$.

Lemma 4.2.16. Define a map $e : \widetilde{I} \to \widetilde{GC(\mathbb{G})}$ by sending a state $x \in S(\mathbb{G}')$ to $x \setminus c$. This yields an isomorphism between $(\widetilde{I}, \widetilde{\partial}_{\widetilde{I}}^{\widetilde{I}})$ and $\widetilde{GC(\mathbb{G})}[1-t]$

Proof. To see why this map commutes with the differential, see [27, Lemma 5.2.5]. By [27, Lemma 5.2.4], we know this map raises $M(x)$ and $Alex(x)$ by 1. Thus:

$$\begin{aligned} gr_t(e(x)) &= M(e(x)) - tAlex(e(x)) \\ &= M(x) + 1 - t(Alex(x) + 1) \\ &= M(x) - tAlex(x) + 1 - t. \end{aligned}$$

Thus the result follows. □

Next, we relate the complexes $\widetilde{I}$ and $\widetilde{N}$. Define:

$$\widetilde{\mathcal{H}}_{X_2}^{\widetilde{I}} : \widetilde{N} \to \widetilde{I},$$

$$\widetilde{\mathcal{H}}_{X_2}^{\widetilde{I}}(x) = \sum_{y \in I(\mathbb{G}')} \sum_{\substack{r \in Rect^o(x,y) \\ r \cap (\mathbb{O} \cup \mathbb{X}) = \{X_2\}}} y.$$

It is proven in [27, Lemma 5.2.6] that this induces an isomorphism on homology, and drops Alexander and Maslov gradings by 1. Thus $\widetilde{\mathcal{H}}_{X_2}^{\widetilde{I}}$ drops the gr_t-grading by $1 - t$. Applying [27, Lemma 5.2.7], we see that $\widetilde{\partial}_{\widetilde{I}}^{\widetilde{N}}$ is zero on homology.

Proposition 4.2.17. There is an isomorphism:

$$\widetilde{GC}(\mathbb{G}') \cong \widetilde{GC}(\mathbb{G}) \oplus \widetilde{GC}(\mathbb{G})[1-t].$$

Proof. Because $\widetilde{\partial}^{\widetilde{N}}_{\widetilde{I}}$ is zero on homology, the long exact sequence of $Cone(\widetilde{\partial}^{\widetilde{N}}_{\widetilde{I}})$ splits as:

$$0 \longrightarrow \widetilde{N} \longrightarrow \widetilde{GH}(\mathbb{G}') \longrightarrow \widetilde{I} \longrightarrow 0.$$

Adding in the isomorphisms we described above we get:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{N} & \longrightarrow & \widetilde{GC}(\mathbb{G}') & \longrightarrow & \widetilde{I} & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle e\circ\widetilde{\mathcal{H}}^{\widetilde{I}}_{X_2}} & & & & \Big\downarrow {\scriptstyle e} & & \\
0 & \longrightarrow & \widetilde{GH}(\mathbb{G}) & \longrightarrow & \widetilde{GH}(\mathbb{G}) \oplus \widetilde{GH}(\mathbb{G})[1-t] & \longrightarrow & \widetilde{GH}(\mathbb{G})[1-t] & \longrightarrow & 0.
\end{array}$$

Finally, note that we are working over $\mathbb{F}$-spaces, so all short exact sequences split, and thus the two isomorphisms induce the middle distinguished isomorphism in the diagram below:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{N} & \longrightarrow & \widetilde{GC}(\mathbb{G}') & \longrightarrow & \widetilde{I} & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle e\circ\widetilde{\mathcal{H}}^{\widetilde{I}}_{X_2}} & & \vdots\downarrow & & \Big\downarrow {\scriptstyle e} & & \\
0 & \longrightarrow & \widetilde{GH}(\mathbb{G}) & \longrightarrow & \widetilde{GH}(\mathbb{G}) \oplus \widetilde{GH}(\mathbb{G})[1-t] & \longrightarrow & \widetilde{GH}(\mathbb{G})[1-t] & \longrightarrow & 0
\end{array}$$

So $(e, e \circ \widetilde{\mathcal{H}}^{\widetilde{I}}_{X_2}) : \widetilde{GH}(\mathbb{G}') \to \widetilde{GH}(\mathbb{G}) \oplus \widetilde{GH}(\mathbb{G})[1-t]$ is an isomorphism. □

Theorem 4.2.18. For each $s \in \mathbb{R}$, the restriction:

$$D_{alg\leq s} : tGC^{\infty}_{alg\leq s}(\mathbb{G}') \to tGC^{\infty}_{alg\leq s}(G) \oplus tGC^{\infty}_{alg\leq s}(\mathbb{G})[1-t],$$

is a quasi-isomorphism.

Proof. First, suppose that $s = 0$. In the case, we have:

$$\frac{tGC^{\infty}_{alg\leq s}(\mathbb{G}')}{v^{\frac{1}{n}} tGC^{\infty}_{alg\leq s}(\mathbb{G}')} \cong \widetilde{GC}(\mathbb{G}'),$$
$$\frac{tGC^{\infty}_{alg\leq s}(G) \oplus tGC^{\infty}_{alg\leq s}(\mathbb{G})[1-t]}{v^{\frac{1}{n}}(tGC^{\infty}_{alg\leq s}(G) \oplus tGC^{\infty}_{alg\leq s}(\mathbb{G})[1-t])} \cong \widetilde{GC}(\mathbb{G}) \oplus \widetilde{GC}(\mathbb{G})[1-t].$$

The quotient map $\overline{D_{alg\leq s}} = D_{alg\leq s}/v^{\frac{1}{n}} D_{alg\leq s}$ is equal to the quasi-isomorphism given in 4.2.17, hence is a quasi-isomorphism. Then by 4.2.15, $D_{alg\leq s}$ is a quasi-isomorphism.

Next, note that for all $s \in \mathbb{R}$, there exists $k \in \mathbb{Z}$ such that:

$$tGC^{\infty}_{alg\leq s}(\mathbb{G}') = tGC^{\infty}_{alg\leq k/n}(\mathbb{G}'),$$
$$tGC^{\infty}_{alg\leq s}(\mathbb{G}) \oplus tGC^{\infty}_{alg\leq s}(\mathbb{G})[1-t] = tGC^{\infty}_{alg\leq k/n}(\mathbb{G}) \oplus tGC^{\infty}_{alg\leq k/n}(\mathbb{G})[1-t]$$

Fix any such s and k. Then we have a commutative diagram:

$$\begin{array}{ccc} tGC^{\infty}_{alg\leq 0}(\mathbb{G}') & \xrightarrow{D_{alg\leq 0}} & tGC^{\infty}_{alg\leq 0}(\mathbb{G}) \oplus tGC^{\infty}_{alg\leq 0}(\mathbb{G})[1-t] \\ \Big\downarrow v^{-k/n} & & \Big\downarrow v^{-k/n} \\ tGC^{\infty}_{alg\leq s}(\mathbb{G}') & \xrightarrow{D_{alg\leq 0}} & tGC^{\infty}_{alg\leq s}(\mathbb{G}) \oplus tGC^{\infty}_{alg\leq s}(\mathbb{G})[1-t] \end{array}$$

Three of the four morphisms in this diagram are quasi-isomorphisms, so $D_{alg\leq s}$ is a quasi-isomorphism. □

What we have really demonstrated above is a natural isomorphism between the filtrations on $tGH^{\infty}(\mathbb{G}')$ and $tGH^{\infty}(\mathbb{G})\oplus tGH^{\infty}(\mathbb{G})[1-t]$. If $\mathcal{W}$ is a 2 dimensional, graded $\mathbb{F}$-space, with one basis element in grading level 0 and one in grading level $1-t$, this natural isomorphism becomes:

$$tGH^{\infty}(\mathbb{G}') \cong_{filt.} tGH^{\infty}(\mathbb{G}) \otimes \mathcal{W}.$$

Thus, we have the following invariance theorem.

Theorem 4.2.19. Suppose $\mathbb{G}$ and $\mathbb{G}'$ are grid diagrams with the same link type. If $\mathbb{G}'$ has rank larger than or equal to that of $\mathbb{G}$, then there is $n \in \mathbb{N}$ for which we have isomorphisms:

$$tGH^{\infty}(\mathbb{G}') \cong tGH^{\infty}(\mathbb{G}) \otimes \mathcal{W}^{\otimes n},$$

$$tGH^{\infty}_{alg\leq s}(\mathbb{G}') \cong tGH^{\infty}_{alg\leq s}(\mathbb{G}) \otimes \mathcal{W}^{\otimes n}.$$

Moreover, these isomorphisms commtue with the inclusion maps corresponding to this filtration.

4.3 Filtered Homology Theories

In this section, we construct $\mathbb{Z}^2$-filtered chain complexes associated to grid diagrams. We have assumed the reader is familiar with these, if not, the basic theory is covered in A.4. In particular, A.4 outlines how to pass between a $\mathbb{Z}^2$-filtration and a pair of $\mathbb{Z}$-filtrations.

4.3.1 $\mathbb{Z}^2$-filtered $\mathcal{GC}^{\infty}(\mathbb{G})$

In this and the subsequent section, we construct some $\mathbb{Z}^2$-filtered chain complexes. Topological invariance will not be discussed, as these chain complexes will be used to construct Υ and Υ^2 by studying their relationships with other chain complexes, whose invariance is already well understood. As usual, fix a grid diagram $\mathbb{G}$ of rank n with markings $\mathbb{O} = \{O_1, \ldots, O_n\}$ and $\mathbb{X} = \{X_1, \ldots, X_n\}$. If ψ is a domain in $\mathbb{G}$, let $O_i(\psi)$ denote the algebraic intersection number of O_i and ψ. Then *multi-variable* $\mathbb{Z}^2$-filtered grid homology is specified by the following.

Definition 4.3.1. Let $\mathcal{GC}^{\infty}(\mathbb{G})$ be the free $\mathcal{R}^{\infty}_n$-module generated over $S(\mathbb{G})$, equipped with the following.

- An endomorphism $\partial^\infty : \mathcal{GC}^\infty(\mathbb{G}) \to \mathcal{GC}^\infty(\mathbb{G})$ given on grid states $x \in S(\mathbb{G})$ by:

$$\partial^\infty(x) = \sum_{y \in S(\mathbb{G})} \sum_{r \in Rect^o(x,y)} V_1^{O_1(r)} \dots V_n^{O_n(r)} y.$$

- Grading $M(V_1^{k_1} \dots V_n^{k_n} x) = M_{\mathbb{O}}(x) - 2\sum_{i=1^n} k_i$
- A pair of $\mathbb{Z}$-filtrations $Alex$ and alg given by, for $x \in S(\mathbb{G})$:

$$Alex(V_1^{k_1} \dots V_n^{k_n} x) = Alex(x) - \sum_{i=1}^{n} k_i,$$

$$alg(V_1^{k_1} \dots V_n^{k_n} x) = -\sum_{i=1}^{n} k_i.$$

These induce a $\mathbb{Z}^2$-filtration:

$$(\{\mathcal{F}_i^{alg}\}_{i \in \mathbb{Z}}, \{\mathcal{F}_j^{Alex}\}_{j \in \mathbb{Z}}).$$

Theorem 4.3.1. The tuple:

$$(\mathcal{GC}^\infty(\mathbb{G}), M, (\{\mathcal{F}_i^{alg}\}_{i \in \mathbb{Z}}, \{\mathcal{F}_j^{Alex}\}_{j \in \mathbb{Z}})),$$

forms a $\mathbb{Z}^2$-filtered complex.

Proof. We begin by observing that, if ∂^- is the boundary map of $\mathcal{GC}^-$ in [27, §13.2], then $\partial^\infty = \partial^- \otimes_{\mathcal{R}_n^-} 1_{\mathcal{R}_n^\infty}$. Abbreviate this as $\partial^\infty = \partial^- \otimes 1$. Noting that $\partial^- \circ \partial^- = 0$, we must have $\partial^\infty \circ \partial^\infty = 0$. Similarly, ∂^- drops M by 1, and 1 preserves the grading, so ∂^∞ drops the grading by 1. Moreover, both ∂^- and 1 are filtered with respect to $Alex$, so ∂^∞ is $Alex$-filtered. Finally, if $x, y \in S(\mathbb{G})$ and $r \in Rect^o(x,y)$, then $alg(x) = 0$ and $alg(V_1^{O_1(r)} \dots V_n^{O_n(r)} y) = -|\mathbb{O} \cap r|$, so ∂^∞ is also alg-filtered, and the result follows. □

Finally, we recover $\mathbb{Z}^2$-filtered $\mathcal{GC}^-$.

Definition 4.3.2. Fix a grid diagram $\mathbb{G}$. Then:

$$\mathcal{GC}^-(\mathbb{G}) := \mathcal{F}_0^{alg}(\mathcal{GC}^\infty(\mathbb{G})).$$

4.3.2 Single Variable $\mathcal{F}_t$-filtered $\mathcal{GC}^\infty$

To ensure our chain complex interacts correctly with t-modified grid complexes, we are interested in the following specialisation of $\mathcal{GC}^\infty$.

Definition 4.3.3. Fix $t \in \mathbb{Q} \cap [0,2]$. Let $\mathcal{GC}_{sv}^\infty(\mathbb{G}) = \frac{\mathcal{GC}^\infty(\mathbb{G})}{V_1 = V_2 = \dots = V_n}$, and replace its $\mathbb{Z}^2$filtration with an $\mathbb{R}$-filtration given on filtered basis elements by:

$$\mathcal{F}_t(x) = \frac{t}{2} Alex(x) + \left(1 - \frac{t}{2}\right) alg(x).$$

We call this *single variable $\mathcal{F}_t$-filtered* grid homology.

Theorem 4.3.2. The tuple:

$$(\mathcal{GC}^{\infty}_{sv}(\mathbb{G}), \partial^{\infty}_{sv}, \mathcal{F}_t),$$

is an $\mathbb{R}$-filtered chain complex.

Proof. We have already checked that $\partial^{\infty} \circ \partial^{\infty} = 0$, so this passes to the quotient and $\partial^{\infty}_{sv} \circ \partial^{\infty}_{sv} = 0$. We have also checked that ∂^{∞} drops the grading by 1, this also passes to the quotient so ∂^{∞}_{sv} drops the grading by 1. Finally, ∂^{∞}_{sv} is clearly filtered with respect to $Alex$ and alg, because ∂^{∞} is, so noting that $\mathcal{F}_t$ is a convex combination of these two filtrations, ∂^{∞}_{sv} is $\mathcal{F}_t$-filtered. □

As with $\mathcal{GC}^{\infty}$, topological invariance properties of $\mathcal{GC}^{\infty}_{sv}$ are not relevant to our computations of Υ and Υ^2, so we omit their discussion.

4.3.3 $\mathcal{CFK}^{\infty}(K)$

In [3], Ozsváth and Szabó construct a $\mathbb{Z}^2$-filtered chain complex $CFK^{\infty}(K)$ by using a doubly marked Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ which represents a knot K. The construction of this chain complex is very long, so for more details see [3]. Fortunately for us, $CFK^{\infty}(K)$ is a *formal knot complex*. Sano and Sato have a discussion of formal knot complexes in [12].

Definition 4.3.4. We call a tuple:

$$(C, \partial, \{C_n\}_{n\in\mathbb{Z}}, \{\mathcal{F}^{Alex}_j\}_{j\in\mathbb{Z}}, \{\mathcal{F}^{alg}_i\}_{i\in\mathbb{Z}}),$$

a *formal knot complex*, if it satisfies the following.

1. C is a chain complex over Λ with decomposition $C = \bigoplus_n C_n$. The grading of a homogeneous element x is denoted $M(x)$ and is called the *Maslov grading* of x.

2. C has a $\mathbb{Z}$-filtration $\{\mathcal{F}^{Alex}_j\}_{j\in\mathbb{Z}}$ called the *Alexander filtration*. The filtration level of an element is denoted $Alex(x)$.

3. C has a $\mathbb{Z}$-filtration $\{\mathcal{F}^{alg}_i\}_{i\in\mathbb{Z}}$ called the *algebraic filtration*. The filtration level of an element x is denoted by $alg(x)$. When we regard C as a $\mathbb{Z}^2$-filtered complex, we use the $\mathbb{Z}^2$ filtration induced by $\Big(\{\mathcal{F}^{alg}_i\}_{i\in\mathbb{Z}}, \{\mathcal{F}^{Alex}_j\}_{j\in\mathbb{Z}}\Big)$.

4. The action of U lowers M by 2 and both $Alex$ and alg by 1.

5. C is a free module with finite rank, and there exists a basis $\{x_k\}_{1\leq k\leq r}$ such that:

   - each x_k is homogeneous with respect to the Maslov grading.
   - $\mathcal{F}^{Alex}_0$ is a free $\mathbb{F}[U]$-module with a basis $\{U^{Alex(x_k)}x_k\}_{1\leq k\leq r}$.
   - $\mathcal{F}^{alg}_0$ is a free $\mathbb{F}[U]$-module with a basis $\{U^{alg(x_k)}x_k\}_{1\leq k\leq r}$.

   We call $\{x_k\}_{1\leq k\leq r}$ a *filtered basis.*

6. There exists a $\mathbb{Z}^2$-filtered homotopy equivalence $\iota : C \to C^r$, where C^r is the same graded chain complex as C, with filtration $\left(\{\mathcal{F}_j^{Alex}\}_{j\in\mathbb{Z}}, \{\mathcal{F}_i^{alg}\}_{i\in\mathbb{Z}}\right)$.

7. Regard Λ as a chain complex with trivial boundary map, and define a homological grading by:
$$\Lambda_n = \begin{cases} \{0, U^{-n/2}\} & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$$
and $\mathcal{F}_i^{alg}(\Lambda) = \mathcal{F}_i^{Alex}(\Lambda) = U^{-i} \cdot \mathbb{F}[U]$. Then there is a $\mathbb{Z}$ filtered chain homotopy equivalence map f_{Alex} (resp. f_{alg}) $f_{Alex} : C \to \Lambda$ over Λ with respect to the Alexander (resp. algebraic) filtration.

This final property is called *global triviality.*

Theorem 4.3.3. If K is a knot, then $CFK^\infty(K)$ is a formal knot complex.

Proof. The tricky part is to check that global triviality holds. Loosely, we prove this by first noting that $CFK^\infty(K)$, equipped only with algebraic filtration, computes $CF^\infty(S^3)$, so is alg-filtered chain homotopic to Λ. Then, we use can property 6 to see that Alex-filtered $CFK^\infty(K)$ is Alg-filtered chain homotopic to Λ. For more details, see [25, §2.1]. □

One subcomplex of $CFK^\infty(K)$ which we are particularly interested in is $\mathcal{F}_0^{alg}(CFK^\infty(K))$. We denote this $CFK^-(K)$. This subcomplex is related to grid homology by the following theorem.

Theorem 4.3.4. If $\mathbb{G}$ is a grid diagram which represents a knot K, then there is a $\mathbb{Z}^2$-filtered chain homotopy equivalence:
$$\mathcal{GC}^-(\mathbb{G}) \simeq CFK^-(K).$$

Proof. See [12, Corollary 4.2]. □

Chapter 5

Υ-Invariant

In this chapter, we develop three different ways of calculating $\Upsilon_K(t)$ for some knot K. The existing methods have used a variety of rings, so, for convenience, we again list out our notations for different rings.

Ring	Notation
$\mathbb{F}[U, U^{-1}]$	Λ
$\mathbb{F}[U]$	Λ^-
$\mathbb{F}[V_1, V_1^{-1}, \ldots, V_n, V_n^{-1}]$	$\mathcal{R}_n^\infty$
$\mathbb{F}[V_1, \ldots, V_n]$	$\mathcal{R}_n^-$
$\mathbb{F}[v^{1/n}, v^{-1/n}]$	$\mathcal{S}_{1/n}$
$\mathbb{F}[v^{1/n}]$	$\mathcal{S}_{1/n}^-$
$\mathbb{F}[v^t, v^{2-t}]$	$\mathcal{S}_t^-$

5.1 t-Modified Construction

We start by discussing torsion in $\mathcal{S}_t^-$-modules, where $\mathcal{S}_t^-$ is the ring used to define $tGC^\infty(\mathcal{G})$.

Definition 5.1.1. Let M be an $\mathcal{S}_t^-$-module. We say an element $x \in M$ is *torsion* if there exists $p \in \mathcal{R}$ such that $p \cdot x = 0$. If no such p exists, then we say x is *non-torsion*.

Because all of our rings are commutative integral domains, we can collect torsion elements into a submodule.

Definition 5.1.2. Let M be an $\mathcal{S}_t^-$-module. Then $Tor(M)$ is the submodule of M consisting of torsion elements. We call $Tor(M)$ the *torsion submodule* of M.

In the case that M is a Maslov graded $\mathcal{S}_t^-$-module (and the action $\mathcal{S}_t^-$ interacts with the grading correctly), a stronger condition governs torsion for homogeneous elements.

Lemma 5.1.1. Let M be an $\mathcal{S}_t^-$-module with a grading such that, if $x \in M$ is homogeneous:

$$gr(v^\alpha x) = gr(x) - \alpha.$$

Then a homogeneous element x of M is torsion if and only if there is a monomial $v^\alpha \in \mathcal{S}_t^-$ such that $v^\alpha \cdot x = 0$.

Proof. One direction is trivial for this lemma. Suppose $x \in M$ is homogeneous and torsion. Then there is some $p = v^{\alpha_1} + \cdots + v^{\alpha_k} \in \mathcal{S}_t^-$ such that $p \cdot x = 0$. Given we are working over $\mathbb{F}_2$, we may assume all the α_i's are distinct. Then clearly for each $i \neq j$, we have:

$$gr(v^{\alpha_i}x) \neq gr(v^{\alpha_j}x).$$

Noting that M splits as $\bigoplus_{d\in\mathbb{Z}} M_d$, the equation:

$$(v^{\alpha_1} + \cdots + v^{\alpha_k})x = 0$$

Implies that $v^{\alpha_1}x = 0$. Thus the result follows. □

It then makes sense to give the following definition.

Definition 5.1.3. Fix a grid diagram $\mathbb{G}$. A homogeneous element $x \in tGH^-\mathbb{G}$ is called *torsion* if there is some $v^\alpha \in \mathcal{S}_t^-$ such that $v^\alpha \cdot x = 0$. If no such v^α exists then we say x is nontorsion.

We use torsion to define the Upsilon invariant.

Definition 5.1.4. Let $\mathbb{G}$ be a grid diagram. Define $\Upsilon_{\mathbb{G}}(t)$ by:

$$\Upsilon_{\mathbb{G}}(t) = \max\{gr_t(x) : x \in tGH^-(\mathbb{G}) \text{ is homogeneous and non-torsion}\}$$

5.1.1 Invariance

Given $tGH^-(\mathbb{G})$ is invariant under commutation moves, we have the following.

Theorem 5.1.2. Let $\mathbb{G}$ and $\mathbb{G}'$ be grid diagrams which differ by commutation move. Then for each $t \in [0, 2]$, $\Upsilon_{\mathbb{G}}(t) = \Upsilon_{\mathbb{G}'}(t)$.

For stabilisation, recall that:

$$tGH^-(\mathbb{G}') \cong tGH^-(\mathbb{G}) \oplus tGH^-(\mathbb{G})[1-t],$$

where $\mathbb{G}'$ is a stabilisation of $\mathbb{G}$. This gives us the following:

Theorem 5.1.3. Let $\mathbb{G}'$ be a stabilisation of $\mathbb{G}$. Then for $t \in [1, 2]$, we have $\Upsilon_{\mathbb{G}'}(t) = \Upsilon_{\mathbb{G}}(t)$.

Proof. By the theorem above we have:

$$tGH^-(\mathbb{G}') \cong tGH^-(\mathbb{G}) \oplus tGH^-(\mathbb{G})[1-t].$$

Clearly, if the maximum grading of a homogeneous, nontorsion element of $tGH^-(\mathbb{G})$ is d, the corresponding grading in $tGH^-(\mathbb{G})[1-t]$ is $d+1-t$. For $t \in [1, 2]$, we have $d+1-t \leq d$, so the result follows. □

Fortunately, $\Upsilon_K(t) = \Upsilon_K(2-t)$(see [20, Proposition 1.2]), so to define our knot invariant, we take a grid representative $\mathbb{G}$ of a knot K and let $\Upsilon_K|_{[1,2]} = \Upsilon_{\mathbb{G}}|_{[1,2]}$, and then extend via $\Upsilon_K(t) = \Upsilon_K(2-t)$.

5.2 Rational t-Modified Construction

In [22] the Livingston uses $\mathcal{S}^-_{1/n}$ as his ring for t-modified homology to check that his construction of Υ_K is equivalent to that of [20]. This chain complex was originally constructed by Ozsváth, Stipsicz, and Szabó in [20], as a means to define the chain complex over a simpler ring than the ring of long power series. We call its homology rational t-modified Heegaard Floer Knot homology. To do this with grid diagrams we use $tGC^\infty(\mathbb{G})$, as its filtration levels are $\mathcal{S}^-_{1/n}$-modules.

Definition 5.2.1. Let $t \in \mathbb{Q} \cap [0, 2]$. If $t \geq 1$, then define:

$$\Upsilon_{\mathbb{G}}(t) = \max\{gr_t(x) : x \in tGH^\infty_{alg\leq 0}(\mathbb{G}) \text{ is homogeneous and nontorsion}\}.$$

If $t \leq 1$, then define $\Upsilon_{\mathbb{G}}(t) = \Upsilon_{\mathbb{G}}(2 - t)$.

Noting the manner in which the isomorphism type of $tGH^\infty(\mathbb{G})$ changes under grid moves, this is clearly a knot invariant. Noting that $tGC^\infty_{alg\leq 0}(\mathbb{G}) = tGC^-(\mathbb{G}) \otimes_{\mathcal{R}^-_t} \mathcal{S}^-_{1/n}$, adapting the proof of [20, Proposition 4.9] implies this definition of the Upsilon invariant is equivalent to the previous definition.

Proposition 5.2.1. Let C be a finitely generated chain complex over $\mathcal{R}^-_t$ and consider the induced chain complex $C \otimes_{\mathcal{R}^-_t} \mathcal{S}^-_{1/n}$. Then the maximal grading of a homogeneous-nontorsion element of $H(C)$ agrees with that of $H(C \otimes_{\mathcal{R}^-_t} \mathcal{S}^-_{1/n})$.

Proof. We begin by noting that $\mathcal{S}^-_{1/n}$ is a free $\mathcal{R}^-_t$-module, treating $\mathcal{R}^-_t$ as a subring of $\mathcal{S}^-_{1/n}$. Next, suppose that M is an $\mathcal{R}_t$-module. Then we have a short exact sequence:

$$0 \longrightarrow Tors(M) \longrightarrow M \longrightarrow M/Tors(M) \longrightarrow 0$$

Noting that $\mathcal{S}^-_{1/n}$ is free, we can take tensor products to form the short exact sequence:

$$0 \longrightarrow Tors(M) \otimes \mathcal{S}^-_{1/n} \xrightarrow{i} M \otimes \mathcal{S}^-_{1/n} \xrightarrow{q} (M/Tors(M)) \otimes \mathcal{S}^-_{1/n} \longrightarrow 0$$

Next, suppose $\sum_{i=1}^n x_i \otimes p_i$ is an element of $Tors(M) \otimes \mathcal{S}^-_{1/n}$. Then for each i there is $\lambda_i \in \mathcal{R}_t$ for which $\lambda_i x_i = 0$. Noting that $\mathcal{R}^-_t$ is commutative, if $\lambda = \prod_{i=1}^n \lambda_i$, then $\lambda x_i = 0$. Thus $\lambda \sum_{i=1}^n x_i \otimes p_i = 0$, so $\sum_{i=1}^n x_i \otimes p_i \in Tors(M \otimes \mathcal{S}^-_{1/n})$, so $Tors(M) \otimes \mathcal{S}^-_{1/n} \subset Tors(M \otimes \mathcal{S}^-_{1/n})$. Thus Im$i \subset Tors(M \otimes \mathcal{S}^-_{1/n})$. By exactness, we have $\ker q \subset Tors(M \otimes \mathcal{S}^-_{1/n})$. Moreover, the $\mathcal{R}^-_t$-module $(M/Tors(M)) \otimes \mathcal{S}^-_{1/n}$ is torsion-free, so $Tors(M \otimes \mathcal{S}^-_{1/n}) \subset \ker q$. Thus $Tors(M \otimes \mathcal{S}^-_{1/n}) = Tors(M) \otimes \mathcal{S}^-_{1/n}$. Then:

$$\begin{aligned}(M \otimes \mathcal{S}^-_{1/n})/Tors(M \otimes \mathcal{S}^-_{1/n}) &= (M \otimes \mathcal{S}^-_{1/n})/(Tors(M) \otimes \mathcal{S}^-_{1/n}) \\ &= (M/Tors(M)) \otimes \mathcal{S}^-_{1/n}.\end{aligned}$$

$\mathcal{S}^-_{1/n}$ is a torsion free module over $\mathcal{R}^-_t$, so by the universal coefficient theorem:

$$H(C \otimes \mathcal{S}^-_{1/n}) \cong H(C) \otimes \mathcal{S}^-_{1/n}.$$

Hence:

$$H(C\otimes\mathcal{S}^-_{1/n})/Tors(H(C\otimes\mathcal{S}^-_{1/n}))\cong(H(C)/Tors(H(C)))\otimes\mathcal{S}^-_{1/n}. \tag{5.1}$$

Elements of $\mathcal{S}^-_{1/n}$ drop the grading or preserve it, so the maximal grading of a homogeneous element of $H(C)/Tors(H(C))$ equals that of $(H(C)/Tors(H(C)))\otimes\mathcal{S}^-_{1/n}$. Applying the isomorphism from 5.1, we see that the maximal grading of a homogeneous element of $(H(C)/Tors(H(C)))\otimes\mathcal{S}^-_{1/n}$ equals that of $H(C\otimes\mathcal{S}^-_{1/n})/Tors(H(C\otimes\mathcal{S}^-_{1/n}))$, so the result follows. □

5.3 Construction from $\mathcal{GC}^\infty_{sv}(\mathbb{G})$

In this section, we perform the same construction as in [22], but insead of working with a double pointed Heegaard diagram, we work with grid diagrams. Fix some $t\in[0,2]$. We will deviate slightly here from the notation for filtered complexes in the appendix, to line up with the notation from [22]. So filtrations level s will be denoted:

$$(\mathcal{GC}^\infty_{sv}(\mathbb{G}),\mathcal{F}_t)_s,$$

Maslov grading level m will be denoted:

$$\mathcal{GC}^\infty_{sv,m}(\mathbb{G}),$$

and their intersection is denoted:

$$(\mathcal{GC}^\infty_{sv,m}(\mathbb{G}),\mathcal{F}_t)_s$$

Fix a grid diagram $\mathbb{G}$ of rank n. Recall that, for each $t\in[0,1]$, $(\mathcal{GC}^\infty_{sv}(\mathbb{G}),\partial^\infty,M,\mathcal{F}_t)$ is a discrete $\mathbb{R}$-filtered, $\mathbb{Z}$-graded chain complex. Associated to $(\mathcal{GC}^\infty_{sv}(\mathbb{G}),\partial^\infty,M,\mathcal{F}_t)$, we have a subset of $\mathbb{Z}\times\mathbb{Z}$ which is the image of the function:

$$\mathcal{GC}^\infty_{sv}(\mathbb{G})\to\mathbb{R}\times\mathbb{R}$$

$$x\mapsto(Alg(x),Alex(x))$$

Denote this subset $L_{\mathbb{G}}$.

Definition 5.3.1. Let s be the smallest number for which the following inclusion is nonzero on homology:

$$(\mathcal{GC}^\infty_{sv,0}(\mathbb{G}),\mathcal{F}_t)_s\oplus(\mathcal{GC}^\infty_{sv,1}(\mathbb{G}),\mathcal{F}_t)_{s+\frac{1}{2}}\longrightarrow\mathcal{GC}^\infty_{sv,0}(\mathbb{G})\oplus\mathcal{GC}^\infty_{sv,1}(\mathbb{G})$$

Then we define $\gamma_{\mathbb{G}}(t)=s$ and $\Upsilon_{\mathbb{G}}(t)=-2s$.

This definition can be made less opaque by studying a particular subset of $L_{\mathbb{G}}$.

Definition 5.3.2. Let $L_0\subset L_{\mathbb{G}}$ consist of the filtration level of nontrivial cycles in grading 0, and L_1 consist of filtration levels of nontrivial cycles in grading 1, shifted down and to the left by $\frac{1}{2}$. Let $L=L_0\cup L_1$.

Consider the line $\mathcal{L}_{t,s}$ given in (x, y)-coordinates by the following:

$$\left(1 - \frac{t}{2}\right) x + \frac{t}{2} y = s$$

Thus the map in 5.3.1 is nonzero precisely when there is a point of L in the half plane down and to the right of $\mathcal{L}_{t,s}$. In particular, $\mathcal{L}_{t,\gamma_{\mathbb{G}}(t)}$ contains no points from L in the interior of the corresponding half plane, and at least one point from L lies on $\mathcal{L}_{t,\gamma_{\mathbb{G}}(t)}$. Noting that there are finitely many lines which intersect more than one point of L (at most $\binom{|L|}{2}$), for all but finitely many values of $t \in [0, 2]$, $\mathcal{L}_{t,\gamma_{\mathbb{G}}(t)}$ intersects precisely one point.

Definition 5.3.3. Denote the points which $\mathcal{L}_{t,\gamma_{\mathbb{G}}(t)}$ passes through by P_t. If P_t contains more than one point of L, we say $t \in [0, 2]$ is a *pivot point* of $\Upsilon_{\mathbb{G}}(t)$.

Clearly, if we choose $\delta > 0$ to be smaller than the distance between any pair of pivot points, then at a pivot point $t \in [0, 2]$, both $P_{t+\delta}$ and $P_{t-\delta}$ must be singletons.

Definition 5.3.4. If t is a pivot point, then we call $p_t^+ \in P_{t+\delta}$ and $p_t^- \in P_{t-\delta}$ the *positive* and *negative pivots* at t, respectively.

Now suppose that P_t is a singleton. Then for sufficiently small $\delta > 0$, all P_x are equal, for $|t - x| < \delta$. Pick some $t' \in (x - \delta, x + \delta) \setminus \{t\}$. Then $P_t = P_{t'}$, so if $P_t = \{(x, y)\}$, then we have equations:

$$y = \frac{2}{t}\gamma_{\mathbb{G}}(t) + \left(1 - \frac{2}{t}\right) x,$$

$$y = \frac{2}{t'}\gamma_{\mathbb{G}}(t) + \left(1 - \frac{2}{t'}\right) x.$$

Hence:

$$\gamma_{\mathbb{G}}(t) - \gamma_{\mathbb{G}}(t') = \left(\frac{t}{2} - \frac{t'}{2}\right)(y - x)$$

So we have the following local equation for $\Upsilon_{\mathbb{G}}$:

$$\Upsilon_{\mathbb{G}}(t) - \Upsilon_{\mathbb{G}}(t') = (t - t')(x - y)$$

So away from points where $|P_t| > 1$, $\Upsilon_{\mathbb{G}}$ is linear. If $t \in [0, 2]$ is a pivot point, then the calculation above yields:

$$\Upsilon_{\mathbb{G}}(t) - \Upsilon_{\mathbb{G}}(t^+) = (t - t^+)(x - y),$$

$$\Upsilon_{\mathbb{G}}(t^-) - \Upsilon_{\mathbb{G}}(t) = (t^- - t)(x' - y'),$$

where $p_t^+ = (x, y)$ and $p_t^- = (x', y')$. So clearly if $x - y \neq x' - y'$, then $\Upsilon_{\mathbb{G}}$ has a singularity at t. Clearly, then, $\Upsilon_{\mathbb{G}}$ is piecewise linear, so all we need now is to compute $\Upsilon_{\mathbb{G}}$ in terms of lattice points is $\Delta\Upsilon'_{\mathbb{G}}(t)$. Clearly if t is not a pivot point, $\Delta\Upsilon'_{\mathbb{G}}(t) = 0$. If t is a pivot point, then setting:

$$p_t^+ = (i, j),$$

$$p_t^- = (x, y).$$

We have:

$$\Upsilon'_{\mathbb{G}}(t^-) = (x-y),$$
$$\Upsilon_{\mathbb{G}}(t^+) = (i-j).$$

So:

$$\Delta\Upsilon'_{\mathbb{G}}(t) = (y-j) - (x-i).$$

The points (i,j) and (x,y) both lie on the line $\mathcal{L}_{t,\gamma(t)}$,so:

$$y = \frac{2}{t}\gamma(t) + \left(1 - \frac{2}{t}\right)x$$

$$j = \frac{2}{t}\gamma(t) + \left(1 - \frac{2}{t}\right)i$$

So:

$$\begin{aligned} \Delta'_{\mathbb{G}}(t) &= \left(\left(1-\frac{2}{t}\right)x - \left(1-\frac{2}{t}\right)i\right) - (x-i) \\ &= \frac{2}{t}(i-x) \\ \Rightarrow |\Delta\Upsilon'_{\mathbb{G}}(t)| &= \frac{2}{t}|i-x| \end{aligned}$$

So we have proven the following.

Theorem 5.3.1. $\Upsilon_{\mathbb{G}}$ satisfies the following conditions:

1. $\Upsilon_{\mathbb{G}}$ is piecewise linear.
2. At nonsingular points of $\Upsilon_{\mathbb{G}}$, $|\Upsilon'_{\mathbb{G}}(t)| = |i-j|$ where (i,j) is some point in $L_{0,1}$.
3. Singularities only occur at $t \in [0,2]$ for which $|P_t| > 1$.
4. At singularities of $\Upsilon_{\mathbb{G}}$, $|\Delta\Upsilon'_{\mathbb{G}}(t)| = \frac{2}{t}|i-i'|$, where i is the x-coordinate of p_t^+, and i' is that of p_t^-.

This gives us a straightforward (although not particularly efficient) way of computing $\Upsilon_{\mathbb{G}}(t)$ from the set L.

Start by computing L. Let P_0 consist of all elements of L with minimal x-coordinate. This x-coordinate is $\gamma_{\mathbb{G}}(0)$. Let $p_0 \in P_0$ be the element with minimal y coordinate. Then:

$$\Upsilon'_{\mathbb{G}}(0) = x(p_0) - y(p_0).$$

This defines the function up until the first singularity. Noting that the slope, m of any $\mathcal{L}_{t,s}$ is equal to $1 - \frac{t}{2}$, we have: Rotate the line $\mathcal{L}_{0,\gamma(0)}$ about p_0 counterclockwise until it hits a second point p_1. Then the slope of this line is given by:

$$m = \frac{y(p_1) - y(p_0)}{x(p_1) - x(p_0)},$$

so:

$$t = 2\left(1 - \frac{y(p_1) - y(p_0)}{x(p_1) - x(p_0)}\right)$$

Finally $\Delta\Upsilon'_{\mathbb{G}}(t) = \frac{2}{t}(x(p_1) - x(p_0))$. We can repeat this process, of calculating slope and rotating $\mathcal{L}_{t,\gamma(t)}$ to find the next pivot point, to fully compute $\Upsilon_{\mathbb{G}}$.

5.4 Equivalence of Definitions

Fix a grid diagram $\mathbb{G}$. Recall we have the chain complexes:

$$(\mathcal{GC}^{\infty}_{sv}(\mathbb{G})) \text{ and } tGC^{\infty}(\mathbb{G})$$

Where:

$$gr_t(x) = M(x) - t(Alex(x) - Alg(x))$$

$$\mathcal{F}_t(x) = \frac{t}{2}Alex(x) + \left(1 - \frac{t}{2}\right)Alg(x)$$

To start with, we construct a few equivalent definitions of $\Upsilon_{\mathbb{G}}(t)$ from $tGC^{\infty}(\mathbb{G})$.

Lemma 5.4.1. $\Upsilon_{\mathbb{G}}(t)$ is equal to the largest grading of an element of $tGH^{\infty}_{alg\leq 0}(\mathbb{G})$ which maps to a nontrivial element in tGH^{∞}.

Proof. Let $[x]$ be a homogeneous, non torsion element of $tGH^{\infty}_{alg\leq 0}(\mathbb{G})$ with grading equal to $\Upsilon_{\mathbb{G}}(t)$. Then for each $v^k \in \mathcal{S}^-_{1/n}$:

$$v^k[x] = [v^k x] \neq 0$$

Now suppose $[y] \in tGH^{\infty}_{alg\leq 0}(\mathbb{G})$ maps to a nontrivial element of $tGH^{\infty}(\mathbb{G})$ and has maximal grading. Suppose $v^k[y] = 0$. Then $[v^k y] = 0$, so $V^k y$ is a boundary of ∂^{∞}_t. But $v^{-k}(Im\partial^{\infty}_t) = Im\partial^{\infty}_t$, so this implies that y is a boundary. Thus we have a contradiction so $[y]$ cannot be torsion. Now suppose $[x]$ maps to a trivial element of $tGH^{\infty}(\mathbb{G})$. Then x is a boundary of ∂^{∞}_t. Given ∂^{∞}_t commutes with v^k for all $v^k \in \mathcal{S}^-_{1/n}$, this implies all $v^k x$ are boundaries of ∂^{∞}_t. Let $\partial^{\infty}_t(y) = x$ and $k = -Alg(y)$. Then $v^k y \in tGC^{\infty}_{alg\leq 0}(\mathbb{G})$, so $[v^k x] = v^k[x] = 0$. Thus we have a contradiction, so the two definitions agree. □

Lemma 5.4.2. $\Upsilon_{\mathbb{G}}(t) = -2s$, where s is the smallest number for which $tGH^{\infty}_{Alg\leq s}(\mathbb{G})$ has an element which is nontrivial in $tGH^{\infty}(\mathbb{G})$ and has grading 0.

Proof. Let $\Upsilon_{\mathbb{G}}(t)$ be defined as per 5.4.1, and choose s as given in the lemma statement. Let $[y] \in tGH^{\infty}_{Alg\leq s}(\mathbb{G})$ be nontrivial in $tGH^{\infty}(\mathbb{G})$ and $[x] \in tGH^{-}(\mathbb{G})$ be nontrivial in $tGH^{\infty}(\mathbb{G})$, and have grading $\Upsilon_{\mathbb{G}}(t)$. Then $[v^{2s}y] \in tGH^{-}(\mathbb{G})$ is nontrivial in $tGH^{\infty}(\mathbb{G})$ and has grading $-2s$. Similarly, $[v^{\Upsilon_{\mathbb{G}}(t)}x] \in tGH^{\infty}_{Alg\leq\frac{-\Upsilon_{\mathbb{G}}(t)}{2}}(\mathbb{G})$. Clearly the two definitions agree by their respective maximality and minimality. □

Now we have a definition of $\Upsilon_{\mathbb{G}}(t)$ which we can relate to that from $H(\mathcal{GC}^{\infty}_{sv}(\mathbb{G}))$. Let Λ act on tGC^{∞} by setting $U = v^2$.

Theorem 5.4.3. Let $\Upsilon_{\mathbb{G}}(t)$ be defined as in 5.4.2. If:

$s := \min\{k : H_0((\mathcal{GC}^{\infty}_{sv}(\mathbb{G}), \mathcal{F}_t)_s) \oplus H_1((\mathcal{GC}^{\infty}_{sv}(\mathbb{G}), \mathcal{F}_t)_{s+\frac{1}{2}}) \to H_0(\mathcal{GC}^{\infty}_{sv}(\mathbb{G})) \oplus H_1(\mathcal{GC}^{\infty}_{sv}(\mathbb{G}))$ is nonzero$\}$

Then $\Upsilon_{\mathbb{G}}(t) = -2s$.

Proof. To begin with, we should describe the relationship between $\mathcal{GC}^{\infty}_{sv,0}(\mathbb{G}) \oplus \mathcal{GC}_{sv,1}(\mathbb{G})$ and $tGC^{\infty}_0(\mathbb{G})$. Partition grid states $S = S(\mathbb{G})$ as $S = E \cup O$, where:

$$E = \{x \in S : M(x) \text{ is even}\}$$

$$O = \{x \in S : M(x) \text{ is odd}\}$$

Thus we have the following filtered bases for $\mathcal{GC}^{\infty}_{sv,0}(\mathbb{G})$ and $\mathcal{GC}^{\infty}_{sv,1}(\mathbb{G})$:

$$B_0 = \{U^{M(x)/2}x : x \in E\}$$

$$B_1 = \{U^{(M(x)-1)/2}x : x \in O\}$$

Similarly $tGC^{\infty}_0(\mathbb{G})$ has a filtered basis:

$$F = \{v^{gr_t(x)}x : x \in S\}$$

Noting that, if $x \in S$ is at filtration level (i, j), we have $gr_t(x) = M(x) - t(j - i)$, so one of the following holds:

$$\text{If } M(x) \text{ is even then: } v^{-t(j-i)}v^{gr_t(x)}x = U^{M(x)/2}x$$

$$\text{If } M(x) \text{ is odd then: } v^{1-t(j-i)}v^{gr_t(x)}x = U^{(M(x)-1)/2}x$$

So we have a bijective correspondence between the two bases. This induces a graded Λ-homomorphism:

$$T : \mathcal{GC}^{\infty}_{sv}(\mathbb{G}) \oplus \mathcal{GC}^{\infty}_{sv}(\mathbb{G})[-1] \to tGC^{\infty}(\mathbb{G}).$$

Which we will analyze. From here, to simplify equations, a filtered basis element y will be said to be at filtration level (i_y, j_y). Fix an element $x \in B_0 \cup B_1$ at filtration level (i, j) of $\mathcal{GC}^{\infty}_{0,1}(\mathbb{G})$. Recall that if we write $\partial^{\infty}(x) = \sum_l y_l$, then $\partial^{\infty}_t(x) = \sum_l V^{\alpha_l}y_l$, where $\alpha_l = t((j - j_y) - (i - i_y))$. If $x \in B_0$:

$$\begin{aligned}
T(\partial^{\infty}(x)) &= T\left(\sum_l y_l\right) \\
&= \sum_l v^{-t(j_{y_l} - i_{y_l})}y_l \\
\partial^{\infty}_t(x) &= \sum_l v^{-t(j-i)}v^{t((j-j_{y_l})-(i-i_{y_l}))}y_l \\
&= \sum_l v^{-t(j_{y_l} - i_{y_l})}y_l
\end{aligned}$$

If $x \in B_1$:

$$T(\partial^\infty(x)) = v \sum_l v^{-t(j_{y_l} - i_{y_l})} y_l$$
$$\partial_t^\infty(T(x)) = v \sum_l v^{-t(j_{y_l} - i_{y_l})} y_l$$

Hence the following diagram must commute:

$$\begin{array}{ccc} \mathcal{GC}_{sv}^\infty(\mathbb{G}) \oplus \mathcal{GC}_{sv}^\infty(\mathbb{G})[-1] & \xrightarrow{\partial^\infty} & \mathcal{GC}_{sv}^\infty(\mathbb{G}) \oplus \mathcal{GC}_{sv}^\infty(\mathbb{G})[-1] \\ T \big\downarrow & & \big\downarrow T \\ tGC^\infty(\mathbb{G}) & \xrightarrow[\partial_t^\infty]{} & tGC^\infty(\mathbb{G}) \end{array}$$

If we recall $U = v^2$ and define $M = \bigcup_{n \in \mathbb{Z}} U^n tGC_0^\infty(\mathbb{G})$, then clearly T maps $\mathcal{GC}^\infty(\mathbb{G}) \oplus \mathcal{GC}^\infty(\mathbb{G})[-1]$ isomorphically onto M. In particular, on the level of homology, we have:

$$H_0(\mathcal{GC}_{sv}^\infty(\mathbb{G})) \oplus H_1(\mathcal{GC}_{sv}^\infty(\mathbb{G})) \xrightarrow{\cong} H_0(tGC^\infty(\mathbb{G}))$$

The final step in our proof is to study the effect of T upon the filtration lattices for $\mathcal{GC}_{sv}^\infty(\mathbb{G}) \oplus \mathcal{GC}_{sv}^\infty(\mathbb{G})[-1]$ and $tGC^\infty\mathbb{G}$. Start with $\mathcal{GC}_{sv}^\infty$. If $x \in B_0$ has filtration level (i, j), then clearly $T(x)$ is at filtration level $(i + \frac{1}{2}(j-i)t, j + \frac{1}{2}(j-i)t)$ So on the filtration lattice for $\mathcal{GC}_{sv}^\infty(\mathbb{G})$, T induces the transformation:

$$(i, j) \mapsto \left(\left(1 - \frac{t}{2}\right) i + \frac{t}{2} j, -\frac{t}{2} + \left(1 + \frac{t}{2}\right) j \right)$$

Note that this sends the line $y = (1 - \frac{2}{t})x + (\frac{2}{t})s$ to the line $x = s$, so we have the restriction:

$$T_0 : (\mathcal{GC}_{sv}^\infty(\mathbb{G}), \mathcal{F}_t)_s \to tGC^\infty(\mathbb{G})_{alg \le s}$$

Similarly, if $x \in B_1$ has filtration level (i, j), then $T(x)$ has filtration level:

$$\left(i + \frac{1}{2}(j-i)t - \frac{1}{2}, j + \frac{1}{2}(j-i)t - \frac{1}{2} \right)$$

Hence T induces the following transformation on the filtration lattice for $\mathcal{GC}^\infty(\mathbb{G})[-1]$:

$$(i, j) \mapsto \left(\left(1 - \frac{t}{2}\right) i + \frac{t}{2} j - \frac{1}{2}, -\frac{t}{2} + \left(1 + \frac{t}{2}\right) j - \frac{1}{2} \right)$$

This sends the line:

$$\left(1 - \frac{t}{2}\right) x + \frac{t}{2} y = s + \frac{1}{2}$$

To the line $x = s$, so we get the restriction:

$$T_1 : (\mathcal{GC}_{sv}^\infty(\mathbb{G})[-1], \mathcal{F}_t)_{s+\frac{1}{2}} \to tGC^\infty(\mathbb{G})_{Alg \le s}$$

Combining these two, we have the commutative square:

$$\begin{array}{ccc} (\mathcal{GC}^\infty_{sv}(\mathbb{G}), \mathcal{F}_t)_s \oplus (\mathcal{GC}^\infty_{sv}(\mathbb{G})[-1], \mathcal{F}_t)_{s+\frac{1}{2}} & \xrightarrow{T_1} & tGC^\infty(\mathbb{G})_{Alg\leq s} \\ \big\downarrow & & \big\downarrow \\ \mathcal{GC}^\infty_{sv}(\mathbb{G}) \oplus \mathcal{GC}^\infty_{sv}(\mathbb{G})[-1] & \xrightarrow{T_1} & tGC^\infty(\mathbb{G}) \end{array} \tag{5.2}$$

Passing down to homology, we obtain:

$$\begin{array}{ccc} H_0((\mathcal{GC}^\infty_{sv}(\mathbb{G}), \mathcal{F}_t)_s) \oplus H_1((\mathcal{GC}^\infty_{sv}(\mathbb{G}), \mathcal{F}_t)_{s+\frac{1}{2}}) & \xrightarrow{\cong} & H_0(tGC^\infty(\mathbb{G}))_{Alg\leq s} \\ {\scriptstyle i_*}\big\downarrow & & {\scriptstyle j_*}\big\downarrow \\ H_0(\mathcal{GC}^\infty_{sv}(\mathbb{G})) \oplus H_1(\mathcal{GC}^\infty_{sv}(\mathbb{G})) & \xrightarrow{\cong} & H_0(tGC^\infty(\mathbb{G})) \end{array}$$

Where i_* and j_* are induced by the corresponding inclusions in 5.2. Clearly i_* is nonzero if and only if j_* is nonzero. Thus $\Upsilon_{\mathbb{G}}(t) = -2s$. □

Corollary 5.4.3.1. $\Upsilon_{\mathbb{G}}(t)$, as defined from $\mathcal{GC}^\infty_{sv}(\mathbb{G})$, is a knot invariant.

Proof. See [21, Theorem 4.3]. □

5.5 Υ^2

As discussed in the introduction, there is also a *secondary* Upsilon invariant. In [25], Kim and Livingston define a function $\Upsilon^2_K : [0,2]^2 \to \mathbb{R} \cup \{-\infty\}$ by the following procedure.

Definition 5.5.1. Let K be a knot, and define $\gamma_K(t) = -\frac{1}{2}\Upsilon_K(t)$. Fix some $t \in [0,2]$. Recall we have positive and negative pivots, so we have sets P_t, $P_{t^\pm}$. Then, let $\mathcal{Z}^\pm$ be the set of cycles in $\mathcal{F}_{t^\pm,\gamma_K(t^\pm)}(CFK^\infty(K))$ which are nontrivial in $H_0(CFK^\infty(K))$. Then we define $\gamma^2_{K,t}(s)$ to be the smallest $r \in \mathbb{R}$ for which there exist $z^+ \in \mathcal{Z}^+$ and $z^- \in \mathcal{Z}^-$ which represent the same homology class in:

$$H_0(\mathcal{F}_{t,\gamma_K(t)}(CFK^\infty(\mathbb{G})) + \mathcal{F}_{s,r}(CFK^\infty(K))).$$

Then $\Upsilon^2_K(t,s) = -2\gamma^2_{K,t}(s) - \Upsilon_K(t)$.

This suggests the following definition of Υ^2_K from $\mathcal{GC}^\infty_{sv}(\mathbb{G})$.

Definition 5.5.2. Let K be a knot and $\mathbb{G}$ be a grid diagram representing K. Fix $t, s \in [0,2]$, and consider the set of points $L_{\mathbb{G}} \subset \mathbb{R}^2$. As above we have sets of points P_t and $P_{t^\pm}$. Let $\mathcal{Z}^\pm$ be the set of cycles which are sent to nontrivial elements on homology by the inclusion in 5.3.1 for $t^\pm$. Then $\gamma^2_{\mathbb{G},t}(s)$ is the smallest $r \in \mathbb{R}$ for which there exist $z^\pm \in Z^\pm$ so that z^- has the same homology class as z^+ in:

$$H_0((\mathcal{GC}^\infty_{sv}(\mathbb{G}), \mathcal{F}_t)_{\gamma_K(t)} + (\mathcal{GC}^\infty_{sv}(\mathbb{G}), \mathcal{F}_s)_r) \oplus H_1((\mathcal{GC}^\infty_{sv}(\mathbb{G}), \mathcal{F}_t)_{\gamma_K(t)+1/2} + (\mathcal{GC}^\infty_{sv}(\mathbb{G}), \mathcal{F}_s)_{r+1/2})$$

Chapter 6

$\mathcal{G}$-sets

In this chapter, we introduce $\mathcal{G}$-sets. These are sets of closed regions in $\mathbb{Z}^2$ which were first introduced by Sato in [14]. We are primarily interested in $\mathcal{G}_0$ and $\mathcal{G}_1$ (Sato notes that he isn't sure if the higher order $\mathcal{G}$-sets are even nonempty), the former computes Υ_K, and the latter gives a bound on Υ_K^2. The first half of the chapter is a survey of results from [12] which outline how one can compute $\mathcal{G}_0(K)$ from a grid diagram $\mathbb{G}$ for K. The core observation Sano and Sato exploit in this section of [12] is that, for very large $s \in \mathbb{Z}_{\geq 0}$, the structures of $U^s CFK_0^\infty(K)$ and $CFK^-_{-2s}(K)$ are identical, so one can perform a *shift trick* and then pass through a chain homotopy to compute $\mathcal{G}_0(K)$ from $\mathcal{GC}^-(\mathbb{G})$. Sano and Sato also construct an algorithm in [12] which we only mention in passing as it is not relevant to the original work in the second half of the chapter. A major advantage of this algorithm is that it avoids computing the boundary map of $\mathcal{GC}^-(\mathbb{G})$ directly, instead using sparse linear systems. This means that the algorithm is reasonably efficient, and allowed Sano and Sato to compute the $\mathcal{G}_0$-type and Υ-invariants of prime knots with up to 11 crossings in around an hour. In the second half, we observe a similar relationship between $U^s CFK_{0,1}^\infty(K)$ and $CFK^-_{-2s,-2s+1}(K)$. Using this shift trick, we find a new way to compute $\mathcal{G}_1(K;-,-)$, along the lines of [12] from $\mathcal{GC}^-(\mathbb{G})$.

6.1 Formal Knot Complexes and ν^+-equivalence

Recall we defined formal knot complexes in 4.3.4. We start by defining a chain homotopy invariant of formal knot complexes.

Definition 6.1.1. Let C be a formal knot complex. Then define:

$$\nu^+(C) := \min\{m \in \mathbb{Z}_{\geq 0} : C_{i\leq 0, j\leq m} \text{ contains a homological generator}\}.$$

We can then use this to define ν^+-equivalence classes.

Definition 6.1.2. Let C and C' be formal knot complexes. Then we say C is *ν^+-equivalent* to C' if:

$$\nu^+(C \otimes C'^*) = \nu^+(C^* \otimes C') = 0,$$

where C^* denotes the dual complex of C.

We also want to work with ν^+-equivalence of subcomplexes, for which the definitions above are not suitable. Fortunately, Sato proves the following in [14] (the partial order is introduced in [14] but we will not discuss details as they are not relevant):

Theorem 6.1.1. Two ν^+-classes satisfy $[C]_{\nu^+} \geq [C']_{\nu^+}$ if and only if there is a $\mathbb{Z}^2$-filtered quasi-isomorphism:

$$f : C \to C'.$$

Thus we can redefine ν^+-equivalence by the following.

Definition 6.1.3. Let C and C' be formal knot complexes. Then C is ν^+-equivalent to C' if there exists a pair of $\mathbb{Z}^2$-filtered quasi-isomorphisms:

$$f : C \to C'$$
$$g : C' \to C$$

We also say $C' \leq_{\nu^+} C$ if there exists a $\mathbb{Z}^2$-filtered quasi-isomorphism:

$$f : C \to C'$$

This definition is easily extended to subcomplexes of formal knot complexes. Moreover, it recovers the original definition of ν^+-equivalence, so this is what we will use as the *definition* from here.

Definition 6.1.4. A *formal knot subcomplex* is a chain complex S which is a $\mathbb{Z}^2$-filtered subcomplex of a formal knot complex C.

The definition above clearly applies to formal knot subcomplexes.

Definition 6.1.5. Let C and D be a pair of formal knot subcomplexes. We say that C is ν^+-equivalent to D if there are filtered quasi-isomorphisms:

$$f : C \to D,$$

$$g : D \to C.$$

We also say that $D \leq_{\nu^+} C$ or equivalently $[D]_{\nu^+} \leq [C]_{\nu^+}$ if there is a filtered quasi-isomorphism:

$$f : C \to D.$$

6.2 $\mathcal{G}_0$

Next, we outline the construction of $\mathcal{G}_0(K)$, and the results which motivate, and are also crucial for, the next section.

Definition 6.2.1. Fix a knot K, and let $C = CFK^\infty(K)$. A *homological generator* of C is a nontrivial cycle $x \in C_0$. Then:

$$\widetilde{\mathcal{G}}_0(K) := \{R \in CR(\mathbb{Z}^2) : R \text{ contains a homological generator}\},$$
$$\mathcal{G}_0(K) := \min \widetilde{\mathcal{G}}_0(K).$$

In [12], Sano and Sato prove the following.

Theorem 6.2.1. For all knots K, $\mathcal{G}_0(K)$ is nonempty and finite.

Theorem 6.2.2. For any $R \in CR(\mathbb{Z}^2)$ (this set is defined in §$A.4.1$), the following holds:

$$R \in \widetilde{\mathcal{G}}_0(K) \iff \exists R' \in \mathcal{G}_0(K), R' \subset R.$$

Proposition 6.2.3. If $[K]_{\nu^+} \leq [J]_{\nu^+}$, then for any $R' \in \mathcal{G}_0(J)$, there exists $R \in \mathcal{G}_0(K)$ such that $R \subset R'$.

Corollary 6.2.3.1. $\mathcal{G}_0(K)$ is a concordance invariant.

Proposition 6.2.4. For any $R \in \mathcal{G}_0(K)$, there exists a homological generator whose closure is equal to R. We call these homological generators *realisers* for R, and collect them in the set $gen_0(C; R)$.

Next, we need to discuss shifts of $\mathcal{G}_0$.

Definition 6.2.2. Let C be a formal knot subcomplex such that $H_n(C) = \mathbb{F}$ for some $n \in \mathbb{Z}$. Then a *homological generator of degree* n is a nontrivial cycle in C_n. Then we can define:

$$\widetilde{\mathcal{G}}_0^{(n)}(C) := \{R \in CR(\mathbb{Z}^2) : R \text{ contains a homological generator of degree } n\},$$
$$\mathcal{G}_0^{(n)} := \min \widetilde{\mathcal{G}}_0^{(n)}(C).$$

To relate these homologically shifted versions of $\mathcal{G}_0$ to $\mathcal{G}_0$, we need to be able to move our closed regions around the plane.

Definition 6.2.3. If $R \in CR(\mathbb{Z}^2)$, then for $s \in \mathbb{R}$:

$$R[s] := \{(i, j) \in \mathbb{Z}^2 : (i + s, j + s) \in R\},$$

and if $S \subset CR(\mathbb{Z}^2)$:

$$S[s] := \{R[s] : R \in S\}.$$

The following results, also from [12], indicate why shifted $\mathcal{G}_0$ sets are crucial to translating $\mathcal{G}_0$ from $CFK^\infty(K)$ to $\mathcal{GC}^-(\mathbb{G})$, where $\mathbb{G}$ is a grid diagram which represents K.

Proposition 6.2.5. For any knot K and $s \in \mathbb{Z}$:

$$\widetilde{\mathcal{G}}_0^{(-2s)}(CFK^\infty(K)) = \widetilde{\mathcal{G}}_0(K)[s]$$
$$\mathcal{G}_0^{(-2s)}(CFK^\infty(K)) = \mathcal{G}_0(K)[s].$$

Theorem 6.2.6. For any knot K, and $s \in \mathbb{Z}_{\geq 0}$:

$$\mathcal{G}_0^{(-2s)}(CFK^-(K)) = \{R[s] : R \in \mathcal{G}_0(K), shiftR \leq s\}.$$

Where $shiftR := \max\{i \in \mathbb{Z} \cup \{\infty\} : \exists j \in \mathbb{Z} \text{ such that } (i, j) \in R\}$.

Defining $shift\mathcal{G}_0(K) = \max\{shiftR : R \in \mathcal{G}_0(K)\}$, we obtain the following corollary.

Corollary 6.2.6.1. If $s \in \mathbb{Z}$ such that $s \geq shift\mathcal{G}_0(K)$, then:

$$\mathcal{G}_0^{(-2s)}(CFK^-(K)) = \mathcal{G}_0(K)[s].$$

6.3 $\mathcal{G}_1$

6.3.1 Pseudo-Holomorphic Construction

We begin by constructing $\mathcal{G}_1$ along the lines of [14].

Definition 6.3.1. Let C be a formal knot complex, and suppose that $\mathcal{G}_0(K)$ contains two distinct elements R_1 and R_2. Define $\mathcal{G}_1(C; R_1, R_2)$ by the following:

- $\widetilde{gen}_1(C; R_1, R_2) := \{x \in C_1 : \exists z_i \in gen_0(C; R_i), \partial x = z_1 + z_2\}$.
- $\widetilde{\mathcal{G}}_1(C; R_1, R_2) := \{R_x : x \in \widetilde{gen}_1(C; R_1, R_2)\}$.
- $\mathcal{G}_1(C; R_1, R_2) := \min \widetilde{\mathcal{G}}_1(C; R_1, R_2)$.

There are only three results about $\mathcal{G}_1$ in [14], these are listed below and can all be found in [14, §5.3.1]. We fix the notation from 6.3.1 for brevity.

Lemma 6.3.1. $\mathcal{G}_1(C; R_1, R_2)$ is nonempty and finite.

Theorem 6.3.2. Suppose $[C]_{\nu^+} \leq [C']_{\nu^+}$ and $\mathcal{G}_0(C) \cap \mathcal{G}_0(C')$ contains two distinct elements R_1 and R_2. Then, for any $R' \in \mathcal{G}_1(C'; R_1, R_2)$, there exists $R \in \mathcal{G}_1(C; R_1, R_2)$ for which $R \subset R'$.

Corollary 6.3.2.1. For any $[C]_{\nu^+} \in C^f$, and distinct pair $R_1, R_2 \in \mathcal{G}_0(C)$, $\mathcal{G}_1(C; R_1, R_2)$ is an invariant of $[C]_{\nu^+}$.

6.3.2 Shifts

The next step in constructing $\mathcal{G}_1$ for $\mathcal{GC}^-(\mathbb{G})$ is to set up shifted versions of $\mathcal{G}_1$.

Definition 6.3.2. Fix a formal knot subcomplex C and an integer $s \in \mathbb{Z}_{\geq 0}$. If R_1 and R_2 are distinct elements of $\mathcal{G}_0^{(-2s)}(C)$, then we construct $\mathcal{G}_1^{(-2s)}(C; R_1, R_2)$ by the following:

- $\widetilde{gen}_1^{(-2s)}(C; R_1, R_2) := \{x \in C_{-2s+1} : \exists z_i \in gen_0^{(-2s)}(C; R_i), \partial x = z_1 + z_2\}$.
- $\widetilde{\mathcal{G}}_1^{(-2s)}(C; R_1, R_2) := \{R_x : x \in \widetilde{gen}_1^{(-2s)}(C; R_1, R_2)\}$.
- $\mathcal{G}_1^{(-2s)}(C; R_1, R_2) := \min \widetilde{\mathcal{G}}_1^{(-2s)}(C; R_1, R_2)$.

To begin with, we want to check that shifted $\mathcal{G}_1$-sets are still invariant under ν^+-equivalence. We mimic the approach of [14, §5.3.1].

Lemma 6.3.3. Let C and D be formal knot subcomplexes. Suppose $f : C \to D$ is a filtered quasi-isomorphism, and that there are distinct $R_1, R_2 \in \mathcal{G}_0^{(-2s)}(C) \cap \mathcal{G}_0^{(-2s)}(D)$. Then if $R' \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$, there exists $R \in \mathcal{G}_1^{(-2s)}(D; R_1, R_2)$ for which $R \subset R'$.

Proof. Let $R' \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$. Then there are realisers $z_i \in gen_0^{(-2s)}(C; R_i)$ and $x \in C_{-2s+1}$ for which $\partial x = z_1 + z_2$ and $R_x = R'$. Given f is a quasi-isomorphism, $R_{f(z_i)}, R_i \in \widetilde{\mathcal{G}}_0^{(-2s)}(D)$. Moreover, f is filtered so $R_{f(z_i)} \subset R_{z_i} = R_i$. By the minimality of R_i in $\widetilde{\mathcal{G}}_0^{(-2s)}(D)$, we must then have $R_{f(z_i)} = R_i = R_{z_i}$. In particular, this implies $f(z_i) \in gen_0^{(-2s)}(D; R_i)$. Noting that $\partial f(x) = f(\partial x) = f(z_1) + f(z_2)$, we find $f(x) \in \widetilde{\mathcal{G}}_1^{(-2s)}(D; R_1, R_2)$, so there is $R \in \mathcal{G}_1^{(-2s)}(D; R_1, R_2)$ with $R \subset R_{f(x)} \subset R_x = R'$. □

Corollary 6.3.3.1. If C and D are ν^+-equivalent formal knot subcomplexes, $s \in \mathbb{Z}$ and there exist distinct $R_1, R_2 \in \mathcal{G}_0^{(-2s)}(C) \cap \mathcal{G}_0^{(-2s)}(D)$, then:

$$\mathcal{G}_1^{(-2s)}(C; R_1, R_2) = \mathcal{G}_1^{(-2s)}(D; R_1, R_2).$$

Proof. Given C and D are ν^+-equivalent, there exist filtered quasi-isomorphisms:

$$f : C \to D,$$

$$g : D \to C.$$

By applying 6.3.3 to f, if $R \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$, then there exists $R' \in \mathcal{G}_1^{(-2s)}(D; R_1, R_2)$ such that $R' \subset R$. Similarly, applying 6.3.3 to g, we obtain $R'' \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$ such that $R'' \subset R' \subset R$. By the minimality of R, we have $R = R' = R''$, so $R \in \mathcal{G}_1^{(-2s)}(D; R_1, R_2)$. Thus $\mathcal{G}_1^{(-2s)}(C; R_1, R_2) \subset \mathcal{G}_1^{(-2s)}(D; R_1, R_2)$. If we exchange the roles of f and g is the argument above, we see that $\mathcal{G}_1^{(-2s)}(D; R_1, R_2) \subset \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$, so the result follows. □

In this setting, we lose global triviality, so in general we no longer have:

$$\mathcal{G}_1^{(-2s)}(C; R_1, R_2) = \mathcal{G}_1(C; R_1[-s], R_2[-s]).$$

Restricting to formal knot complexes recovers this.

Theorem 6.3.4. If C is a formal knot complex, $s \in \mathbb{Z}_{\geq 0}$ and $R_1, R_2 \in \mathcal{G}_0^{(-2s)}$ are distinct, then:

$$\mathcal{G}_1^{(-2s)}(C; R_1, R_2) = \mathcal{G}_1(C; R_1[-s], R_2[-s])[s].$$

Proof. First note that, by the proof of [12, Proposition 3.2], we have:

$$\mathcal{G}_0^{(-2s)}(C) = \mathcal{G}_0(C)[s],$$

so if R_1 and R_2 are distinct elements of $\mathcal{G}_0^{(-2s)}(C)$, then (following the proof of [12, Proposition 3.2]) the corresponding elements of $\mathcal{G}_0(C)$ are $R_1[-s]$ and $R_2[-s]$ respectively. Thus, we can relate realisers by the following:

$$gen_0^{(-2s)}(C; R_i) = U^s gen_0(C; R_i[-s]).$$

From here, it is obvious from the definitions of $\mathcal{G}_1$ and $\mathcal{G}_1^{(-2s)}$ (and property 4 of formal knot complexes) that it is sufficient to prove:

$$\widetilde{gen}_1^{(-2s)}(C; R_1, R_2) = U^s \widetilde{gen}_1(C; R_1[-s], R_2[-s]).$$

Let $x \in \widetilde{gen}_1(C; R_1, R_2)$. Then there are $z_i \in gen_0^{(-2s)}(C; R_i)$ such that $\partial x = z_1 + z_2$. Clearly, $\partial U^{-s}x = U^{-s}z_1 + U^{-s}z_2$. By construction, $U^{-s}x \in C_1$ and $U^{-s}z_i \in gen_0(C; R_i[-s])$, so $U^{-s}x \in \widetilde{gen}_1(C; R_1[-s], R_2[-s])$. Conversely, suppose that $x \in \widetilde{gen}_1(C; , R_1[-s], R_2[-s])$. Then there are $x \in C_1$ and $z_i \in gen_0(C; R_i)$ such that $\partial x = z_1 + z_2$. As above, $\partial U^s x = U^s z_1 + U^s z_2$, so by a nearly identical argument, $U^s x \in \widetilde{gen}_1^{(-2s)}(C; R_1, R_2)$. Thus the result follows. □

This is the first shift result we need. Now we want to extract $\mathcal{G}_1$ from $\mathcal{F}_0^{alg}(C)$.

Proposition 6.3.5. If $f : C \to C'$ is a chain map which preserves filtration levels of elements, for some $s \in \mathbb{Z}$ induces quasi-isomorphisms on C_{-2s} and C_{-2s+1}, and if there exist distinct $R_1, R_2 \in \mathcal{G}_0^{(-2s)}(C) \cap \mathcal{G}_0^{(-2s)}(C')$, then:

$$\widetilde{\mathcal{G}}_1^{(-2s)}(C; R_1, R_2) \subset \widetilde{\mathcal{G}}_1^{(-2s)}(C'; R_1, R_2).$$

Proof. Let f be a map which satisfies the hypotheses above. Let $x \in \widetilde{gen}_1^{(-2s)}(C; R_1, R_2)$. Then there are $z_i \in gen_0^{(-2s)}(C; R_i)$ such that $\partial x = z_1 + z_2$. Given f preserves filtration levels of elements, $R_{f(z_i)} = R_{z_i} = R_i$. Moreover f is a quasi-isomorphism on C_{-2s} and C_{-2s+1}, $f(z_i)$ is a nontrivial cycle, so $f(z_i) \in gen_0^{(-2s)}(C'; R_i)$. Given f is a chain map, $\partial f(x) = f(z_1) + f(z_2)$. Thus $f(x) \in \widetilde{gen}_1^{(-2s)}(C'; R_1, R_2)$, so $R_x = R_{f(x)} \in \widetilde{\mathcal{G}}_1^{(-2s)}(C'; R_1, R_2)$, and the result follows. □

For the following lemmata, fix the following setup. Let C be a formal knot complex, and $\iota : \mathcal{F}_0^{alg}C \to C$ be the standard inclusion. Fix an integer $s \in \mathbb{Z}_{\geq 1}$ and distinct $R_1, R_2 \in \mathcal{G}_0^{(-2s)}(C)$ for which $shiftR_i \leq 0$.

Lemma 6.3.6. If $R \in \widetilde{\mathcal{G}}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2)$, then there exists an element $R' \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$ such that $R' \subset R$ and $shiftR' \leq 0$.

Proof. Recall that, by the global triviality of C, we have a filtered chain homotopy equivalence:

$$\left(C, \{\mathcal{F}_k^{alg}\}_{k\in\mathbb{Z}}\right) \sim \left(\Lambda, \{\mathcal{F}_k^{alg}\}_{k\in\mathbb{Z}}\right).$$

Thus ι induces quasi-isomorphisms on all $\mathcal{F}_0^{alg}C_k$ for all k less than or equal to 0. in particular, ι induces quasi-isomorphisms on $\mathcal{F}_0^{alg}C_{-2s}$ and $\mathcal{F}_0^{alg}(C_{-2s+1})$. Moreover, ι is an inclusion, so preserves filtration levels of elements. Thus by 6.3.5 we have:

$$\widetilde{\mathcal{G}}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2) \subset \widetilde{\mathcal{G}}_{1^{(-2s)}}(C; R_1, R_2).$$

So if $R \in \widetilde{\mathcal{G}}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2)$, then $R \cap \{i \leq 0\} \in \widetilde{\mathcal{G}}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2)$, so $R \cap \{i \leq 0\} \in \widetilde{\mathcal{G}}_{1^{(-2s)}}(C; R_1, R_2)$. Thus there is $R' \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$ such that $R' \subset R \cap \{i \leq 0\}$, so $R' \subset R$ and $shiftR' \leq 0$. □

Lemma 6.3.7. If $R \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$, and $shiftR \leq 0$, then $R \in \widetilde{\mathcal{G}}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2)$.

Proof. Let $R \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$ with $shiftR \leq 0$. Pick realisers $z_i \in gen_0^{(-2s)}(C; R_i)$ and $x \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2; R)$ so that $\partial x = z_1 + z_2$. Then $R_x = R$ and $shiftR \leq 0$, so $x \in C_{R\cap\{i\leq 0\}}$ which implies that $x \in \mathcal{F}_0^{alg}C$. Given $shiftR_i \leq 0$, both R_1 and R_2 lie in $\mathcal{G}_0$, so $z_i \in gen_0(\mathcal{F}_0^{alg}C; R_i)$. Thus $x \in \widetilde{gen}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2)$, so $R = R_x \in \widetilde{\mathcal{G}}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2)$. □

We can now prove the main shifting theorem.

Theorem 6.3.8. For s, R_1, R_2 as defined earlier, we have:

$$\mathcal{G}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2) = \{R \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2) : shiftR \leq 0\}.$$

Proof. Let $R \in \mathcal{G}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2)$. Then $R \in \widetilde{\mathcal{G}}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2)$, so by 6.3.6 there is $R' \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$ such that $R' \subset R$ and $shiftR' \leq 0$. By 6.3.7 $R' \in \widetilde{\mathcal{G}}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2)$. Noting that R is minimal, we must have:

$$R = R' \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2).$$

Now suppose that $R \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$ and $shiftR \leq 0$. By 6.3.7 we must have $R \in \widetilde{\mathcal{G}}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2)$. Assume that $R' \in \widetilde{\mathcal{G}}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1, R_2)$ for which $R' \subset R$. Then by 6.3.6 there is $R'' \in \mathcal{G}_1^{(-2s)}(C; R_1, R_2)$ such that $R'' \subset R' \subset R$. By minimality of R, $R'' = R' = R$, so $R \in \mathcal{G}_1^{(-2s)}(\mathcal{F}_0^{(-2s)}C; R_1, R_2)$. □

As a consequence of 6.3.8 and 6.3.4, we have the following corollary.

Corollary 6.3.8.1. If $s \geq \max\{shiftR : R \in \mathcal{G}_1(C; R_1, R_2)\}$, then:

$$\mathcal{G}_1(C; , R_1, R_2)[s] = \mathcal{G}_1^{(-2s)}(\mathcal{F}_0^{alg}C; R_1[s], R_2[s]).$$

6.3.3 Combinatorialisation

The final step is to relate this to the grid complex $\mathcal{GC}^-(\mathbb{G})$. Recall that we have a filtered chain homotopy equivalence:

$$CFK^-(K) \simeq \mathcal{GC}^-(\mathbb{G}).$$

Where $\mathbb{G}$ is a grid diagram who represents the knot K. Noting that $\mathcal{G}_1$ is invariant under chain homotopies, the corollary above can be combinatorialised to the following:

Corollary 6.3.8.2. If $s \geq \max\{shiftR : R \in \mathcal{G}_1(C; R_1, R_2)\}$, then:

$$\mathcal{G}_1(C; , R_1, R_2)[s] = \mathcal{G}_1^{(-2s)}(\mathcal{GC}^-(\mathbb{G}); R_1[s], R_2[s]).$$

6.4 Recovering Invariants from $\mathcal{G}$-sets

In [12, Theorem 2.20], Sano and Sato give the following formula for Υ_K.

Definition 6.4.1. For $t \in [0, 2]$ and $s \in \mathbb{R}$, set:

$$\{R^t(s) := \{(i, j) \in \mathbb{Z}^2 : (1 - t/2)i + (t/2)j \leq s\}\}.$$

Then:

$$\Upsilon_K(t) = -2 \min\{s \in \mathbb{R} : \exists R \in \mathcal{G}_0(K), R \subset R^t(2)\}.$$

Noting the similarity between the constructions of $\mathcal{G}_1$ and Υ^2_K, one might hope that $\mathcal{G}_1(K;-,-)$ can be used to recover Υ^2_K. Unfortunately, in [14], the authors are only able to demonstrate a bound. They do this by comparing $\mathcal{G}_1$ and γ^2_K. Recall we have $t^\pm$ by choosing a very small $\delta > 0$ and setting $t^\pm = t \pm \delta$. Fix $t \in [0,2]$, and define $\mathcal{G}_0^{t^\pm}(K) := \{R \in \mathcal{G}_0(K) : R \subset R^{t^\pm(\gamma_K(t^\pm))}\}$. Then we set $\mathcal{G}_1^t(K) = \bigcup_{R^\pm \in \mathcal{G}_0^{t^\pm}(K), R^- \neq R^+} \mathcal{G}_1(K; R^-, R^+)$. This culminates with the following bound in [14]:

$$\gamma^2_{K,t}(s) \leq \min\{r \in \mathbb{R} : \exists R \in \mathcal{G}_1^t(K), R \subset (R^t(\gamma_K(t)) \cup R^2(r))\}.$$

Appendix A

Homological Algebra

In this chapter, we outline the preliminary homological algebra needed to work with grid complexes and Knot Floer complexes.

A.1 Representation Theory

For the duration of this section, let L be a unitary ring.

Definition A.1.1. An abelian group M equipped with a homomorphism $\psi_M : L \to End(M)$ is called an *L-module*. We denote the action of $l \in L$ on $x \in M$ by $l \cdot x$ or lx. Note that ψ being a homomorphism is equivalent to the conditions:

1. For all $x, y \in M$ and $l \in L$, $l(x + y) = lx + ly$.
2. For all $x \in M$ and $l, s \in L$, $l(sx) = (ls)x$.
3. For all $x \in M$ and $l, s \in L$, $(l + s)x = lx + sx$.

Modules admit all the usual definitions of subobjects, freeness, coproducts, ect.

Definition A.1.2. Let M be an L-module. Then $N \subset M$ is called a *submodule* of M if N is a subgroup of M which is invariant under the action of each $l \in L$.

We can quotient out by a submodule.

Definition A.1.3. Let M be an L-module and N be a submodule of M. Then the *quotient module* of M modulo N, denoted M/N is given by the set of cosets:

$$M/N := \{x + N : x \in M\},$$

with its induced group and module structure.

Definition A.1.4. Let $\{M_i\}_{i \in I}$ be a collection of L-modules. Then the *product* of this collection is the module with underlying group $\prod_{i \in I} M_i$, and L-action determined by $l \cdot (x_i)_{i \in I} = (l \cdot x_i)_{i \in I}$.

Definition A.1.5. Let $\{M_i\}_{i\in I}$ be a collection of L-modules. Then the *direct sum* of this collection is the module:

$$\bigoplus_{i\in I} M_i := \{x \in \prod_{i\in I} M_i : \text{for all but finitely many } i,\ x_i = 0\}.$$

Definition A.1.6. An L-module M is *free* if there exists a set $B = \{b_i\}_{i\in I} \subset M$ such that:

$$\sum_{k=1}^{n} l_k b_{i_k} = 0 \iff \text{each } l_k = 0,$$

and each $x \in M$ lies in $\{\sum_{k=1}^{n} l_k b_{i_k} : n \in \mathbb{N} \text{ and each } l_k \in L\}$. If B satisfies this property, we call it a *basis*.

Definition A.1.7. An L-module M is *finitely generated* if there exists a finite set $F \subset M$ such that:

$$M = \{\sum_{k=1}^{n} l_k f_k : n \in \mathbb{N}, l_k \in L, f_k \in F\}.$$

Definition A.1.8. Let M be an L-module. An element $x \in M$ is called *torsion* if there exists nonzero $l \in L$ for which $l \cdot x = 0$. If no such l exists we say x is *nontorsion*.

If L is a commutative integral domain (all of the rings we use in the main text are commutative integral domains), then it is immediate that the set of torsion elements forms a submodule of M, denoted $Tor(M)$. Next, we discuss the tensor product. This is a complicated operation in general, so we discuss the finitely generated case (which is all we will need).

Definition A.1.9. Let M and N be finitely generated L-modules. Then their tensor product is the set of formal linear combinations:

$$\{\sum_{i=1}^{n} l_i(m_i \otimes n_i) : n \in \mathbb{N}, l_i \in L, m_i \in M, n_i \in N\},$$

quotiented by the following relations:

1. $\forall l \in L, m \in M, n \in N$, we have $(lm) \otimes n = l(m \otimes n)$.
2. $\forall m_1, m_2 \in M, n \in N$, we have $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$.
3. $\forall m \in M, n_1, n_2 \in N$, we have $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$.

Sometimes we will be working in situations where an object has actions induced by multiple rings. In this case, we denote the tensor product $M \otimes_L N$ to make it clear when the first condition of A.1.9 can be applied. Finally, we discuss morphisms.

Definition A.1.10. Let M and N be L-modules and $f : M \to N$ be a function. Then f is an *L-homomrphism* (or just a homomorphism if the context is clear) if, for each $x \in M$ and $l \in L$, $f(lx) = lf(x)$.

We have a few submodules associated to homomorphisms.

Definition A.1.11. Let $f : M \to N$ be an L-homomorphism. Then the *kernel* of f is the set of elements mapped to 0 in N. This is clearly a submodule, as f is a homomorphism.

Definition A.1.12. Let $f : M \to N$ be a homomorphism. Then the *image* of f, denoted $\mathrm{im}(f)$ is the set $f(M)$. Because M is an L-module and f is an L-homomorphism, it is easy to check that this is a submodule of N.

Definition A.1.13. Let $f : M \to N$ be a homomorphism. Then we say f is an *isomorphism* if it is a bijection.

We finish with the isomorphism theorems. These are well known, proofs can be found in [28, §5.4].

Theorem A.1.1 (First Isomorphism Theorem)**.** Let $f : M \to N$ be an L-homomorphism. Then we can decompose f according to the following diagram:

$$M \twoheadrightarrow M/\ker(f) \xrightarrow[\cong]{} \mathrm{im}(f) \rightarrowtail N$$

The surjection on the left hand side is the canonical quotient, and the injection on the left hand side is the canonical inclusion. Most importantly, we have an isomorphism $M/\ker(f) \cong \mathrm{im}(f)$.

Theorem A.1.2 (Second Isomorphism Theorem)**.** Let M be an L-module and N, P be submodules of M. Then:

- $N + P$ is a submodule of M.
- $N \cap P$ is a submodule of M.
- $\frac{N+P}{P} \cong \frac{P}{N \cap P}$.

Theorem A.1.3 (Third Isomorphism Theorem)**.** Let M be an L-module, N be a submodule of M, and P be a submodule of M containinig N, that is: $N \leq P \leq M$. Then $P/N \leq M/N$, and:

$$\frac{M/N}{P/N} \cong M/P.$$

A crucial tool in homological algebra is the *exact sequence*. We will be interested in two different kinds of exact sequences.

Definition A.1.14. A *short exact sequence* is a diagram:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

in which $\ker g = \mathrm{im} f$, f is an injection, and g is a surjection.

Definition A.1.15. A *long exact sequence* is a diagram:

$$\cdots \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots,$$

in which for each $i \in \mathbb{Z}$, $\ker f_i = \mathrm{im} f_{i-1}$.

To effectively work with exact sequences, we can add a *cancelling differential* to our modules.

A.2 Graded Chain Complexes

Graded complexes are the first sort of chain complexes one encounters when learning algebraic topology, generally as a way of constructing simplicial homology. We will work in a much more general setting. Fix a pair of rings L and R for the remainder of this section.

Definition A.2.1. An *R-graded chain complex over L* consists of a pair (C, ∂), where C is an L-module, and $\partial : C \to C$ is an L-homomorphism with $\partial \circ \partial = 0$. We also require that C has an *R-grading* which is compatible with ∂. That is, we have a splitting $C = \bigoplus_{r \in R} C_r$ so that for each $r \in R$, $\partial(C_r) \subset C_{r-1}$.

Associated to a graded chain complex is its *homology*, which is the L-module $\ker \partial / \mathrm{im}\partial$ and denoted $H(C)$. Note that $H(C)$ inherits the R-grading; if we set $H_r(C) := \ker(\partial|_{C_r})/\mathrm{im}(\partial|_{C_{r-1}})$, then compatibility with the differential confirms that $H(C) = \bigoplus_{r \in R} H_r(C)$.

Definition A.2.2. Let (C, ∂_C) and (D, ∂_D) be graded chain complexes, and $f : C \to D$ be an L-homomorphism. Then we say f is a *chain map* if the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\partial_C} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow[\partial_D]{} & D. \end{array}$$

Moreover, if $f(C_r) \subset D_r$ for all $r \in R$, we say that f is *graded*.

Commutativity of A.2.2 ensures that a chain map f will induce a homomorphism $H(f)$. Clearly, if f is also graded, for each $r \in R$ we have $H_r(f) : H_r(C) \to H_r(D)$.

Definition A.2.3. Let $f : C \to D$ be a chain map and $s \in R$. We say that f is *homogeneous of degree s* if, for each $r \in R$, $f(C_r) \subset D_{r+s}$.

Definition A.2.4. A graded chain map f is a *quasi-isomorphism* if $H(f)$ is an isomorphism.

Definition A.2.5. A *short exact sequence* of chain complexes consists of a short exact sequence of L-modules:

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0,$$

in which all objects are chainn complexes and all arrows are chain maps.

We conclude with the snake lemma.

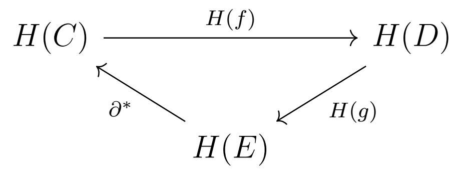
Lemma A.2.1 (Snake Lemma)**.** Suppose we have a short exact sequence of chain complexes:

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0,$$

with the additional property that all arrows are graded. Then for each $r \in R$, there is a long exact sequence:

$$\cdots \xrightarrow{\partial^*_{r+1}} H_r(C) \xrightarrow{H_r(f)} H_r(D) \xrightarrow{H_r(g)} H_r(E) \xrightarrow{\partial^*_r} H_{r-1}(C) \xrightarrow{H_{r-1}(f)} \cdots .$$

Equivalently, we can bundle these into an *exact triangle*.



Where ∂^* is homogeneous of degree -1.

Proof. This is a foundational lemma for homological algebra, a proof can be found in [29, Chapter 1]. The proof essentially involves using the boundary map on E and exactness to send an element of $H_r(E)$ to $H_{r-1}(C)$. □

Definition A.2.6. Let (C, ∂_C) and (D, ∂_D) be a pair of graded chain complexes. Then we form their *tensor product* of L by the tuple $(C \otimes_L D, \partial_C \otimes_L 1 + 1 \otimes_L \partial_D)$, and homological grading, for $r \in R$:

$$(C \otimes_L D)_r = \bigoplus_{s_1+s_2=r} (C_{s_1} \otimes_L D_{s_2}).$$

Definition A.2.7. Let (C, ∂) be a graded chain complex over L. Let $\mathbb{K}$ be a field which acts on L. Let $\mathcal{W}$ be a graded $\mathbb{K}$-space. Then the tensor product of C and $\mathcal{W}$ is given by the tuple $(C \otimes_{\mathbb{K}} \mathcal{W}, \partial \otimes_{\mathbb{K}} 1)$ and for $r \in R$:

$$(C \otimes_{\mathbb{K}} \mathcal{W})_r = \bigoplus_{s+l=r} C_s \otimes_{\mathbb{K}} \mathcal{W}_l.$$

Definition A.2.8. Let $Q \leq L$ be a subring. Let (C, ∂) be an R-graded chain complex over Q. Moreover, assume L admits a grading $\bigoplus_{r \in R} L_r$ so that, if $q \in Q \cap L_r$, and $s \in R$:

$$qC_s \subset C_{s+r}.$$

Then define the tensor product of C and L is given by the tuple $(C \otimes_Q L, \partial \otimes 1)$ and R-grading:

$$(C \otimes_Q L)_r = \bigoplus_{s+l=r} C_s \otimes_Q L_l.$$

A.3 Bigraded Chain Complexes

In this section we outline the general properties of *bigraded* chain complexes. Fix the following rings:

- L, R are unitary rings.
- S is an arbitrary ring.

In what follows, L will be the ring over which out chain complex is a module, R will record our *homological grading*, and S will record our other grading.

Definition A.3.1. An *R, S-bigraded chain complex* over L is a tuple (C, ∂) which satisfies the following conditions.

1. C is an L-module.
2. ∂ is an endomorphism of C which squares to 0.
3. C splits as a direct sum $\bigoplus_{\substack{r \in R \\ s \in S}} C_{r,s}$.
4. The endomorphism ∂ is *homogeneous of degree* $(-1, 0)$ with respect to the R, S-bigrading. That is, $\partial(C_{r,s}) \subset C_{r-1,s}$.

We call the R-grading the *homological grading* of C.

Definition A.3.2. Let (C, ∂) be an R, S-bigraded chain complex. The *homology* of (C, ∂), denoted $H(C)$, is the L-module $\ker(\partial)/\mathrm{im}(\partial)$. It is clear that we also get a splitting:

$$H(C) = \bigoplus_{\substack{r \in R \\ s \in S}} H_{r,s}(C),$$

if we set $H_{r,s}(C) = \ker(\partial|_{C_{r,s}})/\mathrm{im}(\partial|_{C_{r+1,s}})$.

Definition A.3.3. Let C and D be bigraded chain complexes. An L-homomorphism $f : C \to D$ is called a *chain map* if the diagram below commutes.

$$\begin{array}{ccc} C & \xrightarrow{\partial_C} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{\partial_D} & D \end{array}$$

If f also preserves the bigrading (that is: for all $(r, s) \in R \times S$ $f(C_{r,s}) \subset D_{r,s}$) we say f is *bigraded*.

Note that a chain map will always induce a map on $H(C)$, and if it is bigraded, then the induced map on homology will be bigraded. If $f : C \to D$ is a chain map then we denote the map it induces on hommology by $H(f) : H(C) \to H(D)$ or $f_* : H(C) \to H(D)$.

Definition A.3.4. Let C, D be bigraded chain complexes and $f : C \to D$ be a chain map. Let $(m, n) \in R \times S$. Then f is *homogeneous of degree* (m, n) if, for each $(r, s) \in R \times S$, $f(C_{r,s}) \subset D_{r+m,s+n}$.

In particular, note that a map is graded if and only if it is homogeneous.

Definition A.3.5. A *short exact sequence of bigraded chain complexes* is a short exact sequence of chain complexes:

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0,$$

where all the arrows are bigraded.

Note that we can deal with short exact sequences with homogeneous arrows by applying bigrading shifts.

Lemma A.3.1 (Bigraded Snake Lemma)**.** Suppose we have a short exact sequence of bigraded chain complexes:

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0.$$

Then for $r \in R$ and $s \in S$, we have a long exact sequence:

$$\cdots \xrightarrow{\partial^*_{r+1,s}} H_{r,s}(C) \xrightarrow{H_{r,s}(f)} H_{r,s}(D) \xrightarrow{H_{r,s}(g)} H_{r,s}(E) \xrightarrow{\partial^*_{r,s}} H_{r-1,s}(C) \longrightarrow \cdots .$$

Equivalently, we can collect these into an exact triangle:

$$\begin{array}{ccc} H(C) & \xrightarrow{\quad H(f) \quad} & H(D) \\ & {\scriptstyle \partial^*} \nwarrow \quad \swarrow {\scriptstyle H(g)} & \\ & H(E), & \end{array}$$

where ∂^* is homogeneous of degree $(-1, 0)$.

Proof. Note that, for each $s \in S$, A.3.1 restricts to a short exact sequence:

$$0 \longrightarrow C_s \longrightarrow D_s \longrightarrow E_s \longrightarrow 0.$$

Applying to this short exact sequence for each $s \in S$ completes the proof. □

Definition A.3.6. Let $f : C \to D$ be a bigraded chain map. Then f is a *quasi-isomorphism* if the induced map $H(f)$ is an isomorphism of L-modules. Note that because the map is bigraded, all of the restrictions $H(f)|_{H_{r,s}(C)}$ are also isomorphisms.

In general, quasi-isomorphisms are sufficient to study bigraded chain complexes. However, in practise it can sometimes be difficult to directly prove a map is a quasi-isomorphism, so we introduce some homotopy theory of chain complexes. We start with the basic definition of homotopy, and then build up to homotopy equivalence and cones in the usual manner.

Definition A.3.7. Let $f, g : C \to D$ be chain maps who are both homogeneous of degree (m, n). Then we say f is *chain homotopic* (or more often just *homotopic*) to g if there exists an L-homomorphism $h : C \to D$ which is homogeneous of degree $(m + 1, n)$ and satisfies:

$$f - g = \partial_D H + H\partial_C.$$

If f is homotopic to g we write $f \simeq g$. Upon inspection of the right hand side of the equation, it is immediate that this means $H(f) - H(g) = 0$, as $\partial_D H \subset \mathrm{im}\partial_D$ and if $x \in \ker(\partial_C)$, then $H\partial_C(x) = 0$. So if two maps are chain homotopic, they induce the same maps on homology.

This suggests the following definition.

Definition A.3.8. Let C and D be bigraded chain complexes. Suppose $f : C \to D$ and $g : D \to C$ are bigraded chain maps with:

$$gf \simeq 1_C,$$

and,

$$fg \simeq 1_D.$$

Then we say C is *homotopy equivalent* to D.

Keeping the remarks above in mind, $gf \simeq 1_C$ and $fg \simeq 1_D$ imply that, on the level of homology f and g are true inverses for oneanother. Thus if C is homotopy equivalent to D, it is immediate that C is quasi-isomorphic to D. Finally, we introduce a tool which allows us to study homogeneous chain maps $f : C \to D$ as chain complexes.

Definition A.3.9. Let C, D be bigraded chain complexes. Let $f : C \to D$ be a chain map which is homogeneous of degree (m, n). The *mapping cone* of f, denoted $Cone(f)$ or $Cone(f : C \to D)$ (depending on how much context we need), is the bigraded chain complex $(Cone(f), \partial_f)$, where:

- $Cone(f) = C \oplus D$.
- $\partial_f(c, d) = (-\partial_C(c), \partial_D(d) + f(c))$.
- The bigrading is given by:

$$Cone(f)_{r,s} = C_{r-m-1,s-n} \oplus D_{r,s}.$$

Note that the bigrading condition is chosen to ensure that ∂_f is homogeneous of degree $(-1, 0)$.

Associated to a cone, we have a short exact sequence:

$$0 \longrightarrow D \overset{i}{\rightarrowtail} Cone(f) \overset{p}{\twoheadrightarrow} C \longrightarrow 0$$

Where i is the canonical inclusion of D into $Cone(f)$ and p is the canonical projection of $Cone(f)$ onto C. These are all homogenous chain maps, so splitting **??** along the bigrading and applying the bigraded snake lemma, we obtain the long exact sequence:

$$\cdots \longrightarrow H_{r+1,s}(D) \xrightarrow{H(i)} H_{r+1,s}(Cone(f)) \xrightarrow{H(p)} H_{r-m,s-n}(C) \xrightarrow{H(f)} H_{r,s}(D) \longrightarrow \cdots.$$

Or, if the reader prefers it, an exact triangle:

$$\begin{array}{ccc} H(C) & \xrightarrow{H(f)} & H(D) \\ & {\scriptstyle H(p)}\nwarrow \quad \swarrow{\scriptstyle H(i)} & \\ & H(Cone(f)). & \end{array}$$

From exactness, it is immediate that $H(Cone(f)) = 0$ if and only if f is a quasi-isomorphism. We enshrine this in a lemma.

Lemma A.3.2. A bigraded chain map $f : C \to D$ is a quasi-isomorphism if and only if $H(Cone(f)) = 0$.

The long exact sequence of a cone also proves the following lemma.

Lemma A.3.3. If $f : C \to D$ is an injective, homogeneous chain map, then $Cone(f)$ is quasi-isomorphic to $D/f(C)$.

Proof. This is proven in [27, Lemma A.3.9]. The basic idea is to form the short exact sequence:

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{q} D/f(C) \longrightarrow 0.$$

Then one can apply A.3.1 to form a long exact sequence and relate that to the long exact sequence of $Cone(f)$. □

Definition A.3.10. Let (C, ∂_C) and (D, ∂_D) be a pair of bigraded chain complexes. Then we form their *tensor product* of L by the tuple $(C \otimes_L D, \partial_C \otimes_L 1 + 1 \otimes_L \partial_D)$, and bigrading, for $r \in R$ and $s \in S$:

$$(C \otimes_L D)_r = \bigoplus_{\substack{s_1+s_2=r \\ l_1+l_2=s}} (C_{s_1,l_1} \otimes_L D_{s_2,l_2}).$$

Definition A.3.11. Let (C, ∂) be a bigraded chain complex over L. Let $\mathbb{K}$ be a field which acts on L. Let $\mathcal{W}$ be a bigraded $\mathbb{K}$-space. Then the tensor product of C and $\mathcal{W}$ is given by the tuple $(C \otimes_{\mathbb{K}} \mathcal{W}, \partial \otimes_{\mathbb{K}} 1)$ and for $r \in R$, $s \in S$:

$$(C \otimes_{\mathbb{K}} \mathcal{W})_r = \bigoplus_{\substack{s+l=r \\ p+q=s}} C_{s+p} \otimes_{\mathbb{K}} \mathcal{W}_{l+q}.$$

Definition A.3.12. Let $Q \leq L$ be a subring. Let (C, ∂) be an R, S-bigraded chain complex over Q. Moreover, assume L admits a bigrading $\bigoplus_{r \in R} L_r$ so that, if $q \in Q \cap L_{r,s}$, $a \in R$ and $b \in S$:

$$qC_{a,b} \subset C_{a+r,b+s}.$$

Then define the tensor product of C and L is given by the tuple $(C \otimes_Q L, \partial \otimes 1)$ and bigrading:

$$(C \otimes_Q L)_{r,s} = \bigoplus_{\substack{p+q=r \\ a+b=s}} C_{p,a} \otimes_Q L_{q,b}.$$

A.4 Graded, Filtered Chain Complexes

As above, we begin this section by fixing some notation. Let L and $\mathcal{R}$ be unitary rings, and $(P, \leq)$ be a partially ordered ring. L will be our coefficient ring, $\mathcal{R}$ will record the homological grading, and $(P, \leq)$ will induce a filtration. To place a P-filtration on a chain complex (and work with it in the finitely generated case) we need the following definitions.

Definition A.4.1. Let $(P, \leq)$ be a partially ordered set. We say $C \subset P$ is a *closed region* if it id downwardly closed with respect to $\leq$. That is, if $x \in C$, and $y \leq x$, then $y \in C$. We collect these subsets into $CR(P)$.

When doing computations, generally we will only be able to deal with a particular type of closed region.

Definition A.4.2. Let $(P, \leq)$ be a partial order, and $p \in P$. Then the *simple region* corresponding to p is the closed region $\{q \in P : q \leq p\}$. We denote this set R_p. Clearly $R_p \in CR(P)$.

Definition A.4.3. Let $(P, \leq)$ be a partial order, and R be a closed region. We say R is *semi-simple* if thereis some finite set $\{p_1, \ldots, p_n\} \subset P$ for which $R = R_{p_1} \cup R_{p_2} \cup \cdots \cup R_{p_n}$.

These are in a certain sense the *computable* closed regions. We are now prepared to discuss chain complexes.

Definition A.4.4. An *$\mathcal{R}$-graded P-filtered chain complex* over L is a tuple $(C, \partial, \{\mathcal{F}_R\}_{R \in CR(P)})$ which satisfies the following.

1. C is an L-module.
2. $\partial : C \to C$ is an L-homomorphism which squares to 0.
3. C splits as a direct product $\bigoplus_{r \in \mathcal{R}} C_r$, we call this the *homological grading*.
4. $\{\mathcal{F}_R\}_{R \in CR(P)}$ is a collection of L-submodules of C for which $\bigcup_{R \in CR(P)} \mathcal{F}_R C = C$ and $\bigcap_{R \in CR(P)} \mathcal{F}_R C = \emptyset$. That is, the P-filtration *exhausts* C.
5. If $R, S \in CR(P)$ with $R \subset P$, then $\mathcal{F}_R C \subset \mathcal{F}_S C$.
6. The differential ∂ is compatible with the grading and the filtration. That is, if $\mathcal{F}_R C_r = C_r \cap \mathcal{F}_R C$, then for all $r \in \mathcal{R}$, and $R \in CR(P)$, then $\partial(\mathcal{R} C_r) \subset \mathcal{F}_R C_{r-1}$.

We will often just refer to these as graded, filtered chain complexes or even filtered complexes, if the various rings are clear from context.

Definition A.4.5. Let $(C, \partial, \{\mathcal{F}_R C\}_{R \in CR(P)})$ be a graded, filtered chain complex. Then the *homology* of this chain complex is the graded, filtered L-module $H(C) := \ker(\partial)/\mathrm{im}(\partial)$, with grading and filtration given by:

- $H_r(C) := \ker(\partial|_{C_r})/\mathrm{im}(\partial|_{C_{r+1}})$.
- $\mathcal{F}_R H(C) := \ker(\partial|_{\mathcal{F}_R C})/\mathrm{im}(\partial|_{\mathcal{F}_R C})$.

Graded, filtered chain complexes admit very similar morphisms to bigraded chain complexes. The main difference between the two structures is that quasi-isomorphisms are much weaker in the filtered setting.

Definition A.4.6. Let C and D be filtered chain complexes, and $f : C \to D$ be an L-homomorphism. Then f is a *chain map* if the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\partial_C} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{\partial_D} & D. \end{array}$$

If, for each $r \in \mathcal{R}$, $f(C_r) \subset D_r$, then we say f is *graded.* If, for each $R \in CR(P)$, $f(\mathcal{F}_R C) \subset \mathcal{F}_R D$, then we say f is *filtered.*

Note that any chain map f induces an L-homomorphism on homology denoted $H(f)$ or f_*. Moreover, this assignment is functorial. Up until not, we have not used the ring structure on P. This is nessecary for the following definition.

Definition A.4.7. Let $f : C \to D$ be a chain map of filtered complexes. Let $(m, n) \in \mathcal{R} \times P$. Then f is *homogeneous of degree* (m, n) if, for each $R \in CR(P)$ and $r \in \mathcal{R}$, we have:

$$f(\mathcal{F}_R C_r) \subset \mathcal{F}_{R+m} D_{r+n}.$$

Definition A.4.8. A *short exact sequence of filtered chain complexes* is a short exact sequence of chain complexes:

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0.$$

Where C, D, E are filtered complexes and f, g are graded and filtered.

As in the bigraded setting, filtration and grading shifts can be used to extend this to homogeneous chain maps.

Lemma A.4.1 (Filtered Snake Lemma)**.** Suppose we have a short exact sequence of filtered chain complexes:

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0.$$

Then for each $r \in \mathcal{R}$ and each $R \in CR(P)$, there is a long exact sequence:

$$\cdots \xrightarrow{\mathcal{F}_R \partial^*_{r_1}} H_r(\mathcal{F}_R C) \xrightarrow{\mathcal{F}_R H_r(f)} H_r(\mathcal{F}_R D) \xrightarrow{\mathcal{F}_R H_r(g)} H_r(\mathcal{F}_R E) \xrightarrow{\mathcal{F}_R \partial^*_r} H_{r-1}(\mathcal{F}_R C) \longrightarrow \cdots.$$

Or, equivalently, an exact triangle:

$$\begin{array}{ccc} H(C) & \xrightarrow{\quad H(f) \quad} & H(D) \\ & {\scriptstyle \partial^*} \nwarrow \quad \swarrow {\scriptstyle H(g)} & \\ & H(E), & \end{array}$$

Where ∂^* is homogeneous of degree $(-1, 0)$.

Proof. As in the bigraded setting, for each $R \in CR(P)$, A.4.1 restricts to:

$$0 \longrightarrow \mathcal{F}_R C \longrightarrow \mathcal{F}_R D \longrightarrow \mathcal{F}_R E \longrightarrow 0.$$

Applying A.3 to each of these short exact sequences completes the proof. □

Having defined the various types of morphisms between filtered chain complexes, we now discuss a few types of equivalence, along the lines of bigraded chain complexes.

Definition A.4.9. Let $f : C \to D$ be a filtered, graded chain map. We say that f is a *quasi-isomorphism* if the induced map $H(f) : H(C) \to H(D)$ is an isomorphism of L-modules.

As we noted above, quasi-isomorphisms are much weaker in the filtered setting than they are in the graded setting. In the graded setting, a quasi-isomorphism $f : C \to D$ of bigraded complexes induces isomorphisms on each $C_{r,s}$. This is not the case if C and D are filtered.

Example A.4.1. Let $L = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{R} = L$, and $P = \mathbb{Z}$. In a linear order like $\mathbb{Z}$, closed regions are determined by their unique maximum element. Thus we form the triple:

$$(\mathbb{Z}^\infty, 0, \{\mathcal{F}_n\mathbb{Z}^\infty\}_{n\in\mathbb{Z}}),$$

which has trivial homological grading $\mathbb{Z}^\infty_0 = \mathbb{Z}^\infty$, and filtration given by:

$$\mathcal{F}_n\mathbb{Z}^\infty := \{x \in \bigoplus_{i\in\mathbb{Z}} Z : x_i = 0 \text{ for } i \leq n\}\}.$$

Then if we define the shift map $f : \mathbb{Z}^\infty \to \mathbb{Z}$ by $f((x_n)) = (x_{n-1})$. This is clearly filtered, and is an ismorphism of chain complexes, so a quasi-isomorphism. However, f does not map $\mathcal{F}_n\mathbb{Z}^\infty$ surjectively onto itself in $\mathbb{Z}^\infty$, so f does not induce isomorphisms on filtration levels.

One way to solve this is to work with *graded objects*, but for our purposes these will be difficult to work with. Fortunately, homotopy theory in this settining in nearly identical to the bigraded setting.

Definition A.4.10. Let C, D be filtered chain complexes. A pair of chain maps $f, g : C \to D$ which are homogeneous of degree (m, n) are *filtered chain homotopic* if there exists an L-homomorphism $H : C \to D$ which is homogeneous of degree $(m + 1, n)$, and satisfies the following:

$$f - g = \partial_D H + H\partial_C. \tag{A.1}$$

We call H a *filtered chain homotopy* from f to g, and if such an H exists we write $f \simeq g$. As noted above, if $f \simeq g$ then $H(f) = H(g)$.

Because A.1 is checked pointwise, A.1 restricts to, for $R \in CR(P)$ and $r \in \mathcal{R}$:

$$f|_{\mathcal{F}_R C_r} - g|_{\mathcal{F}_R C_r} = \partial_D H|_{\mathcal{F}_R C_r} + H\partial_C|_{\mathcal{F}_R C_r}.$$

Hence $H(f||_{\mathcal{F}_R C_r}) = H(g|_{\mathcal{F}_R C_r})$. This suggests chain homotopies are the correct way to think about equivalence for filtered chain complexes.

Definition A.4.11. Let C, D be filtered chain complexes. If there are graded, filtered chain maps $f : C \to D$ and $g : D \to C$ for which $gf \simeq= 1_C$ and $fg \simeq 1_D$, then we say C is *filtered homotopy equivalent* to D.

Clearly if C is filtered homotopy equivalent to D, for each $R \in CR(P)$ and $r \in \mathcal{R}$, $H(\mathcal{F}_R C_r) \cong H(\mathcal{F}_R D_r)$. Next, we discuss mapping cones in the filtered setting.

Definition A.4.12. Let C, D be filtered chain complexes, and $f : C \to D$ be a chain map which is homogeneous of degree (m, n). Then the *filtered mapping cone* or just *mapping cone* of f, denoted $Cone(f)$, is specified by the triple:

$$(Cone(f), \partial_f, \{\mathcal{F}_R Cone(f)\}_{R\in CR(P)}).$$

The terms in this triple are given by:

1. $Cone(f) = C \oplus D$.
2. $\partial_f(c, d) = (\partial_C(c), \partial_D(d) + f(c))$.
3. $Cone(f)_r := C_{r-m-1} \oplus D_r$.
4. $\mathcal{F}_R Cone(f) = \mathcal{F}_{R-n} \oplus \mathcal{F}_R D$.

It is straightforward to check that this specifies a graded, filtered chain complex. We have analogues of A.3.2 and A.3.3 for filtered chain complexes.

Lemma A.4.2. Let C, D be filtered chain complexes, and $f : C \to D$ be a graded, filtered chain map. If $H(Cone(f)) = 0$ then f is a quasi-isomorphism.

Proof. The argument is formally similar to the way we proved A.3.2. Form the same short exact sequence, note all the maps are filtered, and apply A.4.1. □

Note that there is a slight deviation from A.3.2 here. For each $R \in CRP$, A.3 restricts to:

$$0 \longrightarrow \mathcal{F}_R D \rightarrowtail \mathcal{F}_R Cone(f) \twoheadrightarrow \mathcal{F}_R C \longrightarrow 0.$$

So if $H(Cone(f)) = 0$, then each $f|_{\mathcal{F}_R C}$ is a quasi-isomorphism. Thus, A.4.1 confirms that the converse to this lemma does not hold.

Lemma A.4.3. Let $f : C \to D$ be a filtered, graded, injective chain map. Then there exists a filtered quasi-isomorphism from $Cone(f)$ to $D/f(C)$.

Proof. Again, this is formally equivalent to the proof of A.3.3. The only additional thing to check is that all morphisms are graded and filtered, which is immediate. □

Definition A.4.13. Let (C, ∂_C) and (D, ∂_D) be a pair of filtered chain complexes. Then we form their *tensor product* over L by doing this for the underlying graded complexes, and then extending the filtration by setting:

$$\mathcal{F}_R(C \otimes_L D) = \sum_{\substack{R_1, R_2 \in CR(P) \\ R_1 \cup R_2 = R}} (\mathcal{F}_{R_1} C) \otimes_L (\mathcal{F}_{R_2} D).$$

Definition A.4.14. Let (C, ∂) be a filtered chain complex over L. Let $\mathbb{K}$ be a field which acts on L. Let $\mathcal{W}$ be an $\mathcal{R}, P$-bigraded $\mathbb{K}$-space. Then the tensor product of C and $\mathcal{W}$ is constructed by first taking the tensor product of the underlying graded chain complex and vector space, and then extending the filtration by setting:

$$\mathcal{F}_R(C \otimes_L \mathcal{W}) = \sum_{\substack{R' \in CR(P), p \in P \\ R' + p = R}} (\mathcal{F}_{R'} C) \otimes (\mathcal{W}_p).$$

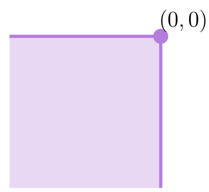
Definition A.4.15. Let $Q \leq L$ be a subring. Let (C, ∂) be an $\mathcal{R}$-graded, P-filtered chain complex over Q. Moreover, assume L admits a bigrading $\bigoplus_{\substack{r \in \mathcal{R} \\ p \in P}} L_{r,p}$ so that, if $q \in Q \cap L_{r,p}$, $s \in \mathcal{R}$ and $R \in CR(P)$:

$$q \mathcal{F}_R C_s \subset \mathcal{F}_{R+p} C_{s+r}.$$

Define the tensor product of C and L over Q by doing this for the underlying graded complex and ring, and then extending the filtration by setting:

$$\mathcal{F}_R(C \otimes_Q L) = \sum_{\substack{S \in CR(P), p \in P \\ p+S \subset R}} \mathcal{F}_S C \otimes_Q L_p.$$

We finish off this chapter by discussing the case where $P = \mathbb{Z}^2$. In this case, the closed regions look like unions of the following set, translated around the plane.



Moreover, we can work interchangeably with our $\mathbb{Z}^2$-filtration, or with a pair of $\mathbb{Z}$-filtrations. For $k \in \mathbb{Z}$, define closed regions $\{i \leq k\} := \{(i,j) \in \mathbb{Z}^2\}$ and $\{j \leq k\} := \{(i,j) \in \mathbb{Z}^2 : j \leq k\}$. Then we can use the to define $\mathbb{Z}$-filtrations $\mathcal{F}^1_n C := \mathcal{F}_{\{i \leq n\}} C$ and $\mathcal{F}^2_n C := \mathcal{F}_{\{j \leq 0\}} C$. Moreover, we can recover the $\mathbb{Z}^2$-filtration by setting, for each $R \in CR(\mathbb{Z}^2)$:

$$\mathcal{F}_R C = \sum_{(i,j) \in R} \mathcal{F}^1_i C \cap \mathcal{F}^2_j C.$$

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